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TSETLIN LIBRARY ON P-COLORED PERMUTATIONS AND Q-ANALOGUE

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Abstract. K. Brown [1] studied the random to top shuffle (the Tsetlin libary) by semigroup method. In this paper, (i) we extend his results to the colored permutation groups, and (ii) we consider a q-analogue of Tsetlin library which is different from what is studied in [1]. In (i), the results also extends those results for the top to random shuffle [4],[5], [6] to arbitrary distribution of choosing cards, but we still have derangement numbers in the multiplicity of each eigenvalues. In (ii), a version of q-analogue of derangement numbers by Chen-Rota [3] appears in the multiplicity of eigenvalues.

1. Introduction

Markov chains generated by card shuffles are formulated in terms of random walks on the symmetric groups, and many studies are done on this subject, including estimates on the mixing times and cut-off. Among them, random to top shuffles (Tsetlin library) can be analyzed clearly by using the left regular band (LRB, in short) [1]. In this paper, we discuss (i) an extension of his results to the colored permutation group, and (ii) a q-analogue, which is different from that in [1].

The first one, that is, the Tsetlin library on p-colored permutation is described as follows. Let $p \in \mathbf{N}$ and suppose that we have certain amount of book covers of p-types. We pick up a book at random from a bookshelf, put a randomly chosen cover, and then put it at the leftmost side of the bookshelf. The corresponding Markov chain (called **the p-Tsetlin library**) is formulated as follows. Let $G_{n,p}$ be the colored permutation group of p-colors which we consider as the state space. We identify $G_{n,p} = C_p \wr_{[n]} \mathfrak{S}_n$, where $[n] := \{1, 2, \cdots, n\}$ and $C_p := \{0, 1, \cdots, p-1\}$. Let $\{v_i\}_{i \in [n]}$, $\{h_j\}_{j \in C_p}$ be distributions on $[n]$ and C_p respectively. Given an element $\big((x_1, q_1), \cdots, (x_n, q_n)\big) \in ([n] \times C_p)^n$ $(x_a \neq x_b$ $(a \neq b))$, we pick up a number $i \in [n]$ with probability v_i, and a number $q \in C_p$ with probability h_q. If $i = x_k$, we pick up (x_k, q_k) change its color into q, and then put it at the top. So the new element is $\big((x_k, q), (x_1, q_1), \cdots, (x_{k-1}, q_{k-1}), (x_{k+1}, q_{k+1}), \cdots, (x_n, q_n)\big)$. The corresponding random walk on $G_{n,p}$ is called the p-Tsetlin library.

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Example $(p=3, n=7)$: For given $\left(\binom{3}{0}\binom{5}{2}\binom{7}{2}\binom{4}{0}\binom{1}{2}\binom{6}{1}\binom{2}{0}\right)$ suppose that we pick up number 7 and color 1. Then