



A simple proof for the existence of V-form fronts in reaction–diffusion equations

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Received: 17 January 2026 / Accepted: 22 September 2026
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Abstract

This paper studies V-form traveling fronts to a bistable reaction–diffusion equation. V-form traveling fronts have been studied so far for many reaction–diffusion equations. The proof of its existence is often based on the multi-scale method. In this paper we present a new simple proof for the existence of V-form traveling fronts based on estimates by inscribed balls.

Keywords Traveling front · V-form · Reaction–diffusion equation

Mathematics Subject Classification Primary 35C07; Secondary 35K57

1 Introduction and the main result

In this paper we study a reaction–diffusion equation

$$u_t = \Delta u + f(u), \quad (x, y, t) \in \mathbb{R}^2 \times (0, \infty), \quad (1)$$

$$u(x, y, 0) = u_0(x, y), \quad (x, y) \in \mathbb{R}^2. \quad (2)$$

We set $u_t = \partial u / \partial t$ and $\Delta = \partial^2 / \partial x^2 + \partial^2 / \partial y^2$. Here, $f \in C^2[0, 1]$ satisfies

$$f(0) = 0, \quad f(1) = 0, \quad f'(0) < 0, \quad f'(1) < 0, \quad (3)$$

$$\int_0^1 f(u) \, du > 0 \quad (4)$$

In honor of Professor Hiroshi Matano.

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and

$$\begin{aligned} f(u) &< 0 & \text{if } u \in (0, a_0), \\ f(u) &> 0 & \text{if } u \in (a_0, 1) \end{aligned}$$

for $a_0 \in (0, 1)$. This f is called a bistable reaction term. Now we extend f as a function of class C^2 on an open interval including $[0, 1]$. Now u_0 is a given bounded and uniformly continuous function from $\mathbb{R}^2$ to $\mathbb{R}$. We write the solution of (1)–(2) as $u(x, y, t; u_0)$. Under the assumptions stated above, there exists $(k, \Phi) \in (0, \infty) \times C^2(\mathbb{R})$ that satisfies

$$\Phi''(\mu) + k\Phi'(\mu) + f(\Phi(\mu)) = 0, \quad \mu \in \mathbb{R}, \quad (5)$$

$$0 < \Phi(\mu) < 1, \quad \mu \in \mathbb{R} \quad (6)$$

with

$$\begin{aligned} \Phi(-\infty) &= 1, \quad \Phi(\infty) = 0, \\ \Phi'(\mu) &< 0, \quad \mu \in \mathbb{R}. \end{aligned}$$

Now we put

$$\mathcal{W}(u) = - \int_0^u f(v) \, dv,$$

and choose $\theta_0 \in (a_0, 1)$ with $\mathcal{W}(\theta_0) > 0$ and choose $s_0 \in (0, a_0)$. We assume $\Phi(0) = \theta_0$ without loss of generality. We denote the solution of (1) and (2) by $u(x, t; u_0)$. This equation is called the Allen–Cahn equation, the scalar Ginzburg–Landau equation or the Nagumo equation. Now $\Phi(y - kt)$ is a planar traveling front solution to (1). For traveling fronts to (1) in general dimensions have been studied by many mathematicians. Among them, the simplest non-planar traveling front is a V-form traveling front. See [5–9] for instance. In this paper we present a simple new proof of the V-form traveling front. Let $c \in (k, \infty)$ be arbitrarily chosen. We put $z = y - ct$ and

$$w(x, z, t) = u(x, z + ct, t), \quad (x, z, t) \in \mathbb{R}^2 \times (0, \infty).$$

Then we have

$$w_t = w_{xx} + w_{zz} + cw_z + f(w), \quad (x, z, t) \in \mathbb{R}^2 \times (0, \infty),$$

$$w(x, z, 0) = u_0(x, z), \quad (x, z) \in \mathbb{R}^2.$$

We write the solution as $w(x, z, t; u_0)$. Hereafter we write z simply as y , and write $w(x, z, t; u_0)$ as $w(x, y, t; u_0)$. Now we have

$$w_t = \Delta w + cw_y + f(w), \quad (x, y, t) \in \mathbb{R}^2 \times (0, \infty), \quad (7)$$

$$w(x, y, 0) = u_0(x, y), \quad (x, y) \in \mathbb{R}^2. \quad (8)$$

The profile equation of (c, v) is given by

$$\Delta v + cv_y + f(v) = 0, \quad (x, y) \in \mathbb{R}^2.$$

Now we define

$$m_* = \frac{\sqrt{c^2 - k^2}}{k} \quad (9)$$

and

$$v_0(x, y) = \Phi\left(\frac{k}{c}(y - m_*|x|)\right) = \max\left\{\Phi\left(\frac{k}{c}(y - m_*x)\right), \Phi\left(\frac{k}{c}(y + m_*|x|)\right)\right\} \quad (10)$$

for $(x, y) \in \mathbb{R}^2$. We define $\Theta \in (0, \pi/2)$ by

$$m_* = \tan \Theta$$

and have $c \cos \Theta = k$. Now we state V-form traveling fronts to (1) given by [8, 9, 13] as follows. See also [3–7, 10].

Theorem 1.1 *Let $c \in (k, \infty)$ be arbitrarily given and let v_0 be given by (10). There exists $V(x, y)$ such that one has*

$$\Delta V + cV_y + f(V) = 0, \quad (x, y) \in \mathbb{R}^2, \quad (11)$$

$$0 < V(x, y) < 1, \quad (x, y) \in \mathbb{R}^2, \quad (12)$$

$$V(-x, y) = V(x, y), \quad (x, y) \in \mathbb{R}^2, \quad (13)$$

$$V_x(x, y) > 0 \quad \text{if } (x, y) \in (0, \infty) \times \mathbb{R}, \quad (14)$$

$$(\nabla V(x, y), \mathbf{t}) < 0, \quad (x, y) \in \mathbb{R}^2, \quad (15)$$

where $\mathbf{t}$ is a unit vector with

$$\mathbf{t} = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}, \quad t_2 > 0, \quad m_*|t_1| \leq t_2. \quad (16)$$

Let $G(x; \theta_0)$ be defined by $V(x, G(x; \theta_0)) = \theta_0$ for $x \in \mathbb{R}$. Then one has

$$\lim_{A \rightarrow \infty} \sup_{(x, y) \in \Omega_0} \left| V(x + A, y + G(A; \theta_0)) - \Phi\left(\frac{k}{c}(y - m_*x)\right) \right| = 0, \quad (17)$$

where Ω_0 is any given compact set in $\mathbb{R}^2$.

