

Saddle-Point Approximation to the False-Vacuum Decay at Finite Temperature in 1D Quantum Mechanics

Koji Harada¹, Shuichiro Tao², and Qiang Yin ^{3,*}

¹*Faculty of Arts and Science, Kyushu University, 744 Motoooka, Nishi-ku, Fukuoka 819-0395, Japan*

²*Institute of Promotion of Education and Campus Life, Okayama University, 2-1-1 Tsushima-naka, Kita-ku, Okayama 700-8530, Japan*

³*Graduate School of Science, Kyushu University, 744 Motoooka, Nishi-ku, Fukuoka 819-0395, Japan*

*E-mail: in.kyo@phys.kyushu-u.ac.jp (Q.Y.), harada@artsci.kyushu-u.ac.jp (K.H.),

tao@okayama-u.ac.jp (S.T.)

Received November 6, 2024; Revised December 25, 2024; Accepted January 6, 2025; Published January 7, 2025

.....
We calculate the false-vacuum decay rate in 1D quantum mechanics on the basis of the saddle-point approximation in the Euclidean path integral at finite temperature. The saddle points are the finite- T and shifted bounce solutions, which are finite-period analogs of the (zero-temperature) bounce solution, and the shot solutions. We reexamine the zero-temperature result by Callan and Coleman and compare it with the zero-temperature limit of our results. We also perform some numerical calculations to illustrate the temperature dependence of the decay rate and compare it with the result by Affleck.
.....

Subject Index A46, A47, A64, B87

1. Introduction

Even though the electroweak phase transition in the standard model seems to be a crossover [1,2], some possible extensions of the Higgs sector may cause a first-order phase transition and would lead to very interesting consequences. A well known example is electroweak baryogenesis [3] (see also Refs. [4–7] for reviews). Recently, much attention has been paid to gravitational waves generated by the collision of bubbles of the true vacuum [8–10] (see also Ref. [11] for a review) and dark matter production due to the filtering effect [12], or to ultrarelativistic bubble expansion [13].

If the phase transition is of the first order, the unstable “false vacuum” decays into the stable “true vacuum” through quantum-mechanical tunneling. The decay rate through such tunneling has been given by Coleman [14,15] by using a saddle-point approximation in the Euclidean path-integral formalism. See Refs. [16,17] for earlier investigations.

A 1D quantum-mechanical system provides a simple example of such a phase transition and the decay rate has been calculated by Callan and Coleman [18]. A classical solution called the “bounce solution” plays the central role in the calculation, and the so-called potential deformation method is employed to get the imaginary part of the (ground-state) energy. See also Ref. [19] (especially the appendices) for useful explanations, and the first few sections of Ref. [20] for an excellent review.

In many applications (in particular in cosmological applications), one would be interested in extending the Callan–Coleman calculation to finite-temperature cases. Affleck [21] provides a

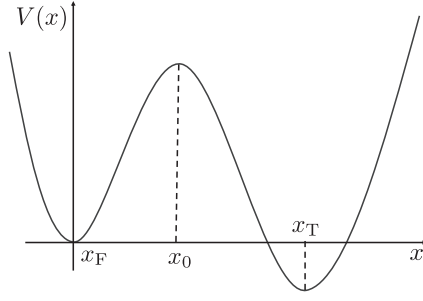


Fig. 1. A typical potential under consideration.

finite-temperature version of the decay rate formula. Linde [22,23] also provides the formula for very high temperature, where the field theory effectively becomes 3D. In those papers, the formulae are given as natural extensions of Coleman's formula, without explicitly mentioning the classical solutions (saddle points) at finite temperature.

In this paper, we derive the false-vacuum decay rate of a 1D quantum-mechanical system (particle) at finite temperature in the saddle-point approximation by explicitly using the (Euclidean) classical solutions. There are several subtleties that, to the best of our knowledge, have not been noticed before.

In the following, we outline how to calculate the vacuum decay rate. Detailed calculations are given in the subsequent sections. Here we want to introduce some basic concepts that will be important later in this paper.

We put a 1D quantum-mechanical system, described by the Hamiltonian

$$H = \frac{1}{2m}p^2 + V(x), \quad (1)$$

in contact with a heat bath, characterized by the temperature T . The potential $V(x)$ (see Fig. 1) has two minima at $x = x_F$ (false vacuum) and at $x = x_T$ (true vacuum), with $V(x_F) = 0 > V(x_T)$. The system is initially in the false-vacuum well, i.e. around $x = x_F$, and decays into the true vacuum. We calculate this decay rate in the equilibrium statistical-mechanics framework. This means that we assume that the system is almost at thermal equilibrium and the false-vacuum decay rate is very small, so that the relaxation time is much shorter than the inverse of the decay rate.

We will see in later sections that the results are independent of the potential shape for $x > x_T$. One may imagine that $V(x) = V(x_T)$ for $x \geq x_T$, so that one does not need to worry about the reflecting waves.

The basic ingredient of the finite-temperature calculation is the partition function Z , or Helmholtz's free energy F ,

$$Z = e^{-\beta F} = \text{tr} (e^{-\beta H}), \quad (2)$$

where $\beta = 1/(k_B T)$.

The trace may be taken in the coordinate space, so that

$$Z = \int dy \langle y | e^{-\beta H} | y \rangle = \int dy \mathcal{N} \int_{x(\pm\hbar\beta/2)=y} \mathcal{D}x(\tau) e^{-S[x]/\hbar}, \quad (3)$$

where the path integral is over all the paths satisfying the boundary condition $x(\pm\hbar\beta/2) = y$,

$$S[x] = \int_{-\hbar\beta/2}^{\hbar\beta/2} d\tau \left[\frac{m}{2} \left(\frac{dx}{d\tau} \right)^2 + V(x) \right] \quad (4)$$

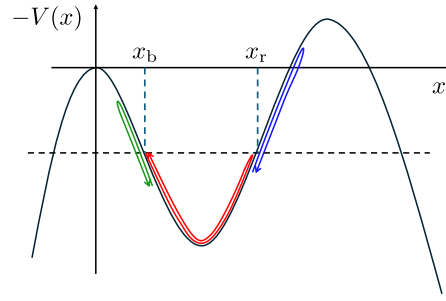


Fig. 2. Three kinds of classical solutions. The red trajectory denotes the finite- T bounce solution. The green and blue trajectories denote shot-F and shot-T solutions, respectively.

is the Euclidean action, and $\mathcal{N}$ is a normalization constant.

We consider the saddle-point approximation to the free energy. A saddle point is the (Euclidean) classical solution $x_{\text{cl}}(\tau)$,

$$m \frac{d^2}{d\tau^2} x_{\text{cl}}(\tau) - V'(x_{\text{cl}}(\tau)) = 0, \quad (5)$$

subject to the boundary conditions

$$x_{\text{cl}}(\pm\beta\hbar/2) = y. \quad (6)$$

We expand the path $x(\tau)$ around the classical solution $x_{\text{cl}}(\tau)$ and write

$$x(\tau) = x_{\text{cl}}(\tau) + r(\tau), \quad (7)$$

where the fluctuation around the classical solution $r(\tau)$ satisfies the Dirichlet boundary conditions,

$$r(\pm\hbar\beta/2) = 0. \quad (8)$$

The saddle-point approximation consists of the exponential factor of the classical action, and the prefactor obtained by the Gaussian integration of the fluctuations.

There are two classes of classical solutions that satisfy the boundary conditions (6). A solution of the first class is “periodic” in the sense that it comes back to the original point with the same velocity after the (Euclidean) time period $\beta\hbar$ and can repeat the same motion again and again. On the other hand, a solution of the second class just returns back to the original position y after $\beta\hbar$, but the velocity is not the same as before (it changes sign).

In the classical paper of Dashen et al. [24], they consider a stationary-phase approximation only for the periodic (real-time) solutions. In Section 2, we will show that our calculations for the periodic classical solutions (after analytic continuation) reproduce their result.

It is amazing that “static solutions” such as the false-vacuum solution $x_{\text{cl}}(\tau) = x_{\text{F}}$ do not contribute to the free energy, since such a solution exists only at a single value of $y = x_{\text{F}}$ so that it has measure zero in the y -integration.

The periodic solutions are (finite- T) *bounce* and *shifted bounce* solutions, which are finite-temperature analogs of the zero-temperature bounce solution. The (usual) bounce solution defined at zero temperature starts at the top ($x = x_{\text{F}}$) of the inverted potential $-V(x)$, moving down and bouncing at $x = x_{\text{R}}$ ($V(x_{\text{R}}) = V(x_{\text{F}})$), and going back to the top. At finite temperature, because it travels during only the finite period $\beta\hbar$, a finite- T bounce starts in the middle of the slope ($x = x_{\text{b}}$) of the inverted potential, with velocity zero, bounces at $x = x_{\text{r}}$ ($V(x_{\text{r}}) = V(x_{\text{b}})$), and goes back to the original point; see Fig. 2.

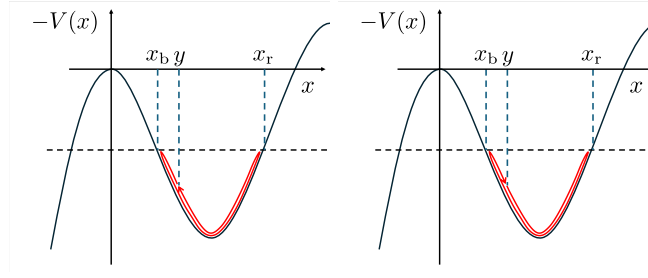


Fig. 3. Shifted bounce solutions. There are two shifted bounce solutions for a given energy (or the period) and a starting/ending point y .

It immediately follows that such a finite- T bounce has a minimum period: as the period decreases, the corresponding finite- T bounce moves only in the vicinity of x_0 , the top of the potential wall. Since the inverted potential there may be well approximated as $-V(x) \sim \frac{1}{2}|V''(x_0)|(x - x_0)^2$, the upper limit of the period is given by $\beta_0\hbar \equiv 2\pi/|V''(x_0)|$. There is no bounce solution with a period smaller than $\beta_0\hbar$. In this paper, we restrict ourselves to temperatures less than β_0^{-1} . Above this, the main cause of the decay would be thermal rather than quantum-mechanical tunneling.

Shifted bounce solutions are time-translated solutions of the finite- T bounce solution; see Fig. 3. At zero temperature, two bounce solutions that are related by time translation are not discriminated. On the other hand, at finite temperature, the time-translated solutions have nonzero initial velocity so that it is useful to distinguish them from the finite- T bounce solution.

The “just-returning-back” solutions are called shot solutions [20]. There are two kinds of shot solutions. At low temperature, shot solutions spend most of their time either near the false-vacuum point x_F (shot-F) or near the true-vacuum point x_T (shot-T). At the zero-temperature limit, the shot-F solution behaves like the (static) false-vacuum solution. Perhaps surprisingly, the shot solutions play a crucial role at finite temperature, though they have not attracted much attention before.

If we include all of these saddle-point contributions in the partition function (3), the result would be real, as it appears. It is the choice of the saddle points and the steepest descent contour that leads to the imaginary part. The choice of the steepest descent contour corresponds to the choice of the boundary condition. (Remember a similar situation in the derivation of the WKB connection formula by the saddle-point method in elementary quantum mechanics.) What we need for calculating the decay rate is the steepest descent contour that corresponds to the Gamow–Siegert [25,26] (GS) boundary condition; i.e. there are only “outgoing (decaying) waves”.

It is easy to infer that the shot-T solutions are not relevant to the false-vacuum decay, since they are related to the true vacuum outside the potential wall. The inclusion of the shot-T solutions in the saddle-point approximation would correspond to the inclusion of the “incoming waves” and the information on the true vacuum.

On the other hand, the shot-F solutions are related to the false vacuum, and the finite- T bounce and shifted bounce solutions are directly related to the tunneling of the potential wall. Note that, at zero temperature, the contributions from the false vacuum and the bounce saddle points lead to the imaginary part of the ground-state energy, the (minus half of the) decay rate [20].

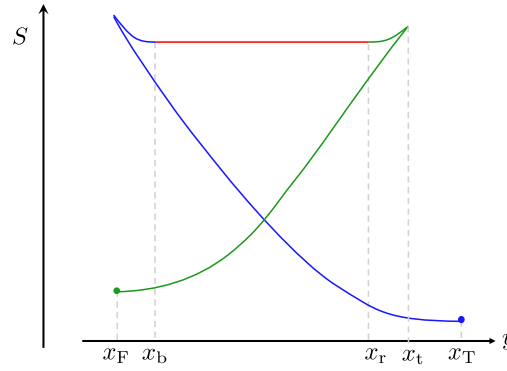


Fig. 4. A schematic presentation for the actions of the classical solutions for various values of y . The horizontal straight red line corresponds to the finite- T bounce and shifted bounce solutions. The green lines show the actions of shot-F solutions. For $y \in (x_r, x_t)$, there are two shot-F solutions. Similarly, the actions of shot-T solutions are represented as blue lines.

There are two shot solutions with y just outside the regions $[x_b, x_r]$, with the same period. They have different values of classical action, and more importantly, one of them supplies the imaginary part to the free energy!

Figure 4 shows that finite- T bounce, shifted bounce, shot-F, and shot-T solutions are intimately connected and form three branches. Ignoring the shot-T solutions (because they are irrelevant to the decay of the false vacuum), there are two solutions for each value of y in the region $[x_b, x_t]$, where x_t is the largest value of y for which the shot-F solutions exist. We call the set of solutions for which the action is larger the *imaginary* branch, consisting of the finite- T bounce and the shifted bounce solutions together with one of the shot-F solutions that supplies the imaginary part to the free energy. The other is called the *real* branch, consisting of the shot-F solutions that give purely real contributions to the free energy.

