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Triangulation in teaching probability: teaching materials for the theoretical foundations of probability in real-world applications

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ABSTRACT

This paper proposes using the concept of *triangulation* with probabilistic models as a means to enhance theoretical inversion for deepening students' understanding of the nature of probability in real-world contexts. Triangulation refers to the combined application of multiple methodologies to investigate the same phenomenon, particularly in the social sciences. Theoretical inversion refers to a shift in focus from surprising outcomes to the theoretical foundations of probability. The paper introduces three types of problem-solving tasks designed to enhance one of four types of triangulations: theory triangulation. Theoretical inversion is expected to emerge through engaging in these tasks. The characteristics of the problems are as follows. Problem 1 promotes students to compare different probabilistic models of events under similar procedures. Problem 2 provides students with an opportunity to simplify an experiment by omitting steps that add no new information. Problem 3 enhances students' ability to recognise how subtle differences in the experimental setup can affect the resulting probability. These tasks are designed to encourage students to view probabilistic reasoning as a form of modelling and to appreciate the importance of assumptions, definitions of elementary events, and clarity in procedural descriptions.

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Probability; triangulation; mathematical modelling; theoretical inversion

1. Introduction

Teaching the concept of probability in school mathematics is challenging due to the multiple, sometimes incompatible, semantic interpretations of probability, such as classical probability and frequentist probability (see Gillies, 2000). Modern mathematics has chosen to axiomatize the theory of probability, thereby avoiding the interpretation of its meaning and neglecting to address the underlying interpretative conflicts. However, it is impossible to ignore the meanings of probability in school mathematics. We need an integrative perspective of these meanings that combines multiple interpretations to foster a deeper understanding of probability. As a foundation for such a perspective, this paper aims to propose the concept of triangulating probabilistic models and to provide teaching materials for a deeper understanding of probability.

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The structure of this paper is as follows: First, we provide an overview of the two fundamental concepts in school mathematics: empirical and theoretical probabilities. Next, we propose a perspective of triangulation for an integrative understanding of the nature of these probabilities. The key idea from this perspective is a triangulation by multiple probabilistic models, which enhances their reliability and students' understanding of the theoretical foundations of probability. Finally, we illustrate three specific examples of teaching materials from this perspective and provide insights into classroom implementation of teaching probability.

2. Empirical and theoretical probabilities

In school mathematics, two perspectives on probability – empirical and theoretical – are often used in conjunction (e.g. Ishibashi, 2022; Kazak & Pratt, 2021). First, empirical probability is defined as the probability 'calculated from observed relative frequencies of different outcomes in repeated trials' (Hawkins & Kapadia, 1984, p. 349). For example, the empirical probability of rolling a 1 when a die is thrown once equals the relative frequency of 1 occurring in many repeated die rolls. In contrast, theoretical probability is defined as the probability 'obtained by making an assumption of equal likelihood in the same space' (Hawkins & Kapadia, 1984, p. 349). When considering theoretical probability, three key concepts – elementary events, equal likelihood, and the principle of insufficient reason – are essential. For instance, the theoretical probability of rolling a multiple of 3 on a die is determined by assuming, based on the principle of insufficient reason, that all faces are equally likely to appear and calculating the ratio of the favourable outcomes to the total possible outcomes (i.e. 2:6). In this context, the principle of insufficient reason means that, given the die is a perfect cube and is thrown without any bias, there is no reason to assume any particular outcome is more likely than the others. Suppose the assumption of equal likelihood based on the principle of insufficient reason holds. In that case, the limiting value of the empirical probability will match the theoretical probability, as per the Law of Large Numbers.

Some prior research considers probabilistic thinking as a form of modelling (e.g. Kazak & Pratt, 2021; Pfannkuch & Ziedins, 2014). For students without this perspective, the principle of insufficient reason might sound somewhat problematic, as it allows for multiple models of the same event under the same conditions. For example, the probability of rolling an even number on a die is usually determined by calculating the ratio of the favourable outcomes to the total possible outcomes (i.e. 3:6). This is the general approach. However, there is no compelling reason to treat each outcome as an elementary event. If our cognitive ability only allows us to recognise 1, 3, and 5 as odd and 2, 4, and 6 as even, then, due to geometric symmetry, the possible outcomes can be grouped into even and odd, which can be considered elementary events. Since these outcomes are equally likely, the probability of rolling an even number is $\frac{1}{2}$ in this model as well (i.e. the ratio of the relevant elementary event to the total events is 1:2). With a modelling perspective, students might not feel uncomfortable even if different modelling approaches lead to the same conclusion. However, students who strongly link events to probabilities might find it unsettling that elementary events can be redefined in multiple ways.