The proof of this theorem has been carried out by the multi-scale methods in [8, 9, 13]. In this paper, we present a simple proof for this theorem based on the estimates by inscribed balls. Now we define a positive constant

$$\beta = \frac{1}{2} \min \{-f'(0), -f'(1)\},$$

and choose $\delta_* \in (0, 1/4)$ so small such that we have

$$-f'(u) \geq \beta \quad \text{if } u \in [-2\delta_*, 2\delta_*] \cup [1 - 2\delta_*, 1 + 2\delta_*].$$

We define

$$M = \max_{u \in [-2\delta_*, 1+2\delta_*]} |f'(u)|. \quad (18)$$

We always assume

$$-\delta_* \leq u_0(x, y) \leq 1 + \delta_*, \quad (x, y) \in \mathbb{R}^2. \quad (19)$$

Then we have

$$-\delta_* \leq u(x, y, t; u_0) \leq 1 + \delta_*, \quad (x, y, t) \in \mathbb{R}^2 \times (0, \infty),$$

$$\sup_{(x, y, t) \in \mathbb{R}^2 \times [1, \infty)} \max\{|u_t(x, y, t; u_0)|, |u_x(x, y, t; u_0)|, |u_y(x, y, t; u_0)|\} < \infty, \quad (20)$$

$$\sup_{(x, y, t) \in \mathbb{R}^2 \times [1, \infty)} \max\{|u_{xx}(x, y, t; u_0)|, |u_{xy}(x, y, t; u_0)|, |u_{yy}(x, y, t; u_0)|\} < \infty. \quad (21)$$

For the proof of the Schauder estimates (20) and (21), see [13]. For the one-dimensional case, (1) and (2) become

$$u_t = u_{xx} + f(u), \quad (x, t) \in \mathbb{R} \times (0, \infty), \quad (22)$$

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}. \quad (23)$$

Here u_0 is a given bounded uniformly continuous function from $\mathbb{R}$ to $\mathbb{R}$. We write the solution of (22)–(23) as $u(x, t; u_0)$. The one-dimensional traveling front $\Phi(x - kt)$ to (22)–(23) is asymptotically stable for an initial perturbation. That is, if

$$\begin{aligned} -\delta_* \leq u_0(x) \leq 1 + \delta_*, \quad x \in \mathbb{R}, \\ \limsup_{x \rightarrow \infty} u_0(x) \leq s_0, \quad \theta_0 \leq \liminf_{x \rightarrow -\infty} u_0(x), \end{aligned}$$

one has, for any given $\varepsilon > 0$,

$$\sup_{x \in \mathbb{R}} |u(x, t; u_0) - \Phi(x - kt - \gamma_0)| < \varepsilon \quad (24)$$

for sufficiently large $t \in (0, \infty)$. Here $\gamma_0 \in \mathbb{R}$ is a constant. See [1, 2, 13] for the proof.

Let $c \in (k, \infty)$ be arbitrarily given and let $\Theta \in (0, \pi/2)$ be defined by $c \cos \Theta = k$. Define v_0 by (10). We write $w(x, y, t; v_0)$ simply as $w(x, y, t)$, and have

$$0 < w(x, y, t) < 1, \quad (x, y, t) \in \mathbb{R}^2 \times (0, \infty). \quad (25)$$

Now we have

$$w(-x, y, t) = w(x, y, t), \quad (x, y, t) \in \mathbb{R}^2 \times (0, \infty) \quad (26)$$

from

$$v_0(-x, y) = v_0(x, y), \quad (x, y) \in \mathbb{R}^2.$$

Also we have

$$w_t(x, y, t) > 0, \quad (x, y, t) \in \mathbb{R}^2 \times (0, \infty). \quad (27)$$

Indeed, putting

$$v_{\text{rgh}}(x, y) = \Phi\left(\frac{k}{c}(y - m_*x)\right), \quad v_{\text{lft}}(x, y) = \Phi\left(\frac{k}{c}(y + m_*x)\right), \quad (x, y) \in \mathbb{R}^2, \quad (28)$$

we have

$$v_0(x, y) = \max \{v_{\text{rgh}}(x, y), v_{\text{lft}}(x, y)\}, \quad (x, y) \in \mathbb{R}^2.$$

Then, using

$$v_{\text{rgh}}(x, y) \leq v_0(x, y), \quad v_{\text{lft}}(x, y) \leq v_0(x, y), \quad (x, y) \in \mathbb{R}^2,$$

we have

$$v_{\text{rgh}}(x, y) \leq w(x, y, t; v_{\text{rgh}}) \leq w(x, y, t), \quad v_{\text{lft}}(x, y) \leq w(x, y, t; v_{\text{lft}}) \leq w(x, y, t)$$

for $(x, y, t) \in \mathbb{R}^2 \times (0, \infty)$. Thus we find

$$v_0(x, y) = \max \{v_{\text{rgh}}(x, y), v_{\text{lft}}(x, y)\} \leq w(x, y, t), \quad (x, y, t) \in \mathbb{R}^2 \times (0, \infty).$$

Now we have

$$w_t(x, y, t) \geq 0, \quad (x, y, t) \in \mathbb{R}^2 \times (0, \infty).$$

Since w_t does not vanish identically in $\mathbb{R}^2$, we obtain (27) by the strong maximum principle. By (20) and (27), there exists $N_0 \in (0, \infty)$ with

$$0 < w_t(x, y, t) \leq N_0, \quad (x, y, t) \in \mathbb{R}^2 \times [1, \infty). \quad (29)$$

Next we have

$$(\nabla w(x, y, t), \mathbf{t}) < 0, \quad (x, y, t) \in \mathbb{R}^2 \times (0, \infty), \quad (30)$$

where $\mathbf{t}$ is a unit vector with (16). Indeed, for every $\sigma \in (0, \infty)$, we have

$$v_0(\mathbf{x}) > v_0(\mathbf{x} + \sigma \mathbf{t}), \quad \mathbf{x} = (x, y) \in \mathbb{R}^2.$$

Then we have

$$w(\mathbf{x}, t; v_0) > w(\mathbf{x} + \sigma \mathbf{t}, t; v_0), \quad \mathbf{x} \in \mathbb{R}^2,$$

that is,

$$w(\mathbf{x} + \sigma \mathbf{t}, t; v_0) - w(\mathbf{x}, t; v_0) < 0, \quad \mathbf{x} \in \mathbb{R}^2,$$

Dividing the both hand sides by $\sigma \in (0, \infty)$ and sending $\sigma \rightarrow 0$, we have

$$\frac{\partial w}{\partial \mathbf{t}}(\mathbf{x}, t; v_0) \leq 0, \quad \mathbf{x} \in \mathbb{R}^2.$$

Since $\partial w / \partial \mathbf{t}$ does not vanish identically in $\mathbb{R}^2$, we obtain (30) from the strong maximum principle. By (30), we have

$$-\frac{\nabla w}{|\nabla w|} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \geq \cos \Theta \quad (x, y, t) \in \mathbb{R}^2 \times (0, \infty)$$

and thus

$$|\nabla w(x, y, t)| \cos \Theta \leq -w_y(x, y, t) \leq |\nabla w(x, y, t)|, \quad (x, y, t) \in \mathbb{R}^2 \times (0, \infty). \quad (31)$$

Here “ $\cdot$ ” means the inner product in $\mathbb{R}^2$. Using (27) and the definition of v_0 , we have

$$\lim_{R \rightarrow \infty} \inf_{(x, t) \in \mathbb{R} \times (0, \infty)} w(x, m_* |x| - R, t) = 1. \quad (32)$$

For any $(\mathbf{x}_0, R) \in \mathbb{R}^2 \times (0, \infty)$, we define $\overline{B(\mathbf{x}_0; R)} = \{\mathbf{x} \in \mathbb{R}^2 \mid |\mathbf{x} - \mathbf{x}_0| \leq R\}$. Now we introduce a useful lemma as follows.

Lemma 1.2 [13, Theorem 5.8] *Let $R \in (0, \infty)$, $T \in (0, \infty)$, $\varepsilon_1 \in (0, 1)$ and $\mathbf{x}_0 \in \mathbb{R}^2$ be arbitrarily given. Let u_0^1 and u_0^2 be bounded and uniformly continuous functions from $\mathbb{R}^2$ to $\mathbb{R}$ that satisfy (19) and*

$$\sup_{\mathbf{x} \in \overline{B(\mathbf{x}_0; 2R)}} |v_2(\mathbf{x}) - v_1(\mathbf{x})| < \varepsilon_1,$$

where $\mathbf{x} = (x, y)$. Then one has

$$\sup_{\mathbf{x} \in \overline{B(\mathbf{x}_0; R)}} |u(\mathbf{x}, T; u_0^2) - u(\mathbf{x}, T; u_0^1)| \leq \varepsilon_1 e^{M_1 T} + 4k_0 e^{M_1 T} \exp\left(-\frac{R^2}{8T}\right).$$

Here k_0 and M_1 are positive constants that depend only on f and are independent of $(R, T, \varepsilon_1, \mathbf{x}_0)$.