Thus, at finite temperature, we calculate the contributions to the partition function from the finite- T bounce and shifted bounce, Z_B , and from the shot-F, Z_{sF} , to obtain

$$Z_{GS} = Z_{sF}^{(R)} + \frac{1}{2} (Z_B + Z_{sF}^{(I)}), \quad (9)$$

where $Z_{sF}^{(R)}$ and $Z_{sF}^{(I)}$ are contributions from the shot-F solutions in the real and imaginary branches, respectively. The reason why the factor $1/2$ appears in the second term is explained by Callan and Coleman [18]. In their case, the corresponding partition function is

$$Z_{CC} = Z_F + \frac{1}{2} Z_B, \quad (10)$$

where Z_F is the contribution of the static false-vacuum solution $x_{cl}(\tau) = x_F$. The shot-F solution plays the role of the static false-vacuum solution in our case.

As we have explained, Z_{GS} is complex; the decay rate may be given as

$$\Gamma = -\frac{2}{\hbar} \text{Im}(F_{GS}) = \frac{2}{\beta \hbar} \tan^{-1} \left(\frac{\text{Im} Z_{GS}}{\text{Re} Z_{GS}} \right). \quad (11)$$

This formula is quite different from the one given by Affleck. We will numerically compare the temperature dependence of the decay rates calculated with two formulae in Section 5.

The rest of the paper is structured as follows: In Section 2, we calculate the finite- T bounce and the shifted bounce contributions, Z_B . Because a shifted bounce has a nonzero initial velocity, its time-derivative of the solution does not satisfy the Dirichlet boundary condition. Thus the usual collective coordinate method does not work. The trace integral over y in Eq. (3) plays

a similar role to the collective coordinate integral. The contributions from the shot-F saddle points, Z_{sF} , will be discussed in Section 3. We then complete the decay rate calculation. In Section 4, we first reexamine the collective coordinate treatment of the zero mode by Callan and Coleman [18]. If one takes the finite time-interval “regularization” seriously, the Dirichlet boundary conditions with which they work are inconsistent with the collective coordinate treatment of the zero mode. The collective coordinate method works only with the periodic boundary conditions. We then calculate the zero-temperature limit of our result. It is different from that of Callan and Coleman. In Section 5, we perform some numerical calculations to see the physical implications of our formula. We compare the zero-temperature limit of our result with that of Callan–Coleman as well as with the WKB result given in Ref. [20], changing the height of the potential barrier. We find that our formula gives a similar result to that of the WKB formula, rather than to that of Callan and Coleman. We then show the temperature dependence of the decay rate and compare it with the result of Affleck [21]. As naively expected, our decay rate increases with increasing temperature. Affleck’s formula, on the other hand, leads to a perplexing result in certain cases: the decay rate *decreases* with increasing temperature to a certain point and then increases. We summarize our result in Section 6.

In the following sections, we employ dimensionless variables for notational simplicity and numerical calculations. In Appendix A, we explain how to introduce them. Superficially, it corresponds to the use of the unit in which $\hbar = 1$ and $m = 1$. In Appendix B, we illustrate the method of calculating the ratio of determinants defined with the periodic boundary conditions, and show that it becomes equal to the ratio defined with the Dirichlet boundary conditions in the limit of the infinite interval.

2. Finite- T bounce and shifted bounce solutions

Let us denote the finite- T bounce solution as $x_{\text{B}}(\tau)$, which satisfies the classical equation of motion (5), with the boundary conditions

$$x_{\text{B}}(\pm\beta/2) = x_{\text{b}}, \quad \dot{x}_{\text{B}}(\pm\beta/2) = 0. \quad (12)$$

It is a periodic solution, satisfying

$$x_{\text{B}}(\tau + \beta) = x_{\text{B}}(\tau) \quad (13)$$

for an arbitrary value of τ . We denote the value of the classical action for the finite- T bounce solution as S_{B} . The finite- T bounce solution has the (Euclidean) energy E ,

$$E = \frac{1}{2} \dot{x}_{\text{B}}^2 - V(x_{\text{B}}), \quad (14)$$

which is equal to $-V(x_{\text{b}})$. The energy is determined as a function of β , or vice versa.

The path integral

$$z_{\text{B}} \approx \int_{x(\pm\beta/2)=x_{\text{b}}} \mathcal{D}x(\tau) e^{-S[x]} \quad (15)$$

may be evaluated by the saddle-point approximation. By substituting Eq. (7), we have

$$z_{\text{B}} \approx e^{-S_{\text{B}}} \int_{r(\pm\beta/2)=0} \mathcal{D}r(\tau) e^{-\Delta S_{\text{B}}[r]}, \quad (16)$$

where

$$\Delta S_{\text{B}}[r] = \frac{1}{2} \int_{-\beta/2}^{\beta/2} d\tau \, r(\tau) \left(-\frac{d^2}{d\tau^2} + V''(x_{\text{B}}(\tau)) \right) r(\tau). \quad (17)$$

The Gaussian integration leads to the determinant of the “fluctuation operator”. But, as is well known, there are two things to be taken into account.

The first is the existence of the zero mode. Actually, by differentiating Eq. (5) with respect to τ , we obtain

$$\left(-\frac{d^2}{d\tau^2} + V''(x_B(\tau))\right) \dot{x}_B(\tau) = 0. \quad (18)$$

Since $\dot{x}_B$ satisfies the boundary conditions $\dot{x}_B(\pm\beta/2) = 0$, it may be regarded as a part of $r(\tau)$. We need to omit the zero eigenvalue from the determinant.

The second is that, since the zero mode $\dot{x}_B$ has a node, there must be a negative mode. The existence of the negative mode causes the imaginary part in the path integral, and leads eventually to the decay rate of the false vacuum. See Refs. [18,19,27].

At zero temperature, a shift of the (Euclidean) time for a bounce solution is related to the zero mode. (See, however, Section 4.1 for the issue of the introduction of a finite time interval.) At finite temperature, on the other hand, the story is not that simple. The shifted bounce solution,

$$x_{sB}(\tau; \tau_c) \equiv x_B(\tau + \tau_c) \quad (0 < \tau_c < \beta), \quad (19)$$

does not satisfy $\dot{x}_{sB}(\pm\beta/2; \tau_c) = 0$ so that it is not regarded as a part of $r(\tau)$ (remember the boundary conditions (8)). There is no zero mode in the Gaussian integration.

We consider the sum of the saddle-point contributions of the shifted bounce solutions,

$$Z_B \approx \mathcal{N} e^{-S_B} \int_{x_b}^{x_r} dy \int_{r(\pm\beta/2)=0} \mathcal{D}r(\tau) e^{-\Delta S_{sB}[r;y]}, \quad (20)$$

where

$$y = x_{sB}(\pm\beta/2; \tau_c) = x_B(\pm\beta/2 + \tau_c) \quad (21)$$

is the starting and ending point, and

$$\Delta S_{sB}[r; y] = \frac{1}{2} \int_{-\beta/2}^{\beta/2} d\tau r(\tau) \left(-\frac{d^2}{d\tau^2} + V''(x_{sB}(\tau; \tau_c))\right) r(\tau). \quad (22)$$

The normalization constant $\mathcal{N}$ will be fixed later.

Note that the shifted bounce solutions exist only for y between x_b and x_r and the value of the classical action for a shifted bounce solution is the same as that for the finite- T bounce solution for any value of y .

There are actually two shifted bounce solutions for any single value of $y \in (x_b, x_r)$; one moves initially to the right and the other to the left, as shown in Fig. 3. Thus Eq. (22) should be understood as an abbreviated one; it is the sum of these two contributions. These two solutions are related by the change of τ_c to $\beta - \tau_c$.

These two contributions can be neatly combined by changing variable from y to τ_c through $y = x_{sB}(\beta/2; \tau_c)$:

$$Z_B \approx e^{-S_B} \int_0^\beta d\tau_c |\dot{x}_{sB}(\beta/2; \tau_c)| \frac{\mathcal{N}}{\sqrt{\det \left[-\frac{d^2}{d\tau^2} + V''(x_B(\tau + \tau_c))\right]}}. \quad (23)$$

We invoke the Gel'fand–Yaglom theorem [28] (see also Appendix 1 of Ref. [19]) to evaluate the determinant. According to the theorem, the ratio of the two determinants can be written as the ratio of the boundary value of the eigenfunctions,

$$\frac{\det \left[-\frac{d^2}{d\tau^2} + V''(x_B(\tau + \tau_c))\right]}{\det \mathcal{O}} = \frac{\psi(\beta/2)}{\psi_{\mathcal{O}}(\beta/2)}, \quad (24)$$

where $\psi(\tau)$ is the eigenfunction of the fluctuation operator,

$$\left[-\frac{d^2}{d\tau^2} + V''(x_B(\tau + \tau_c)) \right] \psi(\tau) = 0, \quad (25)$$

$\mathcal{O}$ is a reference operator of the form

$$\mathcal{O} = -\frac{d^2}{d\tau^2} + W, \quad (26)$$

and $\psi_{\mathcal{O}}(\tau)$ is the eigenfunction of the reference operator,

$$\mathcal{O}\psi_{\mathcal{O}}(\tau) = 0. \quad (27)$$

These eigenfunctions are supposed to satisfy the following boundary conditions:

$$\psi(-\beta/2) = \psi_{\mathcal{O}}(-\beta/2) = 0, \quad (28)$$

$$\dot{\psi}(-\beta/2) = \dot{\psi}_{\mathcal{O}}(-\beta/2) = 1. \quad (29)$$

We assume that the operator $\mathcal{O}$ and its eigenfunction $\psi_{\mathcal{O}}(\tau)$ is known.

Since $\mu(\tau) \equiv \dot{x}_{\text{SB}}(\tau; \tau_c)$ does not satisfy the boundary conditions mentioned above, we need another eigenfunction with eigenvalue zero to construct $\psi(\tau)$. This is provided by the derivative of $x_{\text{SB}}(\tau; \tau_c)$ with respect to the energy. Actually, by differentiating the equation of motion

$$-\frac{d^2}{d\tau^2} x_{\text{SB}}(\tau; \tau_c) + V'(x_{\text{SB}}(\tau; \tau_c)) = 0 \quad (30)$$

with respect to E , we obtain

$$\left[-\frac{d^2}{d\tau^2} + V''(x_B(\tau + \tau_c)) \right] \partial_E x_{\text{SB}}(\tau; \tau_c) = 0, \quad (31)$$

which shows that $\nu(\tau) \equiv \partial_E x_{\text{SB}}(\tau; \tau_c)$ is the one that we need [29].

The eigenfunction $\psi(\tau)$ may be written as a linear combination of $\mu(\tau)$ and $\nu(\tau)$:

$$\psi(\tau) = A\mu(\tau) + B\nu(\tau). \quad (32)$$

By imposing the boundary conditions, we get

$$A\mu(-\beta/2) + B\nu(-\beta/2) = 0, \quad (33)$$

$$A\dot{\mu}(-\beta/2) + B\dot{\nu}(-\beta/2) = 1. \quad (34)$$

The constants A and B are easily obtained by noting that the Wronskian

$$W \equiv \mu(\tau)\dot{\nu}(\tau) - \dot{\mu}(\tau)\nu(\tau) \quad (35)$$

is equal to 1. Actually,

$$W = \dot{x}_{\text{SB}}\partial_E \dot{x}_{\text{SB}} - \ddot{x}_{\text{SB}}\partial_E x_{\text{SB}} = \partial_E \left[\frac{1}{2} \dot{x}_{\text{SB}}^2 - V(x_{\text{SB}}) \right] = 1. \quad (36)$$

Because W is a constant, it may be evaluated at $\tau = -\beta/2$. We obtain $A = -\nu(-\beta/2)$ and $B = \mu(-\beta/2)$.

Thus, the ratio of the determinants (24) has been calculated in terms of μ and ν . We can now rewrite the result in a more useful way.

Because $x_{\text{SB}}(\tau; \tau_c)$ is periodic, $x_{\text{SB}}(\tau + \beta; \tau_c) = x_{\text{SB}}(\tau; \tau_c)$, we obtain

$$\nu(\tau + \beta) + \mu(\tau + \beta) \frac{d\beta}{dE} = \nu(\tau), \quad (37)$$

by differentiating it with respect to E . By setting $\tau = -\beta/2$, we get

$$\nu(\beta/2) - \nu(-\beta/2) = -\mu(\beta/2) \frac{d\beta}{dE}. \quad (38)$$

By using this identity and the periodicity of μ , we rewrite $\psi(\beta/2)$ as

$$\begin{aligned}\psi(\beta/2) &= -v(-\beta/2)\mu(\beta/2) + \mu(-\beta/2)v(\beta/2) \\ &= \mu(-\beta/2)(v(\beta/2) - v(-\beta/2)) \\ &= -[\mu(-\beta/2)]^2 \frac{d\beta}{dE}.\end{aligned}\quad (39)$$

Because the period β is a increasing function of E , $d\beta/dE > 0$, and $\mu(-\beta/2) \neq 0$, we have $\psi(\beta/2) < 0$. This means that the fluctuation operator for x_{SB} does not contain a zero mode, but an odd number of negative modes.