Pfannkuch and Ziedins (2014) refer to only theoretical probability as model probability. However, they also state that both theoretical and empirical probabilities are estimates of an unknown 'true' probability. Therefore, we consider both theoretical and empirical probabilities as models of true probability. In particular, empirical probability, as demonstrated through computer simulation, aligns well with this view. A simulation, by its nature, is a model.

3. Triangulation as a way of increasing the reliability of probability

A modelling perspective is a key to understanding the nature of probability. As Skovsmose (2019) points out, it is not possible to determine the resemblance between a constructed mathematical model and the real-world phenomenon being considered, and there is a risk that mathematical modelling may lead to erroneous judgments. For example, assuming the probability of physically rolling a 1 is $\frac{1}{6}$, based on the principle of insufficient reason, might be a *reasonable* assumption but not necessarily *correct*. The assumptions that the die is a perfect cube and that there is no bias in rolling it are based on both the absence of reasons to doubt and the desire to make a judgment without falling into inaction. Thus, these assumptions may be wrong. Nevertheless, we can keep the mathematical validity of the theory of probability. Suppose we treat the concept of probability purely as a mathematical construct. In that case, it is possible to eliminate the ambiguity surrounding the application of probability in real-world contexts and to establish the mathematical validity of the probability theory. However, such an approach would be insufficient for teaching probability theory as it is typically expected in school mathematics.

The relationship between the treatment of probability, which contains ambiguities in the real world, and probability theory as a rigorous mathematical discipline is similar to the relationship between informal deductive reasoning in the real world and logic as a rigorous mathematical discipline. Logic offers some effective formal models of human deductive reasoning in the real world (see Shapiro, 1991, Chap. 1), and their formality enables mathematical inquiry into the nature and structure of these models. For example, a model for human reasoning in propositional logic demonstrates that Q is derivable from the two premises 'if P then Q' and P. On the other hand, by understanding logical formulas in more detail, a model for human reasoning in predicate logic shows that Q(a) is derivable from 'for all x belonging to D if P(x) then Q(x)' and 'P(a) for some a belonging to D'. However, a single model might not fully describe human deductive reasoning in the real world. The academic discipline of logic provides diverse models related to deductive reasoning, including propositional logic, predicate logic, modal logic, and three-valued logic. It is necessary to use them appropriately, depending on the situation. Similarly, probability theory offers some effective formal models for addressing probability in the real world, and its formality enables mathematical inquiry into the nature and structure of these models. However, a single model may not fully capture a given phenomenon in the real world. Depending on the situation, multiple models are necessary. For example, we use empirical probability when we can repeatedly observe outcomes under the same conditions. In contrast, we use theoretical probability when we can explicitly assume that specific outcomes are equally likely to occur. However, there is one crucial difference between logic and probability theory: in logical models, since the types of logical formulas that can be handled differ for each model, the combined use of multiple models does not necessarily make sense, whereas probabilistic models can all handle probability values, so they can be used together if the situation allows.

In this paper, we argue that the concept of triangulation is beneficial. Triangulation refers to 'the application and combination of several research methodologies in the study of the same phenomenon' (Denzin, 2007, p. 5083). Generally, four types of primary triangulation are known: (1) data triangulation, involving time, space, and people; (2) investigator triangulation, involving multiple observers rather than a single observer; (3) theory triangulation, involving the use of more than one theoretical scheme for interpreting a phenomenon; and (4) methodological triangulation, involving the use of more than one method (Denzin, 2007).

Considering a single phenomenon using multiple probabilistic models can be seen as a form of triangulation. Conducting experiments and observations in multiple environments for the same trial to obtain multiple empirical probabilities corresponds to data triangulation and investigator triangulation. Calculating theoretical probabilities using different schemes in multiple ways corresponds to theory triangulation. Combining empirical and theoretical probabilities corresponds to methodological triangulation.

For example, the Monty Hall problem, which has sparked much debate, involved various probabilistic models before its resolution (see Rosenhouse, 2009). The problem is as follows. The host asks you to choose one of three doors. Behind one of the doors is a prize car. After you choose a door, the host, who knows which door hides the car, opens one of the two doors you did not choose. Then, the host asks you to switch to the other remaining door. Which is more likely to win the car: sticking with your original choice or switching to another door?