See [13] for the proof. Since 0 is an equilibrium solution of (1), we have

$$u(x, y, t; 0) = 0, \quad (x, y, t) \in \mathbb{R}^2 \times (0, \infty).$$

Applying Lemma 1.2 for $u_0^1 = 0$ and $u_0^2 = v_0$, we obtain, for every fixed $t \in (0, \infty)$,

$$\lim_{R \rightarrow \infty} \left(\sup_{x \in \mathbb{R}} u(x, m_*|x| - R, t; v_0) \right) = 1.$$

Using

$$w(x, y - ct, t) = u(x, y, t; v_0), \quad (x, y, t) \in \mathbb{R}^2 \times (0, \infty),$$

we have, for every fixed $t \in (0, \infty)$,

$$\lim_{R \rightarrow \infty} \left(\sup_{x \in \mathbb{R}} w(x, m_*|x| + R, t) \right) = 0. \quad (33)$$

By the Schauder estimate [13, Lemma 2.6], there exists a positive constant B_0 such that

$$\sup_{(x, y, t) \in \mathbb{R}^2 \times [1, \infty)} |\nabla w(x, y, t)| < B_0 \quad (34)$$

holds true. For every $\theta \in (0, \theta_0]$, we choose a positive number $R = R_\theta$ with

$$\mathcal{W}(\theta)R = 1 + B_0\theta. \quad (35)$$

Let $\theta \in (0, 1)$ be arbitrarily given. Using (30), (32) and (33), we define $g(x, t; \theta) \in \mathbb{R}$ by

$$w(x, g(x, t; \theta), t) = \theta \quad (36)$$

for $(x, t) \in \mathbb{R} \times (0, \infty)$. Then $g(x, t; \theta)$ is twice continuously differentiable in $x \in \mathbb{R}$ and $t \in (0, \infty)$ with

$$\begin{aligned} g(-x, t; \theta) &= g(x, t; \theta), & (x, t, \theta) &\in \mathbb{R} \times (0, \infty) \times (0, 1), \\ 0 &\leq g_x(x, t; \theta) < m_*, & (x, t, \theta) &\in [0, \infty) \times (0, \infty) \times (0, 1), \end{aligned} \quad (37)$$

$$m_*|x| < g(x, t; \theta_0), \quad (x, t) \in \mathbb{R} \times (0, \infty). \quad (38)$$

Indeed, we have the first equality from (26). Since $w(x, y, t)$ satisfies

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - c \frac{\partial}{\partial y} - f'(w) \right) w_x = 0, \quad (x, y, t) \in [0, \infty) \times \mathbb{R} \times (0, \infty),$$

$$w_x(0, y, t) = 0, \quad (y, t) \in \mathbb{R} \times (0, \infty),$$

$$w_x(x, y, 0) = (v_0)_x(x, y, 0) > 0, \quad (x, y) \in (0, \infty) \times \mathbb{R},$$

we have

$$w_x(x, y, t) > 0, \quad (x, y, t) \in (0, \infty) \times \mathbb{R} \times (0, \infty). \quad (39)$$

Also we have (37) from (30) and (39), and we have the last inequality from (27) and the definition of v_0 .

The following proposition asserts the lower bound of $|\nabla w|$.

Proposition 1.3 *Let $c \in (k, \infty)$ be arbitrarily given and let v_0 be given by (10). Then, for $w(x, y, t) = w(x, y, t; v_0)$, one has*

$$\inf_{(x, t, \theta) \in \mathbb{R} \times [1, \infty) \times [\delta_*, \theta_0]} |\nabla w(x, g(x, t; \theta), t)| > 0. \quad (40)$$

We will state the proof of Proposition 1.3 in Sect. 4. Now we state the following useful lemma.

Lemma 1.4 *Let $c \in (k, \infty)$ be arbitrarily given and let v_0 be given by (10) and let $w(x, y, t) = w(x, y, t; v_0)$. For any $\mu \in (0, \pi/2 - \Theta]$, there exists $S_\mu \in (0, \infty)$ such that, if $w(\alpha_1, \beta_1, T_1) = \theta_0$ for $(\alpha_1, \beta_1, T_1) \in \mathbb{R}^2 \times [1, \infty)$ one has*

$$\sup_{\sigma \in [S_\mu, \infty)} w(\alpha_1 + \sigma t_1, \beta_1 + \sigma t_2, T_1) \leq s_0. \quad (41)$$

Here $t = (t_1, t_2)$ is an arbitrary unit vector with (16).

Proof This lemma follows from Proposition 1.3 and (30). $\square$

2 The Harnack inequality

Now we state a Harnack inequality that is often used in this paper with its proof for the sake of readers' convenience. Let $h(x, y, t)$ be a bounded and continuous function from $\mathbb{R}^2 \times (0, \infty)$ to $\mathbb{R}$ with

$$M_0 = \sup_{(x, y, t) \in \mathbb{R}^2 \times (0, \infty)} |h(x, y, t)| < \infty. \quad (42)$$

Let $W(x, y, t)$ satisfy

$$\left(\frac{\partial}{\partial t} - \Delta - c \frac{\partial}{\partial y} + h(x, y, t) \right) W(x, y, t) = 0, \quad (x, y, t) \in \mathbb{R}^2 \times (0, \infty), \quad (43)$$

$$W(x, y, 0) = W_0(x, y) \geq 0, \quad (x, y) \in \mathbb{R}^2. \quad (44)$$

Let W_0 be a nonnegative function of class $C^1(\mathbb{R}^2)$ with

$$\|\nabla W_0\|_{L^\infty(\mathbb{R}^2)} \leq v_0 \quad (45)$$

for $v_0 \in (0, \infty)$.

Lemma 2.1 *Let W_0 be a nonnegative function of class $C^1(\mathbb{R}^2)$ with (45) for $v_0 \in (0, \infty)$ and let $W(x, y, t)$ satisfy (43) and (44). If one has $W_0(x_0, y_0) > 0$ for $(x_0, y_0) \in \mathbb{R}^2$, one has*

$$W(x, y, t) \geq \frac{\mu_0^2 e^{-M_0 t}}{8t} W_0(x_0, y_0) \exp \left(-\frac{(|x-x_0|+\mu_0)^2 + (|y-y_0+ct|+\mu_0)^2}{4t} \right) > 0$$

for $(x, y, t) \in \mathbb{R}^2 \times (0, \infty)$. Here $\mu_0 = (2v_0)^{-1} W_0(x_0, y_0) > 0$.