We want to take the fluctuation operator for the finite- T bounce solution as the reference operator: $\mathcal{O} = -d^2/d\tau^2 + V''(x_{\text{B}}(\tau))$. The corresponding eigenfunction that satisfies the boundary conditions at $\tau = -\beta/2$ is $\psi_{\mathcal{O}}(\tau) = \dot{x}_{\text{B}}(\tau)/\ddot{x}_{\text{B}}(-\beta/2)$. As we mentioned earlier, however, there is a zero mode for the operator and $\psi_{\mathcal{O}}$ is actually the zero mode because it satisfies $\psi_{\mathcal{O}}(\beta/2) = 0$. To remove the zero eigenvalue from the determinant, we introduce an arbitrary number λ and consider the determinant, $\det[\mathcal{O} - \lambda]$. Instead of zero, the operator $\mathcal{O} - \lambda$ has the eigenvalue $-\lambda$. We define the zero eigenvalue removed determinant $\det' \mathcal{O}$ as

$$\det' \mathcal{O} = \lim_{\lambda \rightarrow 0} \frac{\det[\mathcal{O} - \lambda]}{-\lambda}.\quad (40)$$

We consider $\psi_{\mathcal{O}\lambda}$, which satisfies

$$[\mathcal{O} - \lambda] \psi_{\mathcal{O}\lambda}(\tau) = 0,\quad (41)$$

with the boundary conditions $\psi_{\mathcal{O}\lambda}(-\beta/2) = 0$ and $\dot{\psi}_{\mathcal{O}\lambda}(-\beta/2) = 1$. Multiplying $\dot{x}_{\text{B}}(\tau)$ with Eq. (41) and integrating over τ , we get

$$\begin{aligned}0 &= \int_{-\beta/2}^{\beta/2} d\tau \dot{x}_{\text{B}}(\tau) [\mathcal{O} - \lambda] \psi_{\mathcal{O}\lambda}(\tau) \\ &= \ddot{x}_{\text{B}}(\beta/2) \psi_{\mathcal{O}\lambda}(\beta/2) - \lambda \int_{-\beta/2}^{\beta/2} d\tau \dot{x}_{\text{B}}(\tau) \psi_{\mathcal{O}\lambda}(\tau),\end{aligned}\quad (42)$$

by integrating by parts and using the boundary conditions. We obtain

$$\frac{\psi_{\mathcal{O}\lambda}(\beta/2)}{-\lambda} = -\frac{\int_{-\beta/2}^{\beta/2} d\tau \dot{x}_{\text{B}}(\tau) \psi_{\mathcal{O}\lambda}(\tau)}{\ddot{x}_{\text{B}}(\beta/2)}.\quad (43)$$

Combining all of them, we have

$$\begin{aligned}\frac{\det\left[-\frac{d^2}{d\tau^2} + V''(x_{\text{B}}(\tau + \tau_c))\right]}{\det' \mathcal{O}} &= \lim_{\lambda \rightarrow 0} \frac{\det\left[-\frac{d^2}{d\tau^2} + V''(x_{\text{B}}(\tau + \tau_c))\right]}{\det[\mathcal{O} - \lambda]/(-\lambda)} \\ &= \lim_{\lambda \rightarrow 0} \frac{\psi(\beta/2)}{\psi_{\mathcal{O}\lambda}(\beta/2)/(-\lambda)} \\ &= -\lim_{\lambda \rightarrow 0} \frac{\ddot{x}_{\text{B}}(\beta/2)}{\int_{-\beta/2}^{\beta/2} d\tau \dot{x}_{\text{B}}(\tau) \psi_{\mathcal{O}\lambda}(\tau)} \psi(\beta/2) \\ &= -\frac{\ddot{x}_{\text{B}}(-\beta/2)\ddot{x}_{\text{B}}(\beta/2)}{\|\dot{x}_{\text{B}}\|^2} \psi(\beta/2),\end{aligned}\quad (44)$$

where we have used $\lim_{\lambda \rightarrow 0} \psi_{\mathcal{O}\lambda}(\tau) = \psi_{\mathcal{O}}(\tau) = \dot{x}_{\text{B}}(\tau)/\ddot{x}_{\text{B}}(-\beta/2)$, and have introduced the notation

$$\|\dot{x}_{\text{B}}\|^2 = \int_{-\beta/2}^{\beta/2} d\tau (\dot{x}_{\text{B}}(\tau))^2.\quad (45)$$

Finally the contribution of the shifted bounce solution (23) can be written as

$$\begin{aligned} Z_B &\approx \mathcal{N} \frac{\beta e^{-S_B}}{\sqrt{\det' \left[-\frac{d^2}{d\tau^2} + V''(x_B(\tau)) \right]}} \sqrt{\frac{||\dot{x}_B||^2}{\ddot{x}_B(-\beta/2)\ddot{x}_B(\beta/2)}} \left[\frac{d\beta}{dE} \right]^{-\frac{1}{2}} \\ &= \mathcal{N} \frac{i\beta e^{-S_B}}{\sqrt{\left| \det' \left[-\frac{d^2}{d\tau^2} + V''(x_B(\tau)) \right] \right|}} \frac{||\dot{x}_B||}{|\ddot{x}_B(\beta/2)|} \left[\frac{d\beta}{dE} \right]^{-\frac{1}{2}}. \end{aligned} \quad (46)$$

Note that the Jacobian in Eq. (23) cancels $[\mu(-\beta/2)]^2$ in Eq. (39) and the factor i appears because $\det' \mathcal{O} < 0$. The integration over τ_c is trivial because the integrand is now independent of τ_c .

The normalization constant $\mathcal{N}$ may be fixed by using the harmonic oscillator potential [18,19]:

$$\frac{\mathcal{N}}{\sqrt{\det \left[-\frac{d^2}{d\tau^2} + \omega^2 \right]}} = \sqrt{\frac{\omega}{2\pi \sinh(\omega\beta)}}. \quad (47)$$

Furthermore, by using the Gel'fand–Yaglom theorem once again, we have

$$\frac{\det \left[-\frac{d^2}{d\tau^2} + \omega^2 \right]}{\det' \left[-\frac{d^2}{d\tau^2} + V''(x_B(\tau)) \right]} = -\frac{|\ddot{x}_B(\beta/2)|^2 \sinh(\omega\beta)}{||\dot{x}_B||^2 \omega}. \quad (48)$$

Note that $\psi_{\text{HO}}(\tau) \equiv (1/\omega) \sinh(\omega(\tau + \beta/2))$ is an eigenfunction of the operator $-d^2/d\tau^2 + \omega^2$ with eigenvalue zero, and satisfies the boundary conditions $\psi_{\text{HO}}(-\beta/2) = 0$ and $\dot{\psi}_{\text{HO}}(-\beta/2) = 1$.

We may now rewrite Eq. (46) in a much simpler form:

$$Z_B \approx \frac{i\beta}{\sqrt{2\pi}} e^{-S_B} \left[\frac{d\beta}{dE} \right]^{-\frac{1}{2}}. \quad (49)$$

At this point, it is instructive to compare Z_B with Eq. (2.12) of Ref. [24],

$$\text{tr } e^{-iHT} = \left(\frac{i}{2\pi\hbar} \right)^{\frac{1}{2}} T \left| \frac{dE_{\text{cl}}}{dT} \right|^{\frac{1}{2}} e^{i(S_{\text{cl}}(T)/\hbar - \pi)}, \quad (50)$$

where T is a time interval, or the period of the classical solution. Changing $T \rightarrow -i\beta$, $E_{\text{cl}} \rightarrow -E$, and $S_{\text{cl}} \rightarrow -iS_B$ formally, we see that it is identical to our result.

The n -cycle finite- T and shifted bounce solutions with period β/n are also classical solutions with period β . We ignore all of these “multiple-bounce” solutions, because (i) we assume that the $n = 1$ contribution is already very small, and (ii) there is a maximum value of n due to the condition $\beta \geq \beta_0$.

3. Shot solutions and decay rate

A shot-F solution $x_{\text{sF}}(\tau; y)$ starts at $x = y$ going toward to $x = x_F$, reaches the highest point $x = x_s$, and returns to the original position after the time lapse β :

$$\begin{aligned} x_{\text{sF}}(\pm\beta/2; y) &= y, \\ x_{\text{sF}}(0; y) &= x_s, \\ \dot{x}_{\text{sF}}(-\beta/2; y) &= -\dot{x}_{\text{sF}}(\beta/2; y). \end{aligned} \quad (51)$$

We denote the corresponding classical action as $S_{\text{sF}}(y)$.

The saddle-point approximation of the contribution of the shot-F solutions may be written as

$$Z_{\text{sF}} \approx \int_{x_b}^{x_t} dy \frac{\mathcal{N} e^{-S_{\text{sF}}(y)}}{\sqrt{\det \left[-\frac{d^2}{d\tau^2} + V''(x_{\text{sF}}(\tau; y)) \right]}}. \quad (52)$$

The upper limit x_t will be discussed shortly.

Note that, since the initial and final velocities are nonzero, $\dot{x}_{\text{sF}}(\tau)$ is not a zero mode of the fluctuation operator, in contrast to the bounce case.

By using Eq. (47) and the Gel'fand–Yaglom theorem, we obtain

$$Z_{\text{sF}} \approx \frac{1}{\sqrt{2\pi}} \int dy \frac{e^{-S_{\text{sF}}(y)}}{\sqrt{\psi_{\text{sF}}(\beta/2; y)}}, \quad (53)$$

where $\psi_{\text{sF}}(\tau; y)$ is the eigenfunction

$$\left[-\frac{d^2}{d\tau^2} + V''(x_{\text{sF}}(\tau; y)) \right] \psi_{\text{sF}}(\tau; y) = 0, \quad (54)$$

with the boundary conditions $\psi_{\text{sF}}(-\beta/2; y) = 0$ and $\dot{\psi}_{\text{sF}}(-\beta/2; y) = 1$.

We can construct this eigenfunction by using $\mu_s(\tau; y) \equiv \dot{x}_{\text{sF}}(\tau; y)$ and $v_s(\tau; y) \equiv \partial_E x_{\text{sF}}(\tau; y)$ in a similar manner to the case of the shifted bounce solutions:

$$\psi_{\text{sF}}(\tau; y) = -v_s(-\beta/2; y)\mu_s(\tau; y) + \mu_s(-\beta/2; y)v_s(\tau; y). \quad (55)$$

Thus, we have

$$\psi_{\text{sF}}(\beta/2; y) = 2\mu_s(-\beta/2; y)v_s(-\beta/2; y), \quad (56)$$

by using Eq. (51) and $v_s(\beta/2; y) = v_s(-\beta/2; y)$.

For the potential depicted in Fig. 1, the initial velocity is negative: $\mu_s(-\beta/2; y) < 0$. On the other hand, the determination of the sign of $v_s(-\beta/2; y)$ requires more careful examination.

If the energy increases, the turning point x_s gets closer to x_F , i.e. shifts to the left. One may naively expect that the starting point also shifts to the left in order for the particle to return back to it after the time lapse β , and thus $v_s(-\beta/2; y) < 0$. If this is the case, $\psi_{\text{sF}}(\beta/2; y) > 0$ so that Z_{sF} is real.

Amazingly, this expectation is not always correct. We numerically find that there is a region of $y \in (x_r, x_t)$, for which there are *two* shot-F solutions for a given time lapse β , one of which has $v_s(-\beta/2; y) < 0$, and the other $v_s(-\beta/2; y) > 0$. The latter shot-F solution leads to an imaginary partition function and thus may be called $x_{\text{sF}}^{(\text{I})}$. The shot-F solution with $v_s(-\beta/2; y) < 0$ may be called $x_{\text{sF}}^{(\text{R})}$.

The eigenfunction (56) may be written in a different way. By differentiating $S_{\text{sF}}(y)$ with respect to the energy E of the shot-F solution, we find

$$\begin{aligned} \frac{dS_{\text{sF}}(y)}{dE} &= \int_{-\beta/2}^{\beta/2} d\tau \left[\dot{x}_{\text{sF}} \frac{\partial \dot{x}_{\text{sF}}}{\partial E} + V'(x_{\text{sF}}) \frac{\partial x_{\text{sF}}}{\partial E} \right] \\ &= \left[\dot{x}_{\text{sF}} \frac{\partial x_{\text{sF}}}{\partial E} \right]_{-\beta/2}^{\beta/2} + \int_{-\beta/2}^{\beta/2} d\tau \left[-\ddot{x}_{\text{sF}} + V'(x_{\text{sF}}) \right] \frac{\partial x_{\text{sF}}}{\partial E} \\ &= -2\mu_s(-\beta/2; y)v_s(-\beta/2; y) \\ &= -\psi_{\text{sF}}(\beta/2; y). \end{aligned} \quad (57)$$

For $x_{\text{sF}}^{(\text{R})}$ it is negative while for $x_{\text{sF}}^{(\text{I})}$ it is positive.

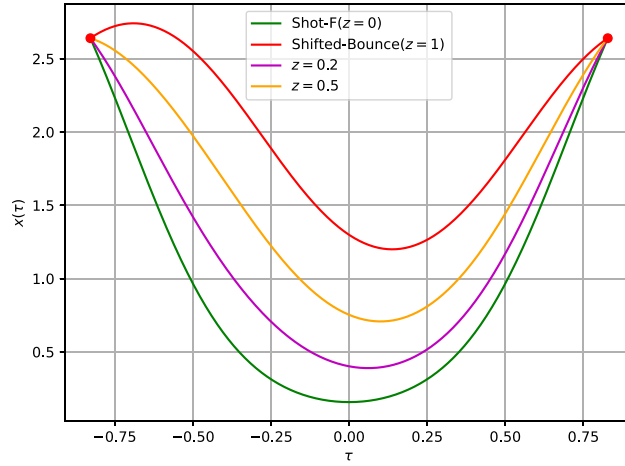


Fig. 5. A one-parameter family of the classical paths $\tilde{x}(\tau; z)$ that interpolates the shifted bounce solution ($z = 1$) and the shot-F solution ($z = 0$), numerically calculated for the potential $V(x) = x^4 - 8x^3 + 15.9x^2$.