From a modelling perspective, Rosenhouse's (2009) explanations for this problem provide models for the probability of the event. Here, we introduce three of his models. The first model is based on classical probability theory. For simplicity, suppose that you choose door A. In this case, the pairs consisting of the door with the car and the door opened by the host are (A, B), (A, C), (B, C), and (C, B). Since the probability that the car is behind any particular door is $\frac{1}{3}$, the probability that (A, B) or (A, C) occurs is $\frac{1}{3}$ in total. Therefore, the probability that the car is behind A is always less than $\frac{1}{3}$, whether the host opens B or C. The second model uses a Monte Carlo simulation approach. The third model is based on the Bayesian probability theory. If you continue to choose the door you initially chose, your chance of winning is $\frac{1}{3}$. However, since the host reduces the two unchosen doors to just one, the remaining door has a probability of $\frac{2}{3}$. As these examples demonstrate, triangulation helps us deepen the understanding of the probability of a particular event.

Steinbring's (1991) concept of theoretical inversion epistemologically supports the importance of triangulation in probability learning. Theoretical inversion refers to a shift in reasoning from treating discrepancies as disconfirmation to treating them as challenges to the theory or its experimental foundation. As the Monty Hall problem suggests, theoretical inversion can also occur between theoretical predictions and intuition. When the theoretical prediction of the first probabilistic model contradicts intuition, theoretical inversion occurs, leading to the generation of a new second probabilistic model from a perspective not considered during the construction of the first model. Through this iterative process, our understanding of the phenomenon deepens, and our intuition is refined.

In this paper, to capture the process of theoretical inversion concisely, we propose the schematic representation shown in Figure 1. Figure 1 illustrates that a discrepancy between a conjecture and the outcomes of experiments, simulations, or computations may trigger

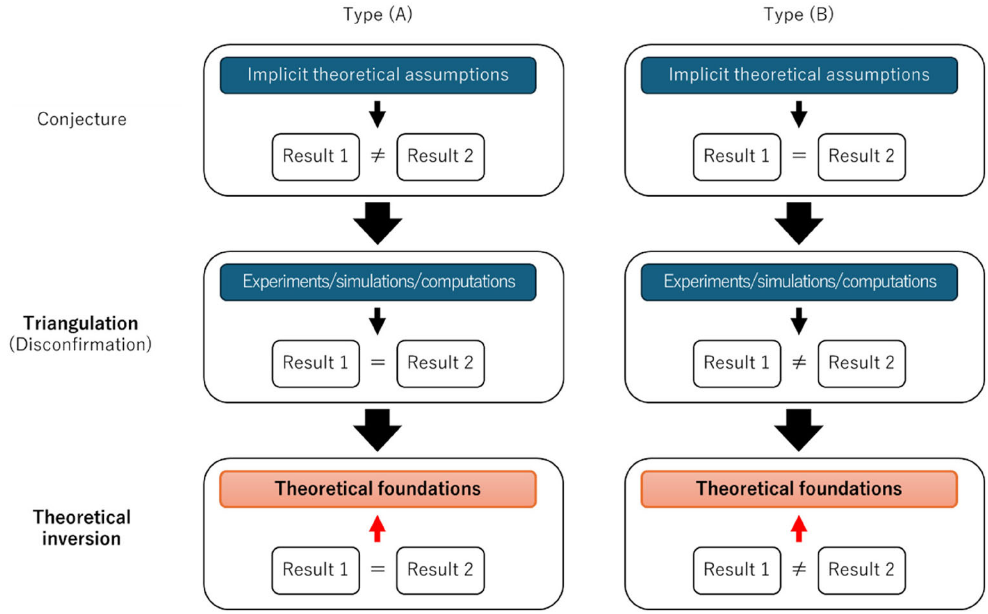

Figure 1. A schematic representation of triangulation and theoretical inversion (Result 1 and Result 2 refer not only to the outcomes of experiments, simulations, or computations but also to intuitively expected results).

a theoretical inversion. Theoretical inversion can be broadly classified into two types, each consisting of three stages.