Proof We define $\underline{w}$ by

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta - c \frac{\partial}{\partial y} + M_0 \right) \underline{w}(x, y, t) &= 0, \quad (x, y, t) \in \mathbb{R}^2 \times (0, \infty), \\ \underline{w}(x, y, 0) &= W_0(x, y), \quad (x, y) \in \mathbb{R}^2, \end{aligned}$$

that is,

$$\underline{w}(x, y, t) = \frac{e^{-M_0 t}}{4\pi t} \int_{\mathbb{R}^2} \exp \left(-\frac{(x-\xi)^2 + (y+ct-\eta)^2}{4t} \right) W_0(\xi, \eta) d\xi d\eta > 0 \quad (46)$$

for $(x, y, t) \in \mathbb{R}^2 \times (0, \infty)$. Now we have

$$0 < \underline{w}(x, y, t) \leq W(x, y, t), \quad (x, y, t) \in \mathbb{R}^2 \times (0, \infty). \quad (47)$$

Indeed, we have

$$\begin{aligned} & \left(\frac{\partial}{\partial t} - \Delta - c \frac{\partial}{\partial y} + h(x, y, t) \right) (\underline{w}(x, y, t) - W(x, y, t)) \\ &= -(M_0 - h(x, y, t)) (\underline{w}(x, y, t) - W(x, y, t)) \leq 0, \quad (x, y, t) \in \mathbb{R}^2 \times (0, \infty), \\ & \underline{w}(x, y, 0) - W(x, y, 0) = 0, \quad (x, y) \in \mathbb{R}^2. \end{aligned}$$

Then we obtain (47) from the maximum principle. It suffices to assume $(x_0, y_0) = (0, 0)$. Since we have

$$W_0(x, y) \geq \frac{1}{2} W_0(0, 0), \quad \text{if } \sqrt{x^2 + y^2} \leq \mu_0,$$

we obtain

$$W(x, y, t) \geq \frac{e^{-M_0 t}}{4t} \exp \left(-\frac{(|x - x_0| + \mu_0)^2 + (|y - y_0 + ct| + \mu_0)^2}{4t} \right) \frac{1}{2} W_0(x_0, y_0) \pi \mu_0^2$$

for $(x, y, t) \in \mathbb{R}^2 \times (0, \infty)$. This completes the proof. $\square$

Now we state the following proposition.

Proposition 2.2 (the Harnack inequality) *Let W_0 be a nonnegative function of class $C^1(\mathbb{R}^2)$ with (45) for $v_0 \in (0, \infty)$ and let $W(x, y, t)$ satisfy (43) and (44) with*

$$\sup_{(x, y, t) \in \mathbb{R}^2 \times [0, \infty)} \|\nabla W_0\|_{L^\infty(\mathbb{R}^2)} \leq v_0.$$

If one has

$$\lim_{i \rightarrow \infty} W(x_i, y_i, t_i) = 0$$

for $(x_i, y_i, t_i) \in \mathbb{R}^2 \times (0, \infty)$ with $\lim_{i \rightarrow \infty} t_i = \infty$, one has

$$\lim_{i \rightarrow \infty} W(x + x_i, y + y_i, t + t_i) = 0$$

uniformly in (x, y, t) on every compact set in $\mathbb{R}^3$.

Proof Lemma 2.1 implies

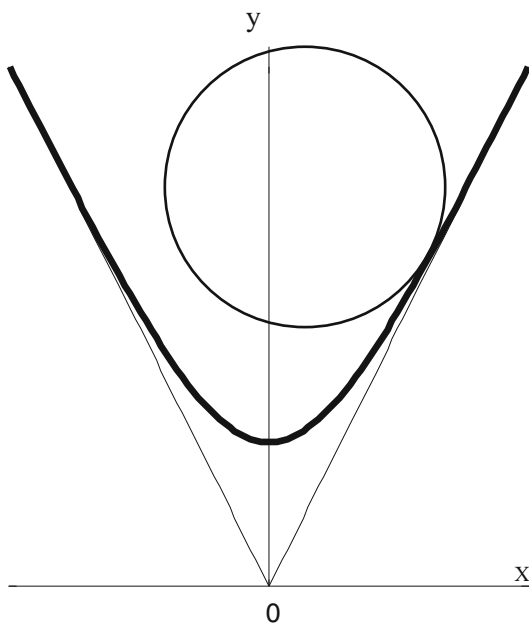
$$\lim_{i \rightarrow \infty} W(x + x_i, y + y_i, t + t_i) = 0$$

uniformly in (x, y, t) on every compact set in $\mathbb{R}^2 \times (-\infty, 0)$. Then, Lemma 1.2 also implies

$$\lim_{i \rightarrow \infty} W(x + x_i, y + y_i, t + t_i) = 0$$

uniformly in (x, y, t) on every compact set in $\mathbb{R}^2 \times [0, \infty)$. This completes the proof. $\square$

Fig. 1 A level set of $\{(x, y) \mid w(x, y, t; v_0) = a_0\}$ and an inscribed ball



3 Proof of Theorem 1.1 by estimates using inscribed balls

Let $c \in (k, \infty)$ be arbitrarily given and let v_0 be given by (10). Let $w(x, y, t) = w(x, y, t; v_0)$. Let θ_0 be as in Sect. 1. Now we consider an unbounded domain

$$\Omega(t) = \{(x, y) \in \mathbb{R}^2 \mid w(x, y, t) \leq \theta_0\} \quad (48)$$

with its boundary

$$\partial\Omega(t) = \{(x, y) \in \mathbb{R}^2 \mid w(x, y, t) = \theta_0\}.$$

We often write $\Omega(t)$ and $\partial\Omega(t)$ simply as Ω and $\partial\Omega$, respectively. For $t \in [1, \infty)$, let $(a, b) \in \Omega(t)$, $(X, Y) \in \partial\Omega(t)$ and $R \in (0, \infty)$ be given, and

$$\overline{B((a, b); R)} = \{(x, y) \in \mathbb{R}^2 \mid |x - a|^2 + |y - b|^2 \leq R^2\}$$

is included in Ω and contacts $\partial\Omega$ at $(X, Y) \in \partial\Omega$. See Fig. 1. Now we also assume

$$b > Y, \quad |a - X| \leq (b - Y) \tan \Theta.$$

The following proposition asserts that, if R can be taken arbitrarily large, then we necessarily have $|a - X| = (b - Y) \tan \Theta$. That is, as R goes to ∞ , $|a - X| / (b - Y)$ has to converge to $\tan \Theta$.

Proposition 3.1 (inscribed ball estimate) *Let $c \in (k, \infty)$ be given and let v_0 be defined by (10). Define $w(x, y, t) = w(x, y, t; v_0)$. Let $\Omega(t)$ be given by (48) for $t \in (0, \infty)$. Assume, for $t \in (0, \infty)$, $\overline{B((a, b); R)}$ is included in $\Omega(t)$ and contacts $\partial\Omega(t)$ at $(X, Y) \in \partial\Omega(t)$ with $R \in (0, \infty)$. Let $\mathbf{v} = (v_x, v_y)$ be a unit vector given by*

$$\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \frac{1}{\sqrt{(a - X)^2 + (b - Y)^2}} \begin{pmatrix} a - X \\ b - Y \end{pmatrix}.$$

Assume

$$v_y \geq \cos(\Theta - \mu_0). \quad (49)$$

for $\mu_0 \in (0, \Theta]$. Then, there exists $L(\mu_0) \in (0, \infty)$ such that one has $R \leq L(\mu_0)$, where $L(\mu_0)$ depends on $\mu_0 \in (0, \Theta]$ and is independent of $t \in (0, \infty)$ and is independent of $(a, b) \in \Omega(t)$ and $(X, Y) \in \partial\Omega(t)$.