It is instructive to construct a one-parameter family of the classical paths $\tilde{x}(\tau; z)$ that interpolates the (shifted) bounce solution $\tilde{x}(\tau; 1) = x_{\text{SB}}(\tau; \tau_c)$ with $y = x_{\text{SB}}(\pm\beta/2; \tau_c)$ and the shot-F solution $\tilde{x}(\tau; 0) = x_{\text{SF}}^{(\text{R})}(\tau; y)$ for each value of $y \in [x_b, x_r]$. We require that it satisfies (i) $\tilde{x}(\pm\beta/2; z) = y$ and (ii) $[d\tilde{x}/dz]_{z=1} = r_-(\tau; \tau_c)$, where $r_-(\tau; \tau_c)$ is the eigenfunction of $-d^2/d\tau^2 + V''(x_{\text{SB}}(\tau; \tau_c))$ with negative eigenvalue. We use a quadratic function of z ,

$$\begin{aligned} \tilde{x}(\tau; z) = & (r_-(\tau; \tau_c) - x_{\text{SB}}(\tau; \tau_c) + x_{\text{SF}}^{(\text{R})}(\tau; y))z^2 \\ & + (2x_{\text{SB}}(\tau; \tau_c) - 2x_{\text{SF}}^{(\text{R})}(\tau; y) - r_-(\tau; \tau_c))z + x_{\text{SF}}^{(\text{R})}(\tau; y). \end{aligned} \quad (58)$$

See Fig. 5. Remember that a similar construction was considered by Callan and Coleman to draw Figs. 5 and 6 in Ref. [18].

One can easily show that the action $S(z) \equiv S[\tilde{x}]$ has a minimum at $z = 0$ (shot-F) and a maximum at $z = 1$ (bounce). Actually,

$$\frac{dS(z)}{dz} = \int_{-\beta/2}^{\beta/2} d\tau \left[\dot{\tilde{x}} \frac{d\tilde{x}}{dz} + V'(\tilde{x}) \frac{d\tilde{x}}{dz} \right] = \int_{-\beta/2}^{\beta/2} d\tau [-\ddot{\tilde{x}} + V'(\tilde{x})] \frac{d\tilde{x}}{dz} \quad (59)$$

shows that it vanishes at $z = 0$ and $z = 1$, because, at these points, $\tilde{x}$ is a solution of the equation of motion. Furthermore, we obtain

$$\begin{aligned} \frac{d^2S(z)}{dz^2} = & \int_{-\beta/2}^{\beta/2} d\tau \left[-\frac{d\ddot{\tilde{x}}}{dz} + V''(\tilde{x}) \frac{d\tilde{x}}{dz} \right] \frac{d\tilde{x}}{dz} + \int_{-\beta/2}^{\beta/2} d\tau [-\ddot{\tilde{x}} + V'(\tilde{x})] \frac{d^2\tilde{x}}{dz^2} \\ = & \int_{-\beta/2}^{\beta/2} d\tau \frac{d\tilde{x}}{dz} \left[-\frac{d^2}{d\tau^2} + V''(\tilde{x}) \right] \frac{d\tilde{x}}{dz} + \int_{-\beta/2}^{\beta/2} d\tau [-\ddot{\tilde{x}} + V'(\tilde{x})] \frac{d^2\tilde{x}}{dz^2}. \end{aligned} \quad (60)$$

The second term vanishes at $z = 0$ and $z = 1$ for the same reason as before. For $z = 0$, because there is no negative mode or zero mode, the first term is positive: $[d^2S(z)/dz^2]_{z=0} > 0$. On the other hand, because of the condition (ii), the first term is negative: $[d^2S(z)/dz^2]_{z=1} < 0$. $S(z)$ is depicted in Fig. 6 for a specific potential.

This implies that the shot-F solution plays a similar role to that of the false-vacuum solution $x = x_{\text{F}}$, that the steepest descent contour may be similar to the Callan–Coleman case, and that the contribution of the bounce solution has the well known factor $1/2$.

For $y \in (x_r, x_t)$, $x_{\text{SF}}^{(\text{I})}$ plays a similar role of the finite- T bounce and shifted bounce solutions.

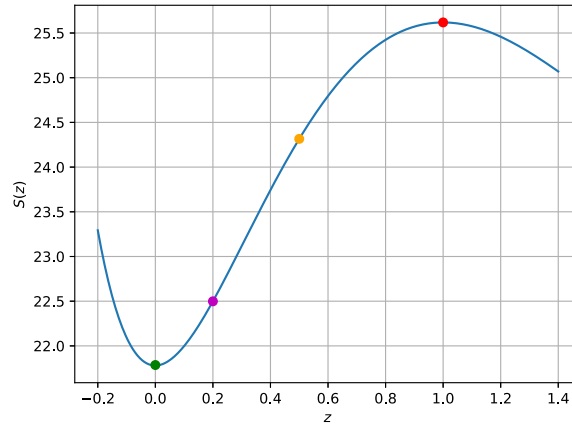


Fig. 6. The classical action $S(z)$ for the parametrized path $\tilde{x}(\tau; z)$. Colored dots correspond to the bounce solution (red), the shot-F solution (green), and the interpolating paths (purple and orange) given in Fig. 5.

From these considerations, we arrive at the formula for the decay rate Γ at finite temperature:

$$\Gamma = -2\text{Im}(F_{\text{GS}}) = \frac{2}{\beta} \tan^{-1} \left(\frac{\text{Im}Z_{\text{GS}}}{\text{Re}Z_{\text{GS}}} \right) = \frac{2}{\beta} \tan^{-1} \left(\frac{Z_{\text{B}} + Z_{\text{sF}}^{(\text{I})}}{2Z_{\text{sF}}^{(\text{R})}} \right), \quad (61)$$

where

$$Z_{\text{GS}} = Z_{\text{sF}}^{(\text{R})} + \frac{1}{2} (Z_{\text{B}} + Z_{\text{sF}}^{(\text{I})}), \quad (62)$$

with

$$Z_{\text{sF}}^{(\text{R})} \approx \frac{1}{\sqrt{2\pi}} \int_{x_{\text{b}}}^{x_{\text{t}}} dy \frac{e^{-S[x_{\text{sF}}^{(\text{R})}(\tau; y)]}}{\sqrt{\left| \frac{dS_{\text{sF}}^{(\text{R})}}{dE} \right|}}, \quad (63)$$

$$Z_{\text{sF}}^{(\text{I})} \approx \frac{i}{\sqrt{2\pi}} \int_{x_{\text{r}}}^{x_{\text{t}}} dy \frac{e^{-S[x_{\text{sF}}^{(\text{I})}(\tau; y)]}}{\sqrt{\left| \frac{dS_{\text{sF}}^{(\text{I})}}{dE} \right|}}. \quad (64)$$

Numerical solutions on the basis of this formula will be discussed in Section 5.

4. Zero-temperature limit

4.1. Zero-temperature result by Callan and Coleman

Before considering the zero-temperature limit of our result, let us recapitulate the results by Callan and Coleman (CC) [18] for zero temperature, emphasizing the finite time interval T (not temperature in this subsection), which is introduced to regularize the theory. We use the notation of Ref. [18] in this subsection, to make the comparison easier.

The CC formula (Eq. (2.24) of Ref. [18]) for the decay rate is given by

$$\Gamma = \left[\frac{B}{2\pi\hbar} \right]^{\frac{1}{2}} e^{-B/\hbar} \left| \frac{\det' [-\partial_t^2 + V''(\bar{x})]}{\det [-\partial_t^2 + \omega^2]} \right|^{-\frac{1}{2}} \times [1 + \mathcal{O}(\hbar)], \quad (65)$$

where $\bar{x}$ denotes the (zero-temperature) bounce solution and B is the bounce action. We would like to emphasize three points on this formula:

- (1) They consider a large but finite time interval T , which is eventually sent to infinity.

- (2) The determinants are supposed to be defined in the finite time interval T with the Dirichlet boundary conditions at the boundaries $t = \pm T/2$.
- (3) The determinant $\det'$ is the one with the zero eigenvalue omitted, and the zero-mode integration is performed as a collective coordinate integration, leading to the factor $(B/2\pi\hbar)^{\frac{1}{2}}T$.

In the saddle-point approximation, they write a general path $x(t)$ satisfying the boundary conditions $x(-T/2) = x_i$ and $x(T/2) = x_f$ as

$$x(t) = \bar{x}(t) + \sum_n c_n x_n(t), \quad (66)$$

where

$$\int_{-T/2}^{T/2} dt x_n(t) x_m(t) = \delta_{nm}, \quad (67)$$

$$x_n(\pm T/2) = 0, \quad (68)$$

and the path-integral measure $[dx]$ is defined as

$$[dx] = \prod_n (2\pi\hbar)^{-\frac{1}{2}} dc_n. \quad (69)$$

The complete set $\{x_n\}$ is chosen as the set of eigenfunctions of the fluctuation operator,

$$\left[-\frac{d^2}{dt^2} + V''(\bar{x}) \right] x_n = \lambda_n x_n. \quad (70)$$

Therefore, the determinant $\det[-\partial_t^2 + V''(\bar{x})] = \prod_n \lambda_n$ is defined with the Dirichlet boundary conditions (68).

Because $\bar{x}(t)$ is a solution of the (Euclidean) equation of motion, $d\bar{x}/dt$ is the eigenfunction with zero eigenvalue. The normalization (67) gives

$$x_1(t) = B^{-\frac{1}{2}} \frac{d\bar{x}}{dt}. \quad (71)$$

The infinitesimal change of the corresponding coefficient c_1 may be related to the infinitesimal change of the time of the classical solution. Thus $dt = dc_1 B^{-\frac{1}{2}}$ and the integration with the measure $(2\pi\hbar)^{-\frac{1}{2}} dc_1$ can be done with the measure $(B/2\pi\hbar)^{\frac{1}{2}} dt$, leading to the factor $(B/2\pi\hbar)^{\frac{1}{2}}T$.

The calculation described above is, however, not satisfactory. First of all, the Dirichlet boundary conditions are not compatible with the time-translational invariance on which the collective coordinate method is based. The translational invariance is broken by the very boundary conditions. Second, the relation $dt = dc_1 B^{-\frac{1}{2}}$ holds only in the vicinity of $t = 0$. A symptom is explained in e.g. Appendix B of Ref. [20]: the range of the thus-defined coefficient c_1 is bounded, incompatible with the integration from $-T/2$ to $T/2$ (eventually, from $-\infty$ to $+\infty$).

In order to have exact translational invariance with a finite T , one should impose the periodic boundary conditions (in the sense of Section 1) on $x(t)$: $x(-T/2) = x(T/2)$, $\dot{x}(-T/2) = \dot{x}(T/2)$.

Consider the time-translated classical solution $\bar{x}(t - t_0)$ and the expansion [20]

$$x(t) = \bar{x}(t - t_0) + \sum_n \bar{c}_n \bar{x}_n(t - t_0), \quad (72)$$

instead of the expansion (66), where the eigenfunctions $\{\bar{x}_n\}$ now satisfy the periodic boundary conditions.

The infinitesimal change of $x(t)$ with respect to the infinitesimal change to t_0 may be related to the infinitesimal change of $\bar{c}_1$,

$$\begin{aligned} dx(t) &= \bar{x}(t - t_0 - dt_0) - \bar{x}(t - t_0) = dt_0 \frac{d\bar{x}}{dt}(t - t_0) \\ &= d\bar{c}_1 \bar{x}_1(t - t_0) = d\bar{c}_1 B^{-\frac{1}{2}} \frac{d\bar{x}}{dt}(t - t_0); \end{aligned} \quad (73)$$

we have $dt_0 = d\bar{c}_1 B^{-\frac{1}{2}}$. This relation holds for any value of $t_0 \in [-T/2, T/2]$. Thus the integration over $\bar{c}_1$ may be replaced by the integration over t_0 . That is, instead of Eq. (69), we have

$$[dx] = (B/2\pi\hbar)^{\frac{1}{2}} dt_0 \prod_{n \neq 1} (2\pi\hbar)^{-\frac{1}{2}} d\bar{c}_n. \quad (74)$$

The integration over t_0 leads to the factor $(B/2\pi\hbar)^{\frac{1}{2}} T$.

It is therefore not the amplitude

$$\langle 0 | e^{-HT/\hbar} | 0 \rangle = \mathcal{N} \int_{x(\pm T/2)=0} [dx] e^{-S[x]/\hbar}, \quad (75)$$

which Callan and Coleman pretended to calculate (with the Dirichlet boundary conditions $x_i = x_f = 0$), but a path integral with the periodic boundary conditions

$$\mathcal{N} \int_{\text{p.b.c.}} [dx] e^{-S[x]/\hbar} \quad (76)$$

that one should be calculating with the method of collective coordinates. However, the operator representation corresponding to Eq. (76) is obscure to us.