In the first stage of a Type (A) theoretical inversion, students, based on their implicit theoretical assumptions, anticipate that Result 1 and Result 2 may fail to coincide for a given trial. Here, Result 1 and Result 2 may refer either to the outcomes of experiments, simulations, or computations, or to intuitively expected results. In the second stage, through actually conducting trials or calculating probabilities, students experience that Result 1 and Result 2 coincide, contrary to their initial expectation. This constitutes a disconfirmation of their conjecture. In the third stage, students engage in a theoretical inversion to make sense of this situation; that is, they reconstruct their underlying theoretical foundations so that Result 1 and Result 2 align.

A Type (B) theoretical inversion proceeds similarly. Only the difference is: Students initially expect Result 1 and Result 2 to coincide in the first stage, and experience in the second stage that they do not. Apart from differences in initial expectations, the two types of theoretical inversion share the same essential three-step structure.

4. Examples of teaching materials

This section presents three examples of teaching materials that facilitate triangulation using probabilistic models. Since deriving empirical probability through simulation can be applied to all examples, we particularly illustrate the use of multiple models for a single phenomenon (i.e. theory triangulation). One well-known problem in which different probabilities arise depending on the model chosen is Bertrand's paradox. However, since this problem involves continuous probability distributions, it may be challenging to address at the secondary education level. For this reason, in this section, we consider probability problems that can be modelled within a discrete sample space using the concept of equal likelihood, making it more suitable as instructional material for secondary education. Note that teaching materials for conditional probability can be more challenging for students. In Example 4.1, conditional probability is not needed. In Example 4.2, it is necessary. In Example 4.3, the concept of conditional probability adds an optional probabilistic model. Additionally, note also that ${}_nP_r$ denotes the number of permutations in which r distinct elements are selected from a set of n distinct elements, and that ${}_nC_r$ denotes the number of combinations in which r distinct elements are selected from a set of n distinct elements. The notation $n(A)$ refers to the number of elementary events constituting event A, and $P(A)$ denotes the probability of event A.

4.1. Drawing two balls simultaneously

Consider the following problem.

[Problem 1]

A bag contains eight red balls and four white balls. We draw two balls simultaneously. Find the probability that one ball is red and the other is white.

Generally, the solution can be approached as follows:

[Solution 1-1]

The number of ways to choose two balls out of 12 is ${}_{12}C_2$. The number of ways to choose one red ball out of eight is ${}_8C_1$. The number of ways to choose one white ball out of four is ${}_4C_1$. Therefore, the number of ways to draw 1 red and 1 white ball is ${}_8C_1 \times {}_4C_1$. Hence, the required probability is:

$$\frac{{}_8C_1 \times {}_4C_1}{{}_{12}C_2} = \frac{8 \times 4}{\frac{12 \cdot 11}{2 \cdot 1}} = \frac{16}{33}$$

However, based on the discussion in this paper, if we intend to question the assumption of equal likelihood and advance the cycle of mathematical modelling, we could pose the following question to the students: 'How were the red and white balls drawn simultaneously?' (see Figure 2). It may be worthwhile to have several students demonstrate this. One group of students may try to grab two balls simultaneously with one hand, while another group may attempt to pick up one ball with each hand. The remaining students may insist that a tool like a cooking ladle is necessary for definitively removing both balls from the bag simultaneously. Various interpretations can be obtained. We can then further ask, 'Did they really draw them simultaneously? If they were not drawn perfectly simultaneously, how would the probability change?'

If we calculate the case where one ball is picked with each hand, we get the following:

[Solution 1-2]

Consider drawing one ball with each hand, starting with the right hand and then the left hand. The number of ways to choose two balls from 12 is ${}_{12}P_2$. The number of ways to draw a red ball with the right hand and a white ball with the left hand is ${}_8C_1 \times {}_4C_1$ (see Figure 3). The number of ways to draw a red ball with the left hand and a white ball with the right