Proof We write $\Omega(t)$ and $\partial\Omega(t)$ simply as Ω and $\partial\Omega$, respectively. Now we define

$$\Gamma = \{(x, y) \in \mathbb{R}^2 \mid y \leq b, |x - a|^2 + |y - b|^2 = R^2\}$$

and have $(X, Y) \in \Gamma$. Then, from Lemma 1.4, there exists $S_\mu \in (0, \infty)$, and we have

$$\sup_{\sigma \in [S_\mu, \infty)} w(x_1 + \sigma t_1, y_1 + \sigma t_2, t) \leq s_0$$

for every $(x_1, z_1) \in \Gamma$, and for every $t \in (0, \infty)$. Here $\mathbf{t} = (t_1, t_2)$ is an arbitrary unit vector with (16). Let $\varepsilon > 0$ be arbitrarily given. We define

$$U_0(\xi) = \begin{cases} s_0 & \text{if } \xi > S_\mu, \\ \theta_0 & \text{if } 0 \leq \xi \leq S_\mu, \\ 1 & \text{if } \xi < 0. \end{cases}$$

We have

$$w(x + X, y + Y, t) \leq U_0\left(\left(\begin{pmatrix} x \\ y \end{pmatrix} \cdot \mathbf{v}\right)\right), \quad (x, y) \in \mathbb{R}^2$$

for $t \in (0, \infty)$. Now a one-dimensional traveling front Φ is asymptotically stable. For any given $T \in (0, \infty)$, if R can be taken sufficiently large, we have, using Lemma 1.2,

$$w(x + X, y + Y, t + T) \leq \Phi\left(\left(\begin{pmatrix} x \\ y + cT \end{pmatrix} \cdot \mathbf{v} - \xi_0 - kT\right)\right) + \varepsilon, \quad (x, y) \in \overline{B((0, 0); 1)}.$$

This inequality holds true for every $t \in (0, \infty)$ and for every $(a, b) \in \Omega(t)$ and $(X, Y) \in \partial\Omega(t)$. Here $\xi_0 \in \mathbb{R}$. Note that $y + cT$ appears because we are using the moving coordinate with speed c . Recalling (27), putting and $(x, y) = (0, 0)$, and using

$$\theta_0 = w(X, Y, t) \leq w(X, Y, t + T),$$

we obtain

$$\theta_0 \leq \Phi(c v_y T - \xi_0 - kT) + \varepsilon = \Phi((c v_y - k)T - \xi_0) + \varepsilon. \quad (50)$$

Now we have

$$c v_y - k = c \cos(\Theta - \mu_0) - k > 0.$$

Recall that θ_0 is a positive constant. Then, taking $\varepsilon > 0$ small enough and taking $T \in (0, \infty)$ large enough, we have a contradiction in (50). This is because the right-hand side of (50) can be arbitrarily small, while its left-hand side is a positive constant. Thus R has an upper bound that depends only on μ_0 , that is, $R \leq L(\mu_0)$. This completes the proof. $\square$

Now we rewrite Proposition 3.1 in the following form.

Corollary 3.2 *Under the the same assumption of Proposition 3.1, let $g(x, t; \theta)$ be defined by (36) and let θ_0 be as in Sect. 1. Then one has*

$$\lim_{x \rightarrow \infty} \inf_{t \in (0, \infty)} g_x(x, t; \theta_0) = m_*. \quad (51)$$

Proof For $t \in (0, \infty)$, assume that $\overline{B((a, b); R)}$ is included in $\Omega(t) = \{(x, y) \in \mathbb{R}^2 \mid y \geq g(x, t; \theta_0)\}$ and contacts $\partial\Omega(t)$ at $(X, g(X, t; \theta_0))$ with $R \in (0, \infty)$. Proposition 3.1 asserts that, as long as $g_x(X, t; \theta_0)$ does not approach to m_* , the radius R has to be uniformly bounded in all $t \in (0, \infty)$. Then we obtain

$$\lim_{X \rightarrow \infty} \inf_{t \in (0, \infty)} g_x(X, t; \theta_0) = m_*,$$

which gives (51). This completes the proof. $\square$

Let $c \in (k, \infty)$ be arbitrarily given and let v_0 be given by (10). We put $w(x, y, t) = w(x, y, t; v_0)$ for $(x, y, t) \in \mathbb{R}^2 \times (0, \infty)$. Differentiating

$$w(x, g(x, t; \theta_0), t) = \theta_0, \quad (x, t) \in \mathbb{R} \times (0, \infty)$$

by t , we have

$$w_t(x, g(x, t; \theta_0), t) + g_t(x, t; \theta_0)w_y(x, g(x, t; \theta_0), t) = 0, \quad (x, t) \in \mathbb{R} \times (0, \infty). \quad (52)$$

Then, from (27), (29) and Proposition 1.3, we have

$$g_t(x, t; \theta_0) = \frac{w_t(x, g(x, t; \theta_0), t)}{-w_y(x, g(x, t; \theta_0), t)} > 0, \quad (x, t) \in \mathbb{R} \times (0, \infty),$$

$$\sup_{(x, t) \in \mathbb{R} \times (0, \infty)} g_t(x, t; \theta_0) < \infty.$$

Then, using (20), (31) and Proposition 1.3, we have

$$0 < \sup_{(x, t) \in \mathbb{R} \times (0, \infty)} g_t(x, t; \theta_0) < \infty.$$

Now we assert the following lemma.

Lemma 3.3 *Let $g(x, t; \theta)$ be defined by (36) and let θ_0 be as in Sect. 1. Then one has*

$$\liminf_{t \rightarrow \infty} g_t(0, t; \theta_0) = 0.$$

Proof We get a contradiction assuming $\liminf_{t \rightarrow \infty} g_t(0, t; \theta_0) > 0$. Let $x_0 \in (0, \infty)$ and $t_0 \in (0, \infty)$ be arbitrarily given. Using (37), we have

$$\limsup_{t \rightarrow \infty} g_t(x_0, t; \theta_0) \geq \liminf_{t \rightarrow \infty} g_t(0, t; \theta_0) > 0. \quad (53)$$

We define $\rho > 0$ as the value of (40) in Proposition 1.3. Let $u_0(x)$ be a uniformly continuous function from $\mathbb{R}$ to $\mathbb{R}$ with

$$0 \leq u_0(x) \leq 1, \quad -u'_0(x) > 0, \quad x \in \mathbb{R}, \quad (54)$$

$$-u'_0(x) > \rho \quad \text{if } x \in \mathbb{R}, \quad \delta_* \leq u_0(x) \leq \theta_0, \quad (55)$$

$$\sup_{x \in \mathbb{R}} (-u'_0(x)) < \infty, \quad (56)$$

where u'_0 means the derivative of u_0 by x . Without loss of generality, we assume $u_0(0) = \theta_0$. Let $u(x, t; u_0)$ be the solution of (22)–(23). Let $\varepsilon \in (0, \infty)$ be arbitrarily given. By (24), there exists $T \in (0, \infty)$ such that we have

$$\inf_{\gamma \in \mathbb{R}} \sup_{(x, t) \in \mathbb{R} \times [T, \infty)} |u(x, t; u_0) - \Phi(x - kt - \gamma)| < \varepsilon.$$

We can choose $T \in (0, \infty)$ uniformly for every u_0 with (54), (55) and (56). Now we change variables

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \frac{1}{\sqrt{1+m_*^2}} \begin{pmatrix} 1 & m_* \\ -m_* & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

with

$$\begin{pmatrix} \xi_0 \\ \eta_0 \end{pmatrix} = \frac{1}{\sqrt{1+m_*^2}} \begin{pmatrix} 1 & m_* \\ -m_* & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ g(x_0, t_0; \theta_0) \end{pmatrix}.$$