Note that Eq. (76) is not the trace

$$\text{tr}(e^{-HT/\hbar}) = \int dy \langle y | e^{-HT/\hbar} | y \rangle = \int dy \mathcal{N} \int_{x(\pm T/2)=y} [dx] e^{-S[x]/\hbar}. \quad (77)$$

Equation (76) does not include a shot-type path $\dot{x}(\pm T/2) \neq 0$, $\dot{x}(T/2) = -\dot{x}(-T/2)$, while Eq. (77) does.

The determinant of the fluctuation operator should also be defined with the periodic boundary conditions. It contains zero eigenvalue even for a finite T .

According to the approach described above, the final result that Callan and Coleman should have obtained is given by the same expression as in Eq. (65) but now the determinants are defined with the periodic boundary conditions, and the $T \rightarrow \infty$ limit is understood.

In this subsection, we pointed out the problem related to the translational invariance and the collective coordinate, and suggested a viable approach to solve it. Even if we take this approach, however, it is not completely satisfactory. The bounce solution at zero temperature does not exactly vanish for a finite T , so that it has somewhere a point where $\dot{\bar{x}}(t)$ is not smooth and thus $\bar{x}(t)$ is not a classical solution there. To have classical solutions that are smooth everywhere, one should consider classical solutions with the period T , which are in essence the finite- T bounce and the shifted bounce solutions considered in Section 2. The best way of obtaining the decay rate at zero temperature is, in our opinion, to obtain the finite-temperature result and take the zero-temperature limit of it. That is our next task.

4.2. Zero-temperature limit of Z_B

In the zero-temperature limit ($\beta \rightarrow \infty$), $x_b \rightarrow x_F = 0$, $x_r \rightarrow x_R$, and $E \rightarrow 0$. Since S_B has a smooth limit (which we call S_B^0), we only need to know how $d\beta/dE$ in Eq. (49) behaves in the zero-temperature limit.

The period β may be written as

$$\beta = 2 \int_{x_b}^{x_r} \frac{dx}{\sqrt{2(V(x) - V(x_b))}}, \quad (78)$$

since $\dot{x} = \pm \sqrt{2(V(x) + E)} = \pm \sqrt{2(V(x) - V(x_b))}$. When β is very large, x_b is very close to x_F , so that, for x close to x_b , the potential $V(x)$ may be well approximated by a parabola,

$$V(x) \approx V(x_b) + V'(x_b)(x - x_b) + \frac{1}{2}V''(x_b)(x - x_b)^2. \quad (79)$$

Having this in mind, we rewrite Eq. (78) as

$$\begin{aligned} \beta = 2 \int_{x_b}^{x_r} dx & \left[\frac{1}{\sqrt{2(V(x) - V(x_b))}} - \frac{1}{\sqrt{2V'(x_b)(x - x_b) + V''(x_b)(x - x_b)^2}} \right] \\ & + 2 \int_{x_b}^{x_r} \frac{dx}{\sqrt{2V'(x_b)(x - x_b) + V''(x_b)(x - x_b)^2}}. \end{aligned} \quad (80)$$

The first integral now has a smooth limit when $x_b \rightarrow x_F$. The second integral is elementary:

$$\int_{x_b}^{x_r} \frac{dx}{\sqrt{2V'(x_b)(x - x_b) + V''(x_b)(x - x_b)^2}} = \frac{1}{\sqrt{V''(x_b)}} \ln \frac{X + \sqrt{X^2 - \left(\frac{V'(x_b)}{V''(x_b)}\right)^2}}{\frac{V'(x_b)}{V''(x_b)}}, \quad (81)$$

where $X = x_r - x_b + V'(x_b)/V''(x_b)$.

For large β , we get

$$\beta \approx \frac{2}{\sqrt{V''(x_F)}} \int_{x_F}^{x_R} dx \left[\frac{\sqrt{V''(x_F)}}{\sqrt{2(V(x) - V(x_F))}} - \frac{1}{x - x_F} \right] + \frac{2}{\sqrt{V''(x_F)}} \ln \left[\frac{2(x_R - x_F)}{x_b - x_F} \right], \quad (82)$$

and thus

$$\lim_{\beta \rightarrow \infty} (x_b - x_F)^2 e^{\omega\beta} = 4(x_R - x_F)^2 e^{2 \int_{x_F}^{x_R} dx \left[\frac{\omega}{\sqrt{2(V(x) - V(x_F))}} - \frac{1}{x - x_F} \right]}, \quad (83)$$

where we introduce $\omega = \sqrt{V''(x_F)}$. Note that the right-hand side of Eq. (83) is a constant. Noting $\ln(x_b - x_F) = \ln((x_b - x_F)e^{\omega\beta/2}) - \omega\beta/2$ and using l'Hôpital's rule, we obtain

$$\lim_{\beta \rightarrow \infty} \frac{d\beta/dx_b}{(x_b - x_F)^{-1}} = \lim_{\beta \rightarrow \infty} \frac{\beta}{\ln(x_b - x_F)} = -\frac{2}{\omega}, \quad (84)$$

where we have taken $\lim_{\beta \rightarrow \infty} \ln((x_b - x_F)e^{\omega\beta/2})$ as finite due to Eq. (83). We thus find

$$\lim_{\beta \rightarrow \infty} (x_b - x_F) \frac{d\beta}{dx_b} = -\frac{2}{\omega}. \quad (85)$$

We also note

$$\frac{d\beta}{dE} = \frac{d\beta}{dx_b} \frac{dx_b}{dE} = -\frac{d\beta}{dx_b} \frac{1}{V'(x_b)}, \quad (86)$$

where we have used $V'(x_b)(dx_b/dE) = -1$, which follows from $V(x_b) = -E$. When β is very large, x_b is very close to x_F , and thus $V(x_b) \approx V(x_F) + \frac{1}{2}\omega^2(x_b - x_F)^2$. It follows that $V'(x_b) \approx \omega^2(x_b - x_F)$.

Therefore, we find

$$\lim_{\beta \rightarrow \infty} \frac{d\beta}{dE} = \lim_{x_b \rightarrow x_F} \frac{2\omega}{\omega^4(x_b - x_F)^2}. \quad (87)$$

By substituting this into Eq. (49), we finally get

$$\lim_{\beta \rightarrow \infty} Z_B/\beta = \lim_{x_b \rightarrow x_F} \frac{i}{\sqrt{2\pi}} e^{-S_B^0} \frac{\omega^2(x_b - x_F)}{\sqrt{2\omega}}. \quad (88)$$

4.3. Zero-temperature limit of Z_{sF}

In the zero-temperature limit ($\beta \rightarrow \infty$), x_t and x_b both go to x_R , so that the imaginary branch solution $x_{sF}^{(I)}$ does not exist, and we only have the real branch contribution. In this subsection, we omit the superscript (R) for notational simplicity. The turning point x_s goes to x_F . The action has a smooth limit, which we call $S_{sF}^0(y)$. Therefore we only need to know how $\sqrt{|dS_{sF}/dE|} = \sqrt{2\mu_s(-\beta/2; y)v_s(-\beta/2; y)}$ behaves to get the limit.

Let us start with $v_s(-\beta/2; y) = \partial_E x_{sF}(-\beta/2; y)$. Since $y = x_{sF}(-\beta/2; y)$, one may write it as

$$v_s(-\beta/2; y) = \frac{dx_s}{dE} \frac{dy}{dx_s} = -\frac{1}{V'(x_s)} \frac{dy}{dx_s}, \quad (89)$$

since $-V(x_s) = E$.

The period β is given by

$$\beta = 2 \lim_{\epsilon \rightarrow 0} \int_{x_s+\epsilon}^y \frac{dx}{\sqrt{2(V(x) - V(x_s))}}, \quad (90)$$

where a small positive number ϵ is introduced. By keeping the period fixed and differentiating it with respect to x_s , we have

$$\begin{aligned} 0 = 2 \lim_{\epsilon \rightarrow 0} & \left[\frac{dy}{dx_s} \frac{1}{\sqrt{2(V(y) - V(x_s))}} - \frac{1}{\sqrt{2(V(x_s + \epsilon) - V(x_s))}} \right] \\ & + 2 \lim_{\epsilon \rightarrow 0} \int_{x_s+\epsilon}^y \frac{V'(x_s) dx}{[2(V(x) - V(x_s))]^{3/2}}. \end{aligned} \quad (91)$$

By adding and substituting $1/\sqrt{2(V(y) - V(x_s))}$, we rewrite it as

$$\left(\frac{dy}{dx_s} - 1 \right) \frac{1}{\sqrt{2(V(y) - V(x_s))}} = \lim_{\epsilon \rightarrow 0} \int_{x_s+\epsilon}^y \frac{V'(x) - V'(x_s)}{[2(V(x) - V(x_s))]^{3/2}} dx, \quad (92)$$

and thus

$$\lim_{\beta \rightarrow \infty} \frac{dy}{dx_s} = \frac{1}{2} \lim_{x_s \rightarrow x_F} \int_{x_s}^y \frac{\sqrt{V(y) - V(x_F)} V'(x)}{(V(x) - V(x_F))^{3/2}} dx + 1 = \lim_{x_s \rightarrow x_F} \sqrt{\frac{V(y) - V(x_F)}{V(x_s) - V(x_F)}}. \quad (93)$$

By substituting this into Eq. (89), we obtain

$$\lim_{\beta \rightarrow \infty} v_s(-\beta/2) = - \lim_{x_s \rightarrow x_F} \frac{1}{V'(x_s)} \sqrt{\frac{V(y) - V(x_F)}{V(x_s) - V(x_F)}}. \quad (94)$$

Let us move to $\mu_s(-\beta/2; y) = \dot{x}_{sF}(-\beta/2; y)$. From the energy conservation, we have

$$\mu_s(-\beta/2; y) = -\sqrt{2(V(y) - V(x_s))}. \quad (95)$$

Note the sign of the right-hand side.

By combining Eqs. (94) and (95), we obtain

$$\begin{aligned} \lim_{\beta \rightarrow \infty} \left| \frac{dS_{sF}(y)}{dE} \right| &= \lim_{\beta \rightarrow \infty} \frac{2\sqrt{2(V(y) - V(x_s))(V(y) - V(x_F))}}{V'(x_s)\sqrt{V(x_s) - V(x_F)}} \\ &= \lim_{\beta \rightarrow \infty} \frac{4(V(y) - V(x_F))}{\omega^3(x_s - x_F)^2}, \end{aligned} \quad (96)$$

and this leads to

$$\lim_{\beta \rightarrow \infty} Z_{sF} = \lim_{x_s \rightarrow x_F} \frac{1}{2} \sqrt{\frac{1}{2\pi\omega}} \int_{x_F}^{x_R} dy \frac{\omega^2(x_s - x_F) e^{-S_{sF}^0(y)}}{\sqrt{V(y) - V(x_F)}}. \quad (97)$$

4.4. Zero-temperature limit of the decay rate

In order to compute the zero-temperature limit of the decay rate, we need to know the relation between $(x_b - x_F)$ in Eq. (88) and $(x_s - x_F)$ in Eq. (97). For this purpose, let us return to Eq. (90). In a parallel way to getting Eq. (83), we have

$$\lim_{\beta \rightarrow \infty} (x_s - x_F)^2 e^{\omega\beta} = 4(y - x_F)^2 e^{2 \int_{x_F}^y dx \left[\frac{\omega}{\sqrt{2(V(x) - V(x_F))}} - \frac{1}{x - x_F} \right]}. \quad (98)$$

Taking the ratio of Eq. (83) to Eq. (98), we obtain

$$\lim_{\beta \rightarrow \infty} \frac{x_b - x_F}{x_s - x_F} = e^{\int_y^{x_R} \frac{\omega dx}{\sqrt{2(V(x) - V(x_F))}}}. \quad (99)$$

Now, we are ready to calculate the zero-temperature limit of the false-vacuum decay rate,

$$\lim_{\beta \rightarrow \infty} \Gamma \approx \lim_{\beta \rightarrow \infty} \frac{1}{\beta} \frac{\text{Im} Z_B}{Z_{SF}} \approx \frac{e^{-S_B^0}}{\int_{x_F}^{x_R} dy \frac{e^{-S_{SF}^0(y)}}{\sqrt{2(V(y) - V(x_F))}} e^{-\int_y^{x_R} \frac{\omega dx}{\sqrt{2(V(x) - V(x_F))}}}}. \quad (100)$$

This is different from the CC formula (65). Before comparing our result with it, let us rewrite it in a convenient form. As we show in Appendix B, the ratio of the determinants in the $T \rightarrow \infty$ limit is the same both for the periodic boundary conditions and the Dirichlet boundary conditions, and is given by Eq. (B29). Thus, ignoring $\mathcal{O}(\hbar)$ and using our notation, Eq. (65) may be written as

$$\Gamma_{CC} = \sqrt{\frac{\omega}{\pi}} \omega(x_R - x_F) e^{-S_B^0} e^{\int_{x_F}^{x_R} dx \left[\frac{\omega}{\sqrt{2(V(x) - V(x_F))}} - \frac{1}{x - x_F} \right]}. \quad (101)$$

Interestingly, these two are not completely unrelated. Consider the integrand of the denominator of Eq. (100),

$$X(y) \equiv \frac{e^{-S_{SF}^0(y)}}{\sqrt{2(V(y) - V(x_F))}} e^{-\int_y^{x_R} dx \left[\frac{\omega dx}{\sqrt{2(V(x) - V(x_F))}} \right]}. \quad (102)$$

In the limit $y \rightarrow x_F$, $S_{SF}^0(y) \rightarrow 0$, $\sqrt{2(V(y) - V(x_F))} \rightarrow \omega(y - x_F)$, we may obtain

$$X^{-1}(x_F) = \omega(x_R - x_F) e^{\int_{x_F}^{x_R} dx \left[\frac{\omega dx}{\sqrt{2(V(x) - V(x_F))}} - \frac{1}{x - x_F} \right]}. \quad (103)$$

Therefore, we find that Γ_{CC} may be written in terms of $X(x_F)$ as

$$\Gamma_{CC} = \frac{e^{-S_B^0}}{\sqrt{\frac{\pi}{\omega}} X(x_F)}. \quad (104)$$

The factor $\sqrt{\pi/\omega}$ corresponds to the y -integration. For a typical potential, $X(y)$ has a peak near $y = x_F$, but the peak is not sharp. In the next section, we compare our decay rate with Γ_{CC} as well as the WKB decay rate Γ_{WKB} obtained in Ref. [20].