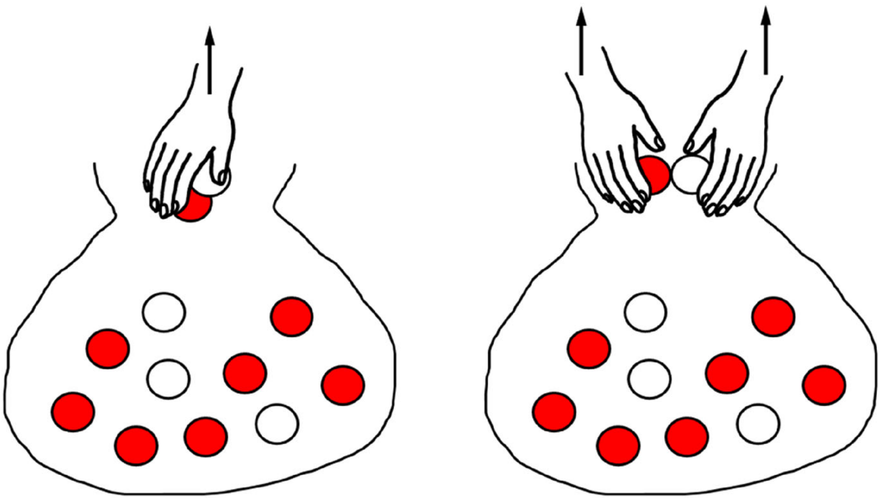

Figure 2. Does 'drawing two balls simultaneously' mean, for example, drawing them in one of these ways?

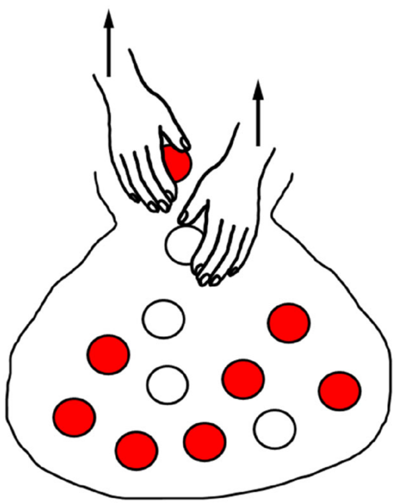

Figure 3. Drawing two balls separately using the right hand and the left hand.

hand is also ${}_8C_1 \times {}_4C_1$. Therefore, the number of ways to draw one red and one white ball is ${}_8C_1 \times {}_4C_1 + {}_4C_1 \times {}_8C_1$. Hence, the required probability is:

$$\frac{{}_8C_1 \times {}_4C_1 + {}_4C_1 \times {}_8C_1}{{}_{12}P_2} = \frac{8 \times 4 + 4 \times 8}{12 \cdot 11} = \frac{16}{33}$$

This indicates that whether the balls are drawn simultaneously or not has no significant impact on the probability in this setting. The assumption that each of the elementary events in Solution 1 is equally likely is based on the principle of insufficient reason. In the context of Problem 1, it is not necessarily intuitive for all students to perceive equal likelihood in this way. Therefore, comparing the two solutions provides students with the opportunity to question the theoretical foundation of equal likelihood and to break down the elementary events. This comparison is a triangulation that leads to a more reliable probability. It becomes clear that simultaneously drawing balls does not fundamentally affect the probability calculation. Additionally, suppose we pursue what it truly means to draw two balls simultaneously from the bag. In that case, numerous interpretive questions arise when attempting to formulate the problem precisely, such as 'Where exactly is the boundary between the inside and outside of the bag?' or 'If two balls are grabbed simultaneously inside the bag but a time lag occurs when pulling them out of the bag, does it still count as drawing them simultaneously?' However, the important point is that regardless of the answers to these questions, they do not affect the way the trial is conducted or how its outcome is observed; therefore, they do not impact the probability of the event in question.

The existence of multiple valid models may appear counterintuitive to beginners. They would conjecture that different models could yield different probabilities. However, this counterintuitive situation can trigger a Type (A) theoretical inversion. It provides students with an opportunity to recognise the theoretical foundation of applying probability to real-world contexts. In particular, we expect that they understand that applying probability involves modelling phenomena. This situation demonstrates that different models yield the same probability under identical mixing conditions. This is also a crucial first step toward appreciating triangulation.

4.2. Hidden first draw

Consider the following problem.

[Problem 2]

From a standard deck of 52 playing cards, two cards are drawn in sequence. After drawing the first card, it is placed face down without being looked at, and the second card is drawn. Find the probability that the second card is a heart.

This is a standard problem involving conditional probability, and generally, we expect the following solution.

[Solution 2-1]

Let event A be that the first card is a heart, and let event B be that the second card is a heart. Event B can occur in either of the following two mutually exclusive cases (see Figure 4):

(i) The first card is a heart, and the second card is also a heart.
(ii) The first card is not a heart, and the second card is a heart.