Putting

$$w(x, y, t) = \psi(\xi, \eta, t), \quad (\xi, \eta) \in \mathbb{R}^2,$$

we have

$$\begin{aligned} \psi_t &= \psi_{\xi\xi} + \psi_{\eta\eta} + \sqrt{c^2 - k^2} \psi_{\xi} + k \psi_{\eta} + f(\psi), \quad (\xi, \eta) \in \mathbb{R}^2, \\ \psi(\xi_0, \eta_0) &= \theta_0 \end{aligned} \quad (57)$$

and

$$\psi_{\xi} = \frac{1}{\sqrt{1+m_*^2}} \nabla \widehat{w} \cdot \begin{pmatrix} 1 \\ m_* \end{pmatrix} > 0, \quad (\xi, \eta) \in \mathbb{R}^2.$$

Using Corollary 3.2, we have

$$\lim_{x_0 \rightarrow \infty} \sup_{t_0 \in (0, \infty)} \psi_{\xi}(\xi_0, \eta_0, t_0) = 0.$$

Then, using Proposition 2.2, we have

$$\lim_{x_0 \rightarrow \infty} \sup_{t_0 \in (0, \infty)} \max_{(\xi, \eta) \in B((\xi_0, \eta_0); R_0)} \psi_{\xi}(\xi, \eta, t_0) = 0 \quad (58)$$

for every $R_0 \in (0, \infty)$. Now we choose

$$R_0 = 2 \left(\sup_{(x, t) \in \mathbb{R} \times (0, \infty)} g_t(x, t; \theta_0) \right) T.$$

By (24), (57), (58) and Lemma 1.2, we have

$$\sup_{(\xi, \eta) \in \overline{B((\xi_0, \eta_0); R_0)}} |\psi(\xi, \eta, t_0 + T) - \Phi(\eta)| < 2\varepsilon \quad (59)$$

for all $t_0 \in (0, \infty)$ and sufficiently large x_0 , say, $x_0 \in [A_0, \infty)$, where A_0 is a positive number. Now we obtain, from (59),

$$\sup_{(x, y) \in \overline{B((x_0, g(x_0, t_0; \theta_0)); R_0)}} |w(x, y, t_0 + T) - v_{\text{rgh}}(x, y)| < 2\varepsilon \quad (60)$$

for $(x_0, t_0) \in [A_0, \infty) \times (0, \infty)$, where v_{rgh} is given by (28). Then, putting $x_0 = A_0$ in (60), we have

$$\sup_{(x, y) \in \overline{B(A_0, g(A_0, t_0 + T; \theta_0); R_0/2)}} |w(x, y, t_0 + T) - v_{\text{rgh}}(x, y)| < 2\varepsilon \quad (61)$$

for every $t_0 \in (0, \infty)$. While, from (52), we have

$$w_t(A_0, g(A_0, t_0 + T; \theta_0), t_0 + T) = -g_t(A_0, t_0 + T; \theta_0) w_y(A_0, g(A_0, t_0 + T; \theta_0)). \quad (62)$$

Since $\varepsilon \in (0, \infty)$ can be taken arbitrarily small, $t_0 \in (0, \infty)$ can be taken arbitrarily and $T \in (0, \infty)$ can be taken arbitrarily large, (61), (62) and Proposition 1.3 contradict (53) for

$t = t_0 + T$. Indeed, as $t = t_0 + T$ goes to infinity, the left-hand side of (62) remains in a neighborhood of zero and the right-hand side of (62) does not remain in the neighborhood of zero. This completes the proof. $\square$

Now we state the proof of Theorem 1.1 as follows.

Proof of Theorem 1.1 We define

$$V(x, y) = \lim_{t \rightarrow \infty} w(x, y + g(0, t; \theta_0), t) \quad \text{on every compact set in } \mathbb{R}^2 \quad (63)$$

and have

$$V(0, 0) = \theta_0.$$

We have

$$\left(\Delta + c \frac{\partial}{\partial y} \right) w(x, y + g(0, t; \theta_0), t) + f(w(x, y + g(0, t; \theta_0), t)) = w_t(x, y + g(0, t; \theta_0), t) \quad (64)$$

for $(x, y, t) \in \mathbb{R}^2 \times (0, \infty)$. Using (52) and Lemma 3.3, we have

$$\lim_{i \rightarrow \infty} w_t(0, g(0, \tau_i; \theta_0), \tau_i) = 0.$$

Here $(\tau_i)_{i \in \mathbb{N}}$ satisfies

$$\begin{aligned} 0 < \tau_i < \tau_{i+1}, \quad i \in \mathbb{N}, \\ \lim_{i \rightarrow \infty} \tau_i = \infty. \end{aligned}$$

Using this equality and Proposition 2.2, we find

$$\lim_{i \rightarrow \infty} w_t(x, y + g(0, \tau_i; \theta_0), \tau_i) = 0$$

uniformly on every compact set in $\mathbb{R}^2$. Using this equality and (64), we obtain (11). By (25), we have

$$0 \leq V(x, y) \leq 1, \quad (x, y) \in \mathbb{R}^2.$$

Since V is not identically 0 or 1, we obtain (12). From Proposition 1.3, we have

$$\inf \{ |\nabla V(x, y)| \mid (x, y) \in \mathbb{R}^2, \delta_* \leq V(x, y) \leq \theta_0 \} > 0. \quad (65)$$

Then, from (26) and (30), we obtain (13) and (15). For every $\theta \in (0, 1)$, we define $G(x; \theta)$ by

$$V(x, G(x; \theta)) = \theta, \quad x \in \mathbb{R}.$$

Using (39), we have

$$V_x(x, y) \geq 0, \quad (x, y) \in [0, \infty) \times \mathbb{R}$$

and obtain (14). Then we have

$$G(-x; \theta) = G(x; \theta), \quad (x, \theta) \in \mathbb{R} \times (0, 1), \quad (66)$$

$$0 < G_x(x; \theta) < m_*, \quad (x, \theta) \in (0, \infty) \times (0, 1), \quad (67)$$

$$\lim_{x \rightarrow \infty} |G_x(x; \theta_0) - m_*| = 0, \quad \lim_{x \rightarrow -\infty} |G_x(x; \theta_0) + m_*| = 0,$$

For $A \in (0, \infty)$, we change variables

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A \\ G(A; \theta_0) \end{pmatrix} + \frac{1}{\sqrt{1 + m_*^2}} \begin{pmatrix} 1 & -m_* \\ m_* & 1 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

and put

$$V(x, y) = \Psi(\xi, \eta), \quad (\xi, \eta) \in \mathbb{R}^2.$$

Now we have

$$\begin{aligned} \Psi_{\xi\xi} + \Psi_{\eta\eta} + \sqrt{c^2 - k^2}\Psi_{\xi} + k\Psi_{\eta} + f(\Psi) &= 0, \quad (\xi, \eta) \in \mathbb{R}^2, \\ \Psi(0, 0) &= \theta_0 \end{aligned}$$

and

$$\Psi_{\xi} = \frac{1}{\sqrt{1+m_*^2}} \nabla V \cdot \begin{pmatrix} 1 \\ m_* \end{pmatrix} > 0, \quad (\xi, \eta) \in \mathbb{R}^2.$$

Using Proposition 2.2 and sending $A \rightarrow \infty$, we have

$$\lim_{A \rightarrow \infty} \max_{B(0; R_0)} \Psi_{\xi} = 0$$

for every $R_0 \in (0, \infty)$. Recalling $\Phi(0) = \theta_0$, we obtain

$$\lim_{A \rightarrow \infty} \sup_{(\xi, \eta) \in B(0; R_0)} |\Psi(\xi, \eta) - \Phi(\eta)| = 0 \quad (68)$$

for every $R_0 \in (0, \infty)$. Now we obtain (17). This completes the proof. $\square$

Now we define

$$\sigma_1 = 1 + \frac{\beta + M}{\beta \min\{-V_y(x, y) \mid (x, y) \in \mathbb{R}, \delta_* \leq V(x, y) \leq 1 - \delta_*\}}.$$

Here M is given by (18). The following lemma asserts the uniqueness of V .