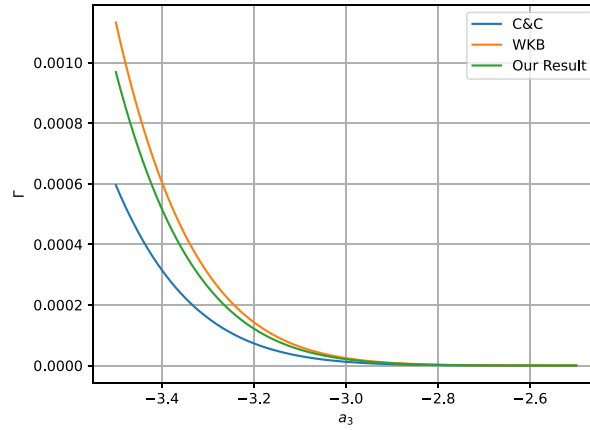


Fig. 7. Decay rates for the potential $V_1(x) = a_3x^3 + 8x^2$. The zero-temperature limits of our result (100), Γ_{CC} , and Γ_{WKB} are plotted as functions of the parameter a_3 , which controls the potential barrier.

5. Numerical solutions and comparison with previous work

In this section, we perform some numerical calculations for several potentials.

5.1. Decay rates as a function of the potential in the zero-temperature limit

We first compare our zero-temperature limit (100) with Γ_{CC} (101) and NLO WKB decay rate Γ_{WKB} (Eq. (2.19) in Ref. [20]), which may be written as

$$\Gamma_{WKB} = \frac{e^{-2 \int_a^b dx \pi(x)}}{\int_{x_F}^a \frac{dx}{p(x)} + \int_a^b \frac{dy}{\pi(y)} e^{-2 \int_a^y dx \pi(x)}}, \quad (105)$$

where $p(x) = \sqrt{2(E_0 - V(x))}$, $\pi(x) = \sqrt{2(V(x) - E_0)}$, and E_0 is the “ground-state” energy, $\omega/2$. The classical turning points a and b ($x_F < a < b < x_R$) satisfy $V(a) = V(b) = E_0$.

We consider the potential

$$V_1(x) = a_3x^3 + 8x^2 \quad x \in [x_R, x_F], \quad (106)$$

which satisfies $V(0) = V'(0) = 0$, $V''(0) = 16$ ($\omega = 4$) and the parameter a_3 controls the potential barrier. Note that the three formulae depend only on the shape of the potential for $x \in [x_F, x_R]$. It is necessary that the height of the potential $V_{\max}$ is much larger than the energy $\omega/2$, which would be the ground-state energy if the tunneling were turned off. The range of a_3 must be chosen to satisfy this constraint. In Fig. 7, we compare the three decay rates. The decay rates decrease as the potential height increases, as expected. Our result is similar to the WKB result in both magnitude and slope, but quite different from the CC result.

Figure 8 shows the decay rates for another potential,

$$V_2(x) = x^4 - 8x^3 + a_2x^2, \quad (107)$$

where the parameter a_2 is close to 16; i.e. the potential is close to the symmetric case $V_{\text{sym}}(x) = x^2(x - 4)^2$. Again, the zero-temperature limit of our result is close to the WKB result, but different from the CC result.

In Fig. 9, we show the decay rates for yet another potential of nonpolynomial type,

$$V_{\text{np}} = \cosh(0.2x) + \frac{10}{\cosh(x + a_0)}. \quad (108)$$

The result is very similar to that of Fig. 8, for a wide range of the parameter a_0 .

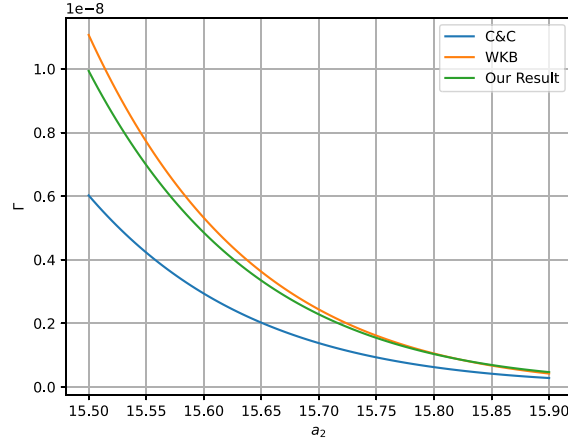


Fig. 8. Decay rates for the potential $V_2(x) = x^4 - 8x^3 + a_2x^2$. The zero-temperature limits of our result (100), Γ_{CC} , and Γ_{WKB} are plotted as functions of the parameter a_2 , when the potential is close to the symmetric case.

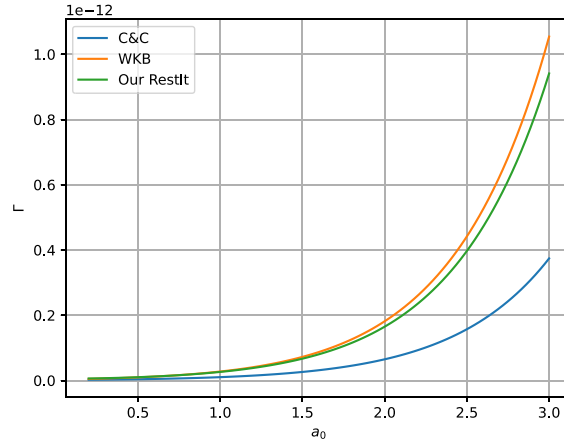


Fig. 9. Decay rates for the potential $V_{np}(x) = \cosh(0.2x) + 10/\cosh(x + a_0)$. The zero-temperature limits of our result (100), Γ_{CC} , and Γ_{WKB} are plotted as functions of the parameter a_0 .

5.2. Temperature dependence of the decay rate

In this subsection, we numerically compute our result (61) as a function of β^{-1} and compare it with that of Affleck's formula, obtained from Eq. (14) in Ref. [21] with $\Gamma = (2/\hbar) \text{Im}F$. It is given in our notation by

$$\Gamma_{\text{Affleck}} = \frac{||\dot{x}_B||}{\sqrt{2\pi}} \left| \frac{\det \left[-\frac{d^2}{d\tau^2} + \omega^2 \right]}{\det' \left[-\frac{d^2}{d\tau^2} + V''(x_B(\tau)) \right]} \right|^{\frac{1}{2}} e^{-S_B}, \quad (109)$$

where the determinants are defined in the finite interval β . Note that the zero-temperature limit of Γ_{Affleck} is Γ_{CC} .

The ratio of the determinants defined with the periodic boundary conditions is given by Eq. (B27). By substituting it, Eq. (109) may be written as

$$\Gamma_{\text{Affleck}} = \frac{1}{\sqrt{2\pi}} 2 \sinh(\omega\beta/2) e^{-S_B} \left[\frac{d\beta}{dE} \right]^{-\frac{1}{2}}. \quad (110)$$

We first describe the method of the numerical calculations briefly.

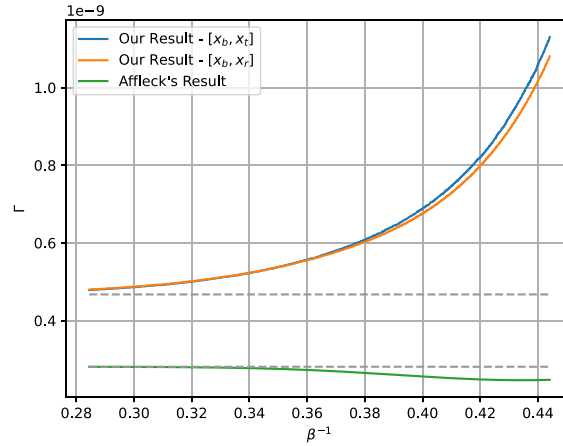


Fig. 10. Decay rates for the potential $V_3(x) = x^4 - 8x^3 + 15.9x^2$. Our result (61) with full contributions from $y \in [x_b, x_t]$, our result with contributions from $y \in [x_b, x_r]$ only, and Affleck's result (109) are plotted as functions of β^{-1} . The dashed lines denote the zero-temperature limits.

In order to obtain Z_B , we start with x_b (and x_r obtained by $V(x_b) = V(x_r)$) and solve Eq. (78) to determine the period β . In this way, we have sets of (x_b, β) . The energy E is also determined by x_b through $E = -V(x_b)$. A small change Δx_b leads to small changes $\Delta\beta$ and ΔE , so that we can calculate $\Delta\beta/\Delta E$ numerically. Of course one may use a more elaborate method to calculate $d\beta/dE$.

The calculation of $Z_{\text{SF}}^{(\text{R})}$ and $Z_{\text{SF}}^{(\text{I})}$ is more complex. Starting with x_s , one needs to find y that satisfies Eq. (90) for a given β . The energy E is determined by x_s through $E = -V(x_s)$. As explained in Section 3, there are two solutions for $y \in (x_r, x_t)$. These two solutions have different values of the action, $S_{\text{SF}}^{(\text{R})}(y)$ and $S_{\text{SF}}^{(\text{I})}(y)$.

Numerical calculations get more difficult as we go closer to the zero-temperature limit, because $d\beta/dE$ diverges exponentially.

Figure 10 shows how the decay rates change with temperature for the potential

$$V_3(x) = x^4 - 8x^3 + 15.9x^2, \quad (111)$$

which is close to the symmetric potential. Our result shows a monotonous increase while Affleck's result shows a dip before it starts to increase. We think that there is no physical reason for the dip. It occurs when the potential is very close to the symmetric double-well potential. Figure 11 shows Affleck's result for the potential

$$V_4(x) = x^4 + a_3x^3 + 16x^2. \quad (112)$$

We see that, as $V_4(x) \rightarrow V_{\text{sym}}(x)$ ($a_3 \rightarrow -8$), the dip gets deeper.

We understand why this happens. When the potential is symmetric, the system spends a lot of time not only near x_b but also near x_r . Thus $d\beta/dE$ is much larger than that for a generic case. Actually, for a symmetric double-well potential $V(x)$, the period β may be written as

$$\beta = 4 \int_{x_b}^{x_c} \frac{dx}{\sqrt{2(V(x) - V(x_b))}}, \quad (113)$$

where $x_c = (x_F + x_R)/2$. Thus, instead of Eq. (85), we have

$$\lim_{\beta \rightarrow \infty} (x_b - x_F) \frac{d\beta}{dx_b} = -\frac{4}{\omega}, \quad (114)$$

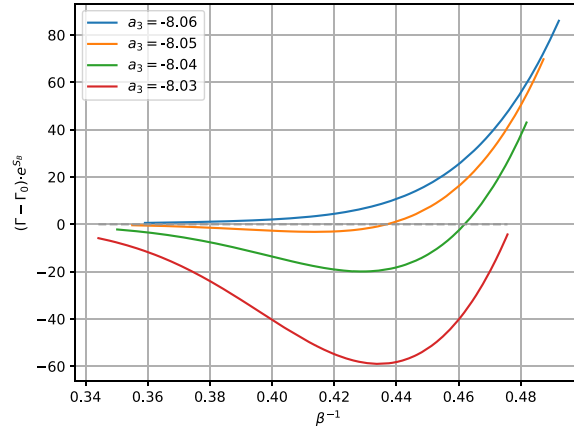


Fig. 11. Affleck's decay rates for the potential $V_4(x) = x^4 + a_3x^3 + 16x^2$ are plotted as a function of β^{-1} . Γ_0 is the zero-temperature limit of Γ_{Affleck} . The parameter of the potential is close to the symmetric case $a_3 = -8$.

and it leads to

$$\lim_{\beta \rightarrow \infty} \frac{d\beta}{dE} = \lim_{\beta \rightarrow \infty} \frac{e^{\omega\beta/2}}{\omega^3(x_C - x_F)^2} e^{-2 \int_{x_F}^{x_C} dx \left[\frac{\omega}{\sqrt{2(V(x) - V(x_F))}} - \frac{1}{x - x_F} \right]}; \quad (115)$$

thus, for large β ,

$$\Gamma_{\text{Affleck}} \sim \sinh(\omega\beta/2) \left[\frac{d\beta}{dE} \right]^{-\frac{1}{2}} \sim e^{\omega\beta/4}. \quad (116)$$

It diverges in the zero-temperature limit. Since the zero-temperature limit of Γ_{Affleck} is Γ_{CC} , the decay rate of Callan and Coleman is also divergent for a symmetric potential.