Therefore, the probability $P(B)$ that the second card is a heart is:

$$P(B) = \frac{13}{52} \times \frac{12}{51} + \frac{39}{52} \times \frac{13}{51} = \frac{1}{4}$$

In this solution, first, the trial of drawing the first card is considered as a single sample space, assuming that each of the 52 cards is equally likely to be drawn. Then, separate sample spaces are considered depending on whether the first card is a heart or not, assuming that each of the remaining 51 cards is equally likely to be drawn in each case. In this solution, a total of three different sample spaces are implicitly involved. However, based on the discussion in this paper, if we intend to question the assumption of equal likelihood and

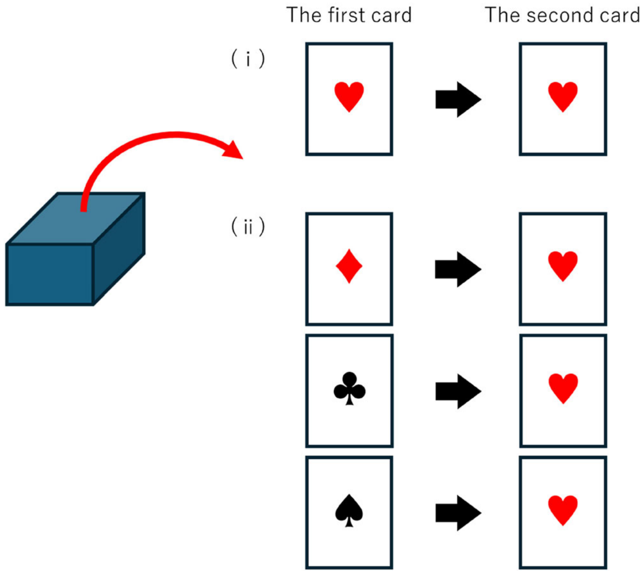

Figure 4. (i) the case where both the first and second cards are hearts, and (ii) the case where the first card is not a heart but the second card is a heart.

advance the cycle of mathematical modelling, we could pose the following question to the students: 'Where did you place the first card you drew and placed face down?' It is expected that various interpretations will emerge, such as placing it face down on the table or hiding it in a box so that no one can see it. We can then summarise by asking, 'If you do not see what card was drawn, does it matter where you put it?' If this convinces the students, the teacher can then ask, 'Then, is it okay to leave the first card on top of the deck?' This means that after touching the top card of the well-shuffled deck with your finger, you ultimately do not draw it; instead, you only draw the second card from the top of the deck. Based on this idea, the following solution can be considered.

[Solution 2-2]

Since we do not know what was drawn for the first card, we ignore it (Figure 5(a)). Thus, the probability of drawing a heart is:

$$\frac{13}{52} = \frac{1}{4}$$

In other words, if we do not look at the first card drawn, it is essentially the same as not drawing it. 'Drawing the first card' does not necessarily mean drawing the top card of the deck; there is no problem in drawing the second card from the top as the first card (Figure 5(b)). Therefore, not looking at the first card and drawing the second card is equivalent to drawing only the second card from the top without considering the top card. This involves discarding operations that have no significant impact on the trial under consideration and considering a more straightforward sample space. Since elementary events can be defined in different ways, redefining them in a more straightforward manner is itself a form of modelling. This process can also be viewed as a form of triangulation, where different models converge to the same conclusion.

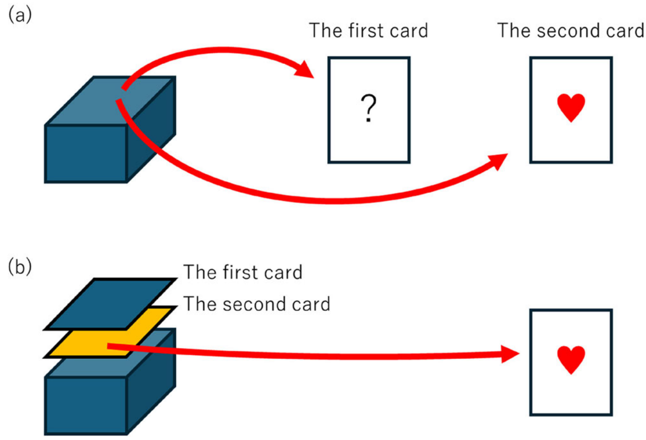

Figure 5. Drawing the second card without looking at the first.

Problem 1 maintains the procedure and models the concept of 'simultaneous drawing' in two ways. Problem 2 takes a different approach. It replaces the original procedure with a simpler one and models that instead. This is a more radical form of triangulation. While they may initially expect that simplifying the trial procedure would lead to a different probability from that obtained in the original experiment, beginners may be surprised to learn that both approaches yield the same probability. This surprise can lead to a Type (A) theoretical inversion. It also supports Devlin's (2014) idea that probability reflects what we know, not just what happens. Problem 2 demonstrates that we can disregard steps that do not provide new information.