Lemma 3.4 *Let $c \in (k, \infty)$ be given and let V be given by (63). Let U satisfy (11) with*

$$\begin{aligned} 0 < U(x, y) < 1, \quad (x, y) \in \mathbb{R}^2, \\ \lim_{R \rightarrow \infty} \sup_{x^2 + y^2 \geq R^2} |U(x, y) - V(x, y)| &= 0. \end{aligned} \quad (69)$$

Then one has $U \equiv V$.

Proof We use the arguments in [1, 8, 13]. Let $\delta \in (0, \delta_*)$ be arbitrarily given. Using (17) and (69), we have

$$V(x, y) - \delta \leq U(x, y) \leq V(x, y) + \delta$$

if $x^2 + y^2$ is large enough, say, $x^2 + y^2 \geq R_1^2$ for $R_1 \in (0, \infty)$. If $x^2 + y^2 \leq R_1^2$, we have

$$V(x, y + \lambda) \leq U(x, y) \leq V(x, y - \lambda)$$

for sufficiently large $\lambda \in (0, \infty)$. Combining the two inequalities together, we obtain

$$V(x, y + \lambda) - \delta \leq U(x, y) \leq V(x, y - \lambda) + \delta, \quad (x, y) \in \mathbb{R}^2.$$

Then we have

$$V(x, y + \lambda + \sigma_1 \delta (1 - e^{-\beta t})) - \delta e^{-\beta t} \leq U(x, y) \leq V(x, y - \lambda - \sigma_1 \delta (1 - e^{-\beta t})) + \delta e^{-\beta t}$$

for $(x, y, t) \in \mathbb{R}^2 \times (0, \infty)$. Now $V(x, y + \lambda \mp \sigma_1 \delta (1 - e^{-\beta t})) \pm \delta e^{-\beta t}$ are a supersolution and a subsolution, respectively. See [13] for the proof. Taking the limit of $t \rightarrow \infty$, we find

$$V(x, y + \lambda + \sigma_1 \delta) \leq U(x, y) \leq V(x, y - \lambda - \sigma_1 \delta), \quad (x, y) \in \mathbb{R}^2.$$

Now we define

$$\Lambda = \inf \left\{ \lambda_0 \mid U(x, y) \leq V(x, y - \lambda_0), \quad (x, y) \in \mathbb{R}^2 \right\}.$$

Then we have $\Lambda \in [-\lambda - \sigma_1 \delta, \lambda + \sigma_1 \delta]$ and

$$V(x, y) \leq U(x, y - \Lambda), \quad (x, y) \in \mathbb{R}^2.$$

By (17) and (69), we have $\Lambda \geq 0$, that is, $\Lambda \in [0, \lambda + \sigma_1 \delta]$. If $\Lambda = 0$, we have $U = V$ and obtain this lemma. If we have $V(x, y) = U(x, y - \Lambda)$ for all $(x, y) \in \mathbb{R}^2$, we find $\Lambda = 0$ using (17) and (69), and we obtain this lemma. Thus, it suffices to get a contradiction by assuming $\Lambda \in (0, \lambda + \sigma_1 \delta]$,

$$V(x, y) \leq U(x, y - \Lambda), \quad (x, y) \in \mathbb{R}^2.$$

and

$$\sup_{(x, y) \in \mathbb{R}^2} (U(x, y - \Lambda) - V(x, y)) > 0.$$

Now we take $R \in (0, \infty)$ large enough with

$$\sup_{|y - m_*| |x| \geq R - \Lambda - 1} (-V_y(x, y)) < \frac{1}{2\sigma_1}.$$

Taking $h \in (0, \min \{\delta_*, 1/(2\sigma_1)\})$ small enough, we have

$$V(x, y - \Lambda + 2\sigma_1 h) - V(x, y - \Lambda) = 2\sigma_1 h \int_0^1 V_y(x, y - \Lambda + 2\sigma_1 \varphi h) d\varphi > -h \quad (70)$$

if $|y - m_*| |x| \geq R$. Using (69) and

$$\lim_{R \rightarrow \infty} \sup_{x^2 + y^2 \geq R^2} |V(x, y - \Lambda + 2\sigma_1 h) - v_0(x, y - \Lambda + 2\sigma_1 h)| = 0,$$

we have

$$U(x, y) < V(x, y - \Lambda + 2\sigma_1 h) \quad \text{if } |y - m_*| |x| \leq R \quad (71)$$

taking h smaller, if necessary. Now we have (70) for $|y - m_*| |x| \geq R$, and have (71) for $|y - m_*| |x| \leq R$. Then we obtain

$$U(x, y) \leq V(x, y - \Lambda + 2\sigma_1 h) + h, \quad (x, y) \in \mathbb{R}^2.$$

Taking $t \rightarrow \infty$ in

$$U(x, y) \leq V(x, y - \Lambda + 2\sigma_1 h - \sigma_1 h(1 - e^{-\beta t})) + h e^{-\beta t}, \quad (x, y, t) \in \mathbb{R}^2 \times (0, \infty),$$

we obtain

$$U(x, y) \leq V(x, y - \Lambda + \sigma_1 h), \quad (x, y) \in \mathbb{R}^2.$$

This contradicts the definition of Λ . This completes the proof. $\square$

4 Proof of Proposition 1.3

In this section, we state the proof of Proposition 1.3. We apply the arguments of [11, 12, 14] to $w(x, y, t)$ that changes its shape with time t . For any $\theta \in (0, \theta_0]$, let R_θ be a positive number given by (35) and define

$$R = \max_{\delta_*/2 \leq \theta \leq \theta_0} R_\theta.$$

Using

$$w_t - \Delta w - cw_y - f(w) = 0, \quad (x, y, t) \in \mathbb{R}^2 \times (0, \infty)$$

and using

$$w_t(x, y, t) > 0, \quad w_y(x, y, t) < 0, \quad (x, y, t) \in \mathbb{R}^2 \times (0, \infty),$$

we have

$$\Delta w + f(w) > 0, \quad (x, y, t) \in \mathbb{R}^2 \times (0, \infty).$$

Multiplying the both sides of this inequality by $-w_y$ and using

$$w_y \Delta w = \operatorname{div}(w_y \nabla w) - \frac{1}{2} D_y(|\nabla w|^2),$$

we have

$$-\operatorname{div}(w_y \nabla w) + \frac{1}{2} D_y(|\nabla w|^2) - f(w)w_y > 0, \quad (x, y, t) \in \mathbb{R}^2 \times (0, \infty). \quad (72)$$

Here $D_y = \partial/\partial y$. Let θ_1 and s_1 satisfy

$$0 < s_1 < \frac{\delta_*}{2} \leq \theta_1 \leq \theta_0.$$

For every $x_0 \in \mathbb{R}$, we define a domain in $\mathbb{R}^2$ by

$$\Omega_1(t) = \{(x, y) \mid |x - x_0| < R, s_1 < w(x, y, t) < \theta_1\}$$

for $t \in (0, \infty)$ with its boundary

$$\partial\Omega_1(t) = \Gamma_{\text{bot}}(t) \cup \Gamma_{\text{top}}(t) \cup \Gamma_{\text{sid}}(t),$$

where

$$\begin{aligned} \Gamma_{\text{bot}}(t) &= \{(x, y) \in \mathbb{R}^2 \mid |x - x_0| \leq R, w(x, y, t) = \theta_1\}, \\ \Gamma_{\text{top}}(t) &= \{(x, y) \in \mathbb{R}^2 \mid |x - x_0| \leq R, w(x, y, t) = s_1\}, \\ \Gamma_{\text{sid}}(t) &= \{(x, y) \in \mathbb{R}^2 \mid |x - x_0| = R, s_1 \leq w(x, y, t) \leq \theta_1\}. \end{aligned}$$