When the potential is close to (but not) symmetric, Γ_{Affleck} is finite but large in the zero-temperature limit. As the temperature increases from zero, Γ_{Affleck} first decreases since $\Gamma_{\text{Affleck}} \sim e^{\omega\beta/4}$ is approximately correct.

Our result also shows a steep increase when $\beta^{-1} = 0$, but it does not diverge. Let us look at the denominator of Eq. (100),

$$\text{Denom.} = \int_{x_F}^{x_R} dy \frac{e^{-S_{\text{SF}}^0(y)}}{\sqrt{2(V(y) - V(x_F))}} e^{-\int_y^{x_R} \frac{\omega dx}{\sqrt{2(V(x) - V(x_F))}}}. \quad (117)$$

By integrating by parts, we have

$$\begin{aligned} \text{Denom.} &= \frac{1}{\omega} \left[e^{-S_{\text{SF}}^0(y)} e^{-\int_y^{x_R} \frac{\omega dx}{\sqrt{2(V(x) - V(x_F))}}} \right]_{x_F}^{x_R} + \frac{1}{\omega} \int_{x_F}^{x_R} dy \frac{dS_{\text{SF}}^0(y)}{dy} e^{-S_{\text{SF}}^0(y)} e^{-\int_y^{x_R} \frac{\omega dx}{\sqrt{2(V(x) - V(x_F))}}} \\ &= \frac{1}{\omega} e^{-S_B^0} + \frac{1}{\omega} \int_{x_F}^{x_R} dy \frac{dS_{\text{SF}}^0(y)}{dy} e^{-S_{\text{SF}}^0(y)} e^{-\int_y^{x_R} \frac{\omega dx}{\sqrt{2(V(x) - V(x_F))}}}, \end{aligned} \quad (118)$$

where we have used $S_{\text{SF}}^0(x_R) = S_B^0$ and $e^{-\omega\beta} \rightarrow 0$. Because of the singularity at x_R , the integral in the exponential diverges except for $y = x_R$, and

$$\left. \frac{dS_{\text{SF}}^0(y)}{dy} \right|_{y=x_R} = 0, \quad (119)$$

so we see that the second term of Eq. (118) vanishes. Thus, for a symmetric double-well potential, we have

$$\lim_{\beta \rightarrow \infty} \Gamma = \omega. \quad (120)$$

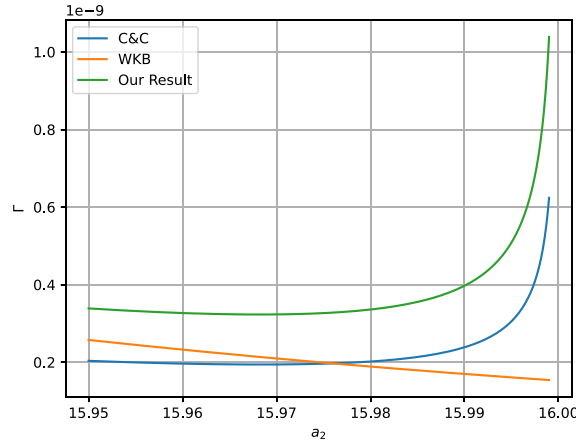


Fig. 12. Decay rates for the potential $V_2(x) = x^4 - 8x^3 + a_2x^2$. The zero-temperature limit of our result (100), Γ_{CC} , and Γ_{WKB} are plotted as functions of the parameter a_2 , when the potential is very close to the symmetric limit ($a_2 \rightarrow 16$). See Fig. 8.

This is of course an unphysical result.

At zero temperature, our decay rate and $\Gamma_{CC} = \lim_{\beta \rightarrow \infty} \Gamma_{\text{Affleck}}$ both show a steep increase when the potential becomes very close to the symmetric one, as shown in Fig. 12. There, our method somehow breaks down and fails to give a reasonable answer.

The failure of the calculations seems to be due to the splitting of a bounce into a pair of an instanton and an anti-instanton when $V'(x_R) = 0$. Namely, the particle travels from $x = x_F$ to $x = x_R$, spends an arbitrarily long time there, and comes back to the original position. Thus the negative eigenvalue decreases in magnitude and becomes a quasi-zero mode (with the magnitude of the eigenvalue being exponentially small), associated with the distance between the instanton and the anti-instanton. Note that the (original) zero mode is now associated with the center of the pair. On the other hand, Callan and Coleman (as well as the present authors) did not take into account the quasi-zero mode.

As we showed in Figs. 10 and 11, this affects the finite-temperature behavior of Γ_{Affleck} even when the potential is not very close to the symmetric one. However, it does not seem to affect our decay rate. The difference probably comes from the fact that Γ_{Affleck} diverges as $\beta \rightarrow \infty$, while the limit of our decay rate stays finite.

6. Summary

In this paper, we considered a 1D quantum-mechanical system in a potential that has a metastable false vacuum at finite temperature, and calculated the decay rate in the saddle-point approximation.

Because of the trace, the static solutions do not contribute and the shot-F solution plays an important role. The trace integral combines all the contributions from the shifted bounce solutions.

There are two shot-F solutions for $y \in (x_r, x_t)$, one of which is imaginary. It plays a similar role to the finite- T bounce and shifted bounce solutions in the calculation of the decay rate.

We then calculated the zero-temperature limit of our result. We also commented on the finite time “regularization” and the collective coordinate method of Callan and Coleman.

Finally, we performed some numerical calculations. We first showed the potential dependence of our decay rate and compared it with that of Callan and Coleman, and with that obtained by the NLO WKB approximation. We find that our result is similar to the WKB one. Next, we showed the temperature dependence of our decay rate and compared it with that of Affleck. We showed that Affleck's decay rate first decreases as temperature increases and then increases when the potential is close to a symmetric double-well potential. This behavior seems to have no physical cause, and seems to be a consequence of the divergence of the decay rate for a symmetric double-well potential at the zero-temperature limit. The zero-temperature limit of our decay rate also shows a steep increase when the potential is very close to a symmetric double well; this does not seem to affect the temperature dependence when the potential is not symmetric.

It would be interesting to extend our results to quantum field theory in which contributions from shot-type solutions have not been included. Investigation in this direction is now under consideration.

Funding

The work of Q.Y. was supported by the Kyushu University Innovator Fellowship Program (Quantum Science Area). Open Access funding: SCOAP³.

Appendix A. Dimensionless variables

In order to simplify the expressions and to do numerical calculations, we use dimensionless variables. In this appendix, we summarize how they are defined.

The Hamiltonian of the 1D quantum mechanics of a particle of mass m is given by

$$H = \frac{p^2}{2m} + V(x), \quad (\text{A1})$$

where the momentum p and the coordinate x have physical dimensions. The inverse temperature $\beta = 1/k_B T$ has the inverse dimension of energy.

We introduce an energy scale ϵ_0 and, by using it, we define the dimensionless variables.

We define a dimensionless coordinate $\mathcal{X} = \frac{\sqrt{m\epsilon_0}}{\hbar}x$ and a dimensionless momentum $\mathcal{P} = \frac{1}{\sqrt{m\epsilon_0}}p$, as well as a dimensionless (Euclidean) time $\mathcal{T} = \frac{\epsilon_0}{\hbar}\tau$. They satisfy the following relation:

$$\frac{d\mathcal{X}}{d\mathcal{T}} = \mathcal{P}. \quad (\text{A2})$$

The Hamiltonian may be written as

$$H = \epsilon_0 \mathcal{H}(\mathcal{P}, \mathcal{X}), \quad (\text{A3})$$

$$\mathcal{H} = \frac{1}{2}\mathcal{P}^2 + \mathcal{V}(\mathcal{X}), \quad (\text{A4})$$

where $\mathcal{H}$ is the dimensionless Hamiltonian, and we have defined a dimensionless potential

$$\mathcal{V}(\mathcal{X}) = \epsilon_0^{-1} V\left(\frac{\hbar}{\sqrt{m\epsilon_0}}\mathcal{X}\right). \quad (\text{A5})$$

We can also define a dimensionless inverse temperature,

$$\mathcal{B} = \beta\epsilon_0. \quad (\text{A6})$$

A dimensionless free energy defined as $\mathcal{F} = \epsilon_0^{-1}F$, may be expressed as

$$\mathcal{F} = -\frac{1}{\mathcal{B}} \ln \text{tr} \left(e^{-\mathcal{B}\mathcal{H}} \right). \quad (\text{A7})$$

The physical decay rate Γ may be obtained from the imaginary part of a certain combination of contributions to $\mathcal{F}$, by multiplying by a factor $\epsilon_0/\hbar$.

Appendix B. Ratio of the determinants defined with periodic boundary conditions

In this appendix, we explain how to calculate the ratio of the determinants defined with the periodic boundary conditions, which is useful to understand Eqs. (65) and (109).

The basic formula is given by Forman [30,31], which is a generalization of the Gel'fand–Yaglom theorem. See also Ref. [32].

We consider two differential operators

$$\mathcal{O}^{(i)} = -\frac{d^2}{d\tau^2} + W^{(i)}(\tau) \quad (i = 1, 2) \quad (\text{B1})$$

in the interval $\tau \in [-\beta/2, \beta/2]$, and the eigenvalue problems

$$\mathcal{O}^{(i)}\phi_n^{(i)}(\tau) = \lambda_n^{(i)}\phi_n^{(i)}(\tau). \quad (\text{B2})$$

Now the eigenfunctions are supposed to satisfy the conditions

$$M \begin{pmatrix} \phi_n^{(i)}(-\beta/2) \\ \dot{\phi}_n^{(i)}(-\beta/2) \end{pmatrix} + N \begin{pmatrix} \phi_n^{(i)}(\beta/2) \\ \dot{\phi}_n^{(i)}(\beta/2) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (i = 1, 2), \quad (\text{B3})$$

where M and N are a 2×2 matrix. For the periodic boundary conditions,

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (\text{B4})$$

The determinant $\det \mathcal{O}^{(i)}$ is (naively) an infinite product of the eigenvalues,

$$\det \mathcal{O}^{(i)} = \prod_n \lambda_n^{(i)}. \quad (\text{B5})$$

As we did in Section 2 for the Gel'fand–Yaglom theorem, we consider the zero-eigenvalue problem for each operator,

$$\mathcal{O}^{(i)}\psi_j^{(i)}(\tau) = 0 \quad (j = 1, 2), \quad (\text{B6})$$

where $\psi_j^{(i)}$ are two independent solutions, and also consider the 2×2 matrices

$$H^{(i)}(\tau) = \begin{pmatrix} \psi_1^{(i)}(\tau) & \psi_2^{(i)}(\tau) \\ \dot{\psi}_1^{(i)}(\tau) & \dot{\psi}_2^{(i)}(\tau) \end{pmatrix}, \quad (\text{B7})$$

and

$$Y^{(i)}(\tau) = H^{(i)}(\tau)H^{(i)-1}(-\beta/2). \quad (\text{B8})$$

Forman's theorem says that

$$\frac{\det \mathcal{O}^{(1)}}{\det \mathcal{O}^{(2)}} = \frac{\det_{2 \times 2}(M + NY^{(1)}(\beta/2))}{\det_{2 \times 2}(M + NY^{(2)}(\beta/2))}, \quad (\text{B9})$$

where, on the right-hand side, $\det_{2 \times 2}$ denotes the determinant for a 2×2 matrix.

If we choose $\psi_j^{(i)}(\tau)$ so that $H^{(i)}(-\beta/2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then we have $Y^{(i)}(\tau) = H^{(i)}(\tau)$ and the expression is simplified considerably. We assume that this choice of $\psi_j^{(i)}(\tau)$ has been done.

Note that for the Dirichlet boundary conditions,

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad (\text{B10})$$

so that

$$\frac{\det \mathcal{O}^{(1)}}{\det \mathcal{O}^{(2)}} = \frac{\psi_2^{(1)}(\beta/2)}{\psi_2^{(2)}(\beta/2)}, \quad (\text{B11})$$

with the boundary conditions $\psi_2^{(i)}(-\beta/2) = 0$ and $\dot{\psi}_2^{(i)}(-\beta/2)$. This is nothing but the Gel'fand–Yaglom theorem for the Dirichlet boundary conditions, considered in Section 2.