4.3. *How mixing affects probability*

Problem 3 was inspired by Feller (1968). Consider the following problem.

[Problem 3]

Six balls are distributed into three boxes labeled P, Q, and R. The boxes may be empty. Let event A be the event that precisely one ball is placed in box P. Find the probability $P(A)$ using the following two methods.

[Experiment 1]

As shown in Figure 6, divide the ground into three regions of 120 degrees each, labeled P, Q, and R, and place a cone at the center of the three regions. Then, drop each ball from directly above the cone and determine which box to place the ball in based on the region in which it lands.

[Experiment 2]

Prepare six cards marked with "○" and two cards marked with "|". Shuffle them thoroughly and lay them out in a row. Place as many balls in box P as there are "○" cards between the left end and the first "|" card. Then, place as many balls in box Q as there are "○" cards between the first "|" card and the second "|" card. Place the remaining balls in box R. For example, if the cards are shuffled and arranged, as shown in Figure 7, there will be two balls in box P, one ball in box Q, and three balls in box R.

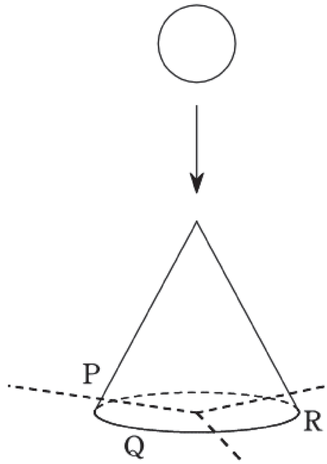

Figure 6. The way of mixing balls in Experiment 1.
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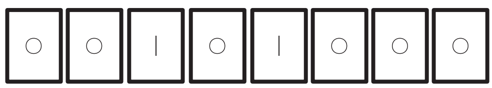

Figure 7. The way of mixing balls in Experiment 2: the case where two balls can be placed in box P, one in box Q, and three in box R.
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Suppose students do not adequately grasp the concept of setting equally likely elementary events and only have a notion of mixing randomly. In that case, they may mistakenly believe that the probability of event A is the same in both experiments. However, the actual probabilities differ.

[Solution for Experiment 1]

For Experiment 1, $n(U)$ is calculated by considering that each ball has three possible choices, giving $n(U) = 3^6$.

The number of ways for event A, where precisely one ball is placed in box P, is 6 (for which ball goes into P) times 2^5 (ways to distribute the remaining five balls into boxes Q and R).

Thus,

$$P(A) = \frac{6 \times 2^5}{3^6} = \frac{64}{243}$$

[Solution for Experiment 2]

For Experiment 2, $n(U)$ is the number of ways to arrange the eight cards, which is a permutation with repetitions, given by $n(U) = \frac{8!}{6!2!} = 28$.

For $n(A)$, if the cards are arranged as "○," "|" from the left, the arrangement of the remaining six cards does not matter, giving $n(A) = \frac{6!}{5!} = 6$. (Figure 8)

Hence,

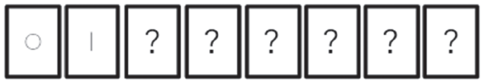

Figure 8. If the first card drawn is the '○' card and the second is the '|' card, then regardless of the order of the remaining cards, only one ball is placed in box P.

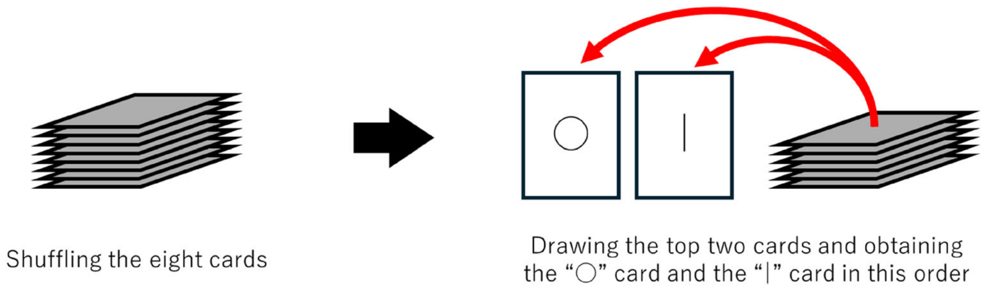

Figure 9. An alternative interpretation of Experiment 2.