Integrating the both sides of (72) over $\Omega_1(t)$ and using the Gauss divergence theorem, we obtain

$$\begin{aligned} 0 &< \int_{\Omega_1(t)} \left(-\operatorname{div}(w_y \nabla w) + \frac{1}{2} D_y(|\nabla w|^2) - f(w)w_y \right) dx dy \\ &= \int_{\partial\Omega_1(t)} \left(-w_y (\nabla w, \mathbf{v}) + \frac{1}{2} v_y |\nabla w|^2 + \mathcal{W}(w)v_y \right) ds, \end{aligned}$$

where $\mathbf{v} = (v_x, v_y)$ is the outward unit normal vector on $\partial\Omega_1(t)$ and ds is the line element of $\partial\Omega_1(t)$. Now we have

$$J_{\text{bot}}(t) + J_{\text{top}}(t) + J_{\text{sid}}(t) > 0, \quad t \in (0, \infty), \quad (73)$$

where

$$\begin{aligned} J_{\text{bot}}(t) &= \int_{\Gamma_{\text{bot}}(t)} \left(-w_y (\nabla w, \mathbf{v}) + \frac{1}{2} v_y |\nabla w|^2 + \mathcal{W}(w)v_y \right) ds, \\ J_{\text{top}}(t) &= \int_{\Gamma_{\text{top}}(t)} \left(-w_y (\nabla w, \mathbf{v}) + \frac{1}{2} v_y |\nabla w|^2 + \mathcal{W}(w)v_y \right) ds, \end{aligned}$$

$$J_{\text{sid}}(t) = \int_{\Gamma_{\text{sid}}(t)} \left(-w_y (\nabla w, \mathbf{v}) + \frac{1}{2} v_y |\nabla w|^2 + \mathcal{W}(w) v_y \right) ds$$

for $t \in (0, \infty)$. First we calculate $J_{\text{bot}}(t)$. Using

$$\mathbf{v} = \frac{\nabla w}{|\nabla w|} \quad \text{on } \Gamma_{\text{bot}}(t),$$

we have

$$J_{\text{bot}}(t) = \int_{\Gamma_{\text{bot}}(t)} \left(-\frac{1}{2} w_y |\nabla w| + \mathcal{W}(\theta_1) v_y \right) ds.$$

Using

$$ds = \frac{|\nabla w|}{-w_y} dx,$$

we obtain

$$J_{\text{bot}}(t) = \int_{x_0-R}^{x_0+R} \left(\frac{1}{2} |\nabla w(x, g(x, t; \theta_1))|^2 - \mathcal{W}(\theta_1) \right) dx. \quad (74)$$

Similarly we obtain

$$J_{\text{top}}(s_1) = \int_{x_0-R}^{x_0+R} \left(-\frac{1}{2} |\nabla w(x, g(x, t; s_1))|^2 + \mathcal{W}(s_1) \right) dx. \quad (75)$$

Secondly, using

$$v_y = 0 \quad \text{on } \Gamma_{\text{sid}}(t),$$

we have

$$J_{\text{sid}}(t) = \int_{\Gamma_{\text{sid}}(t)} (-w_y) (\nabla w, \mathbf{v}) ds.$$

Using (34) and $|\mathbf{v}| = 1$, we have

$$|J_{\text{sid}}(t)| \leq B_0 \int_{\Gamma_{\text{sid}}(t)} (-w_y) ds = B_0 \int_{\Gamma_{\text{sid}}(t)} (-w_y) dy = 2B_0(\theta_1 - s_1). \quad (76)$$

Combining (73), (74), (75) and (76), we obtain

$$\begin{aligned} & \int_{x_0-R}^{x_0+R} \left(\frac{1}{2} |\nabla w(x, g(x, t; \theta_1))|^2 - \mathcal{W}(\theta_1) \right) dx \\ & > \int_{x_0-R}^{x_0+R} \left(\frac{1}{2} |\nabla w(x, g(x, t; s_1))|^2 - \mathcal{W}(s_1) \right) dx - 2B_0(\theta_1 - s_1). \\ & \geq \int_{x_0-R}^{x_0+R} (-\mathcal{W}(s_1)) dx - 2B_0(\theta_1 - s_1). \end{aligned}$$

Sending $s_1 \rightarrow 0$, we find

$$\int_{x_0-R}^{x_0+R} \left(\frac{1}{2} |\nabla w(x, g(x, t; \theta_1))|^2 - \mathcal{W}(\theta_1) \right) dx \geq -2B_0\theta_1,$$

that is

$$\frac{1}{2} \int_{x_0-R}^{x_0+R} |\nabla w(x, g(x, t; \theta_1))|^2 dx \geq 2(\mathcal{W}(\theta_1)R - B_0\theta_1) \geq 2, \quad t \in (0, \infty).$$

Now we obtained

$$\int_{x_0-R}^{x_0+R} |\nabla w(x, g(x, t; \theta_1))|^2 dx \geq 4, \quad \text{for all } (\theta_1, t) \in (\delta_*/2, \theta_0] \times (0, \infty). \quad (77)$$

Let N_0 be as in (29). We define

$$t_* = \min \left\{ \frac{1}{2}, \frac{\delta_*}{2N_0} \right\}.$$

Let $(\alpha_0, \beta_0, t_0) \in \mathbb{R}^2 \times [1, \infty)$ satisfy

$$w(\alpha_0, \beta_0, t_0) = \theta \quad (78)$$

for $\theta \in [\delta_*, \theta_0]$. Then we have $t_0 - t_* \in [1/2, 1)$ and

$$\frac{\delta_*}{2} \leq w(\alpha_0, \beta_0, t_0 - t_*) < \theta.$$

Now we define

$$D = \{(x, y) \in \mathbb{R}^2 \mid |x - \alpha_0| \leq R, |y - \beta_0| \leq m_* R\}.$$

and put $\theta_1 = w(\alpha_0, \beta_0, t_0 - t_*)$. Using (31) and applying (77) for $t = t_0 - t_*$, we have

$$\max_{(x,y) \in D} (-w_y(x, y, t_0 - t_*)) \geq \frac{\sqrt{2} \cos \Theta}{\sqrt{R}}.$$

Then, applying Lemma 2.1, we obtain

$$\inf \{ -w_y(\alpha_0, \beta_0, t_0) \mid (\alpha_0, \beta_0, t_0, \theta) \in \mathbb{R}^2 \times [1, \infty) \times [\delta_*, \theta_0] \text{ with (78)} \} > 0.$$

This completes the proof of Proposition 1.3.

We proved the existence of V-form traveling fronts using estimates based on inscribed balls. This method is a simple one and will be applicable to show the existence of V-form traveling fronts, pyramidal traveling fronts or polyhedral entire solutions ([4, 10, 15]) for various kinds of reaction–diffusion equations or systems.

Acknowledgements The second author is supported by JSPS Grant-in-Aid for Scientific Research (C) Grant Number 26K06866 and Grant Number 22K03288.

Funding Open Access funding provided by Okayama University.

Data availability Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Conflict of interest The authors state that there is no conflict of interest.

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