For the periodic boundary conditions (B4), Eq. (B9) can be written as

$$\left. \frac{\det \mathcal{O}^{(1)}}{\det \mathcal{O}^{(2)}} \right|_{\text{p.b.c.}} = \left. \frac{(1 - \psi_1^{(1)})(1 - \dot{\psi}_2^{(1)}) - \dot{\psi}_1^{(1)}\psi_2^{(1)}}{(1 - \psi_1^{(2)})(1 - \dot{\psi}_2^{(2)}) - \dot{\psi}_1^{(2)}\psi_2^{(2)}} \right|_{\tau=\frac{\beta}{2}}. \quad (\text{B12})$$

Let us now consider the specific example $\mathcal{O}^{(1)} = -\frac{d^2}{d\tau^2} + \omega^2$ and $\mathcal{O}^{(2)} = -\frac{d^2}{d\tau^2} + V''(x_B(\tau))$. One can easily get

$$\psi_1^{(1)}(\tau) = \cosh(\omega(\tau + \beta/2)), \quad (\text{B13})$$

$$\psi_2^{(1)}(\tau) = \omega^{-1} \sinh(\omega(\tau + \beta/2)). \quad (\text{B14})$$

From these, we immediately obtain

$$(1 - \psi_1^{(1)})(1 - \dot{\psi}_2^{(1)}) - \dot{\psi}_1^{(1)}\psi_2^{(1)} \Big|_{\tau=\frac{\beta}{2}} = -4 \sinh^2(\omega\beta/2). \quad (\text{B15})$$

For the operator $\mathcal{O}^{(2)}$, $\psi_j^{(2)}(\tau)$ may be written as linear combinations of $\mu(\tau)$ and $\nu(\tau)$ as in Section 2 (but defined with $\tau_c = 0$). We get

$$\psi_1^{(2)}(\tau) = \dot{\nu}(-\beta/2)\mu(\tau) - \dot{\mu}(-\beta/2)\nu(\tau), \quad (\text{B16})$$

$$\psi_2^{(2)}(\tau) = -\nu(-\beta/2)\mu(\tau) + \mu(-\beta/2)\nu(\tau). \quad (\text{B17})$$

With the periodic boundary conditions, $\det \mathcal{O}^{(2)}$ contains a zero eigenvalue to be removed. The way of removing the zero mode is similar to the case with Dirichlet boundary conditions considered in Section 2. We define $\det' \mathcal{O}^{(2)} = \lim_{\lambda \rightarrow 0} \det[\mathcal{O}^{(2)} - \lambda]/(-\lambda)$, and consider $\psi_{\lambda j}^{(2)}$ ($j = 1, 2$), which satisfies

$$[\mathcal{O}^{(2)} - \lambda] \psi_{\lambda j}^{(2)}(\tau) = 0, \quad (\text{B18})$$

and the initial value conditions $H_{\lambda}^{(2)}(-\beta/2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. As we did in Eq. (42), we consider

$$\begin{aligned} 0 &= \int_{-\beta/2}^{\beta/2} d\tau \dot{x}_B(\tau) [\mathcal{O}^{(2)} - \lambda] \psi_{\lambda j}^{(2)}(\tau) \\ &= \left[-\dot{x}_B(\tau) \dot{\psi}_{\lambda j}^{(2)} + \ddot{x}_B(\tau) \psi_{\lambda j}^{(2)}(\tau) \right]_{-\beta/2}^{\beta/2} - \lambda \int_{-\beta/2}^{\beta/2} d\tau \dot{x}_B(\tau) \psi_{\lambda j}^{(2)}(\tau). \end{aligned} \quad (\text{B19})$$

By substituting the initial values and using the periodicity, we obtain

$$\begin{pmatrix} \psi_{\lambda 1}^{(2)}(\beta/2) - 1 & -\dot{\psi}_{\lambda 1}^{(2)}(\beta/2) \\ \psi_{\lambda 2}^{(2)}(\beta/2) & 1 - \dot{\psi}_{\lambda 2}^{(2)}(\beta/2) \end{pmatrix} \begin{pmatrix} \ddot{x}_B(\beta/2) \\ \dot{x}_B(\beta/2) \end{pmatrix} = \lambda \begin{pmatrix} \int_{-\beta/2}^{\beta/2} d\tau \dot{x}_B(\tau) \psi_{\lambda 1}^{(2)}(\tau) \\ \int_{-\beta/2}^{\beta/2} d\tau \dot{x}_B(\tau) \psi_{\lambda 2}^{(2)}(\tau) \end{pmatrix}, \quad (\text{B20})$$

and thus

$$\begin{pmatrix} \ddot{x}_B(\beta/2) \\ \dot{x}_B(\beta/2) \end{pmatrix} = \frac{-\lambda}{-D} \begin{pmatrix} 1 - \dot{\psi}_{\lambda 2}^{(2)}(\beta/2) & \dot{\psi}_{\lambda 1}^{(2)}(\beta/2) \\ -\psi_{\lambda 2}^{(2)}(\beta/2) & \psi_{\lambda 1}^{(2)}(\beta/2) - 1 \end{pmatrix} \begin{pmatrix} \int_{-\beta/2}^{\beta/2} d\tau \dot{x}_B(\tau) \psi_{\lambda 1}^{(2)}(\tau) \\ \int_{-\beta/2}^{\beta/2} d\tau \dot{x}_B(\tau) \psi_{\lambda 2}^{(2)}(\tau) \end{pmatrix}, \quad (\text{B21})$$

where

$$-D = \left(1 - \psi_{\lambda 1}^{(2)}(\beta/2)\right) \left(1 - \psi_{\lambda 2}^{(2)}(\beta/2)\right) - \psi_{\lambda 1}^{(2)}(\beta/2) \psi_{\lambda 2}^{(2)}(\beta/2) \quad (\text{B22})$$

is the combination that appears in Eq. (B12).

Therefore, we obtain

$$\frac{\det \mathcal{O}^{(1)}}{\det' \mathcal{O}^{(2)}} \Big|_{\text{p.b.c.}} = \lim_{\lambda \rightarrow 0} \frac{\det \mathcal{O}^{(1)}}{\det [\mathcal{O}^{(2)} - \lambda] / (-\lambda)} \Big|_{\text{p.b.c.}} = \left[-4 \sinh^2(\omega\beta/2) \right] \lim_{\lambda \rightarrow 0} \frac{-\lambda}{-D}. \quad (\text{B23})$$

The limit may be written as

$$\begin{aligned} \frac{\ddot{x}_B(\frac{\beta}{2})}{\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} d\tau \dot{x}_B(\tau) \left[(1 - \psi_2^{(2)}(\frac{\beta}{2})) \psi_1^{(2)}(\tau) + \psi_1^{(2)}(\frac{\beta}{2}) \psi_2^{(2)}(\tau) \right]} &= - \frac{\ddot{x}_B(\beta/2)}{[\ddot{x}_B(\beta/2)]^2 \|\dot{x}_B\|^2 \nu(\beta/2)} \left[\frac{d\beta}{dE} \right]^{-1} \\ &= \frac{1}{\|\dot{x}_B\|^2} \left[\frac{d\beta}{dE} \right]^{-1}, \end{aligned} \quad (\text{B24})$$

where we have used

$$\psi_1^{(2)}(\beta/2) = [\ddot{x}_B(\beta/2)]^2 \frac{d\beta}{dE}, \quad (\text{B25})$$

$$\psi_2^{(2)}(\beta/2) = 1, \quad (\text{B26})$$

and $\ddot{x}_B(\beta/2)\nu(\beta/2) = -1$. We finally obtain

$$\frac{\det \mathcal{O}^{(1)}}{\det' \mathcal{O}^{(2)}} \Big|_{\text{p.b.c.}} = - \frac{4 \sinh^2(\omega\beta/2)}{\|\dot{x}_B\|^2} \left[\frac{d\beta}{dE} \right]^{-1}. \quad (\text{B27})$$

To conclude this appendix, we will show that the zero-temperature limit of the ratio of the determinants with the periodic boundary conditions is equal to that with the Dirichlet boundary conditions,

$$\lim_{\beta \rightarrow \infty} \frac{\det \mathcal{O}^{(1)}}{\det' \mathcal{O}^{(2)}} \Big|_{\text{p.b.c.}} = \lim_{\beta \rightarrow \infty} \frac{\det \mathcal{O}^{(1)}}{\det' \mathcal{O}^{(2)}} \Big|_{\text{d.b.c.}}. \quad (\text{B28})$$

In the zero-temperature limit, $\|\dot{x}_B\|^2 \rightarrow S_B^0$, and from Eqs. (87) and (83), we see

$$\begin{aligned} \lim_{\beta \rightarrow \infty} \frac{\det \mathcal{O}^{(1)}}{\det' \mathcal{O}^{(2)}} \Big|_{\text{p.b.c.}} &= - \lim_{\beta \rightarrow \infty} \frac{e^{\omega\beta} \omega^4 (x_b - x_F)^2}{S_B^0 2\omega} \\ &= - \frac{2\omega^3}{S_B^0} (x_R - x_F)^2 e^{2 \int_{x_F}^{x_R} dx \left[\frac{\omega}{\sqrt{2(V(x) - V(x_F))}} - \frac{1}{x - x_F} \right]}. \end{aligned} \quad (\text{B29})$$

On the other hand, from Eq. (48), by using $|\ddot{x}_B(\beta/2)|^2 = |V'(x_b)|^2 = \omega^4 (x_b - x_F)^2$, we have

$$\begin{aligned} \lim_{\beta \rightarrow \infty} \frac{\det \mathcal{O}^{(1)}}{\det' \mathcal{O}^{(2)}} \Big|_{\text{d.b.c.}} &= - \lim_{\beta \rightarrow \infty} \frac{\omega^4 (x_b - x_F)^2 e^{\omega\beta}}{S_B^0 2\omega} \\ &= - \frac{2\omega^3}{S_B^0} (x_R - x_F)^2 e^{2 \int_{x_F}^{x_R} dx \left[\frac{\omega}{\sqrt{2(V(x) - V(x_F))}} - \frac{1}{x - x_F} \right]}. \end{aligned} \quad (\text{B30})$$

This completes the proof.

References

- [1] K. Kajantie, M. Laine, K. Rummukainen, and M. Shaposhnikov, Phys. Rev. Lett. **77**, 2887 (1996).
- [2] M. D'Onofrio and K. Rummukainen, Phys. Rev. D **93**, 025003 (2016) [arXiv:1508.07161[hep-ph]] [Search INSPIRE].
- [3] V. A. Kuzmin, V. A. Rubakov, and M. E. Shaposhnikov, Phys. Lett. B **155**, 36 (1985).

- [4] A. G. Cohen, D. B. Kaplan, and A. E. Nelson, *Ann. Rev. Nucl. Part. Sci.* **43**, 27 (1993) [[arXiv:hep-ph/9302210](#)] [[Search INSPIRE](#)].
- [5] V. A. Rubakov and M. E. Shaposhnikov, *Usp. Fiz. Nauk* **166**, 493 (1996) [[arXiv:hep-ph/9603208](#)] [[Search INSPIRE](#)].
- [6] K. Funakubo, *Prog. Theor. Phys.* **96**, 475 (1996) [[arXiv:hep-ph/9608358](#)] [[Search INSPIRE](#)].
- [7] D. E. Morrissey and M. J. Ramsey-Musolf, *New J. Phys.* **14**, 125003 (2012) [[arXiv:1206.2942](#)] [[hep-ph](#)] [[Search INSPIRE](#)].
- [8] A. Kosowsky, M. S. Turner, and R. Watkins, *Phys. Rev. D* **45**, 4514 (1992).
- [9] M. Kamionkowski, A. Kosowsky, and M. S. Turner, *Phys. Rev. D* **49**, 2837 (1994).
- [10] S. J. Huber and T. Konstandin, *J. Cosmol. Astropart. Phys.* **0809**, 022 (2008) [[arXiv:0806.1828](#)] [[hep-ph](#)] [[Search INSPIRE](#)].
- [11] D. J. Weir, *Phil. Trans. R. Soc. Lond. A* **376**, 20170126 (2018); **381**, 20230212 (2023) [erratum] [[arXiv:1705.01783](#)] [[hep-ph](#)] [[Search INSPIRE](#)].
- [12] M. J. Baker, J. Kopp, and A. J. Long, *Phys. Rev. Lett.* **125**, 151102 (2020) [[arXiv:1912.02830](#)] [[hep-ph](#)] [[Search INSPIRE](#)].
- [13] A. Azatov, M. Vanvlasselaer, and W. Yin, *J. High Energy Phys.* **2103**, 288 (2021) [[arXiv:2101.05721](#)] [[hep-ph](#)] [[Search INSPIRE](#)].
- [14] S. Coleman, *Phys. Rev. D* **15**, 2929 (1977).
- [15] S. Coleman, *Phys. Rev. D* **16**, 1248 (1977).
- [16] I. Yu. Kobzarev, L. B. Okun, and M. B. Voloshin, *Yad. Fiz.* **20**, 1229 (1974).
- [17] J. S. Langer, *Ann. Phys.* **41**, 108 (1967).
- [18] C. G. Callan and S. Coleman, *Phys. Rev. D* **16**, 1762 (1977).
- [19] S. Coleman, *The uses of instantons*, in *Aspects of Symmetry* (Cambridge University Press, Cambridge, UK, 1985), p. 265.
- [20] A. Andreassen, D. Farhi, W. Frost, and M. D. Schwartz, *Phys. Rev. D* **95**, 085011 (2017).
- [21] I. Affleck, *Phys. Rev. Lett.* **46**, 388 (1981).
- [22] A. D. Linde, *Phys. Lett. B* **100**, 37 (1981).
- [23] A. D. Linde, *Nucl. Phys. B* **216**, 421 (1983); **223**, 544 (1983) [erratum].
- [24] R. F. Dashen, B. Hasslacher, and A. Neveu, *Phys. Rev. D* **10**, 4114 (1974).
- [25] G. Gamow, *Z. Phys.* **51**, 204 (1928).
- [26] A. J. F. Siegert, *Phys. Rev.* **56**, 750 (1939).
- [27] S. R. Coleman, *Nucl. Phys. B* **298**, 178 (1988).
- [28] I. M. Gel'fand and A. M. Yaglom, *J. Math. Phys.* **1**, 48 (1960).
- [29] M. Mariño, *Instantons and Large N: An Introduction to Non-Perturbative Methods in Quantum Field Theory* (Cambridge University Press, Cambridge, UK, 2015).
- [30] R. Forman, *Invent. Math.* **88**, 447 (1987).
- [31] R. Forman, *Invent. Math.* **108**, 453 (1992).
- [32] G. Dunne, *Functional Determinants in Quantum Field Theory* (Lecture at the 14th WE Heraeus Saalburg Summer School in Worfersdorf, Thuringia) (2008).