$$P(A) = \frac{6}{28} = \frac{3}{14}$$

Of course, Experiment 1 can also be understood as a series of repeated trials. Experiment 2 can be thought of as drawing two cards in sequence from a shuffled deck of eight cards, with '○' and '|' appearing in that order. Namely, $\frac{6}{8} \times \frac{2}{7}$ (Figure 9). Although different sample spaces can be considered for each experiment, the two experiments do not yield identical conclusions about the sample spaces. Although it may be difficult to recognise, this problem illustrates that the notion of equally likely outcomes depends on how the items are mixed. Even if students initially believe that the probabilities are the same in both experiments after deciding on the experimental method, by analyzing each experiment using multiple models and triangulating, it is expected that their understanding of the differences between the two trials will be deepened.

Problem 2 illustrates a Type (A) theoretical inversion where students recognise that they can omit steps that provide no new information. Problem 3 involves a Type (B) theoretical inversion. It shows that even if two events sound the same, slight differences in the experiment conditions can lead to different probabilities. Although students may initially expect the two experiments to yield the same result, they find that the results differ. This experience prompts them to reconsider their underlying theoretical assumptions: a precise description of the experimental procedure is essential for the application of probability. Saying, 'randomly assign six balls to three boxes', is insufficient.

5. Conclusion

In this paper, we proposed the idea of triangulation by probabilistic models as a perspective for integrating the meanings of probability in school mathematics. The application of probability to real-world phenomena as a form of modelling implies that the difference between probabilistic models lies in frameworks rather than philosophies. By using both empirical and theoretical probabilities or by modelling based on different theoretical schemes, we can achieve a more comprehensive and in-depth understanding of the probability of a single event. Triangulation originates from methodologies in the social sciences and is epistemologically supported by Steinbring's (1991) concept of theoretical inversion as an approach to probability education. Furthermore, in this paper, we presented three examples of teaching materials that facilitate triangulation: *Drawing Two Balls Simultaneously*, *Hidden First Draw*, and *How Mixing Affects Probability*.

These teaching materials aim to stimulate theoretical inversion through theory triangulation. They help students understand how probability connects to real-world applications. Problem 1 demonstrates that different models, based on the same procedure, yield the same probability for a given event. Problem 2 demonstrates that if a step in the procedure provides no additional information, it can be omitted without affecting the outcome. Problem 3 shows that when the procedure is not clearly described, the probability cannot be uniquely defined. This paper focused on theoretical probability and proposed materials designed for theory triangulation. However, it is noteworthy that we can also apply the other three types of triangulations (data triangulation, investigator triangulation, and methodological triangulation) in actual classrooms, where empirical probability is also important.

This paper has proposed teaching materials from a theoretical perspective. The first and third authors of this paper are high school mathematics teachers and use these materials in their instruction on probability. Based on their teaching experience, they feel confident in the effectiveness of these materials. However, determining whether these materials genuinely trigger theoretical inversion in students will require substantial further work, including the development of a theoretical framework capable of capturing instances of theoretical inversion in classroom interactions and in individual students, as well as the refinement of methodological approaches for data collection. Several issues must therefore be examined before such investigations can be carried out systematically.

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This paper expands on the core content of a paper presentation delivered at the 53rd Research Conference of the Japan Academic Society of Mathematics Education, with new theoretical considerations added. The original presentation manuscript (in Japanese only) is available at https://ir.lib.hiroshima-u.ac.jp/00050267. Figures 6 and 7 are reproduced from the original presentation manuscript of the authors' earlier work, with copyright retained by the authors.

Author contributions

CRediT: **Yusuke Uegatani:** Conceptualization, Project administration, Writing – original draft; **Ippo Ishibashi:** Conceptualization, Writing – review & editing; **Aya Sakota:** Conceptualization, Writing – review & editing

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Declaration of AI tool use

As non-native English speakers, we used AI-based tools solely to improve the clarity and fluency of the English expressions in this manuscript. Specifically, we employed **ChatGPT-5.1 (OpenAI)** for language refinement and **Grammarly (version 1.2)** for grammar and style suggestions. These tools were not used for generating content or analyzing data, and all intellectual contributions remain our own.

Data availability statement

No empirical data were collected or analyzed; therefore, no datasets are associated with this work.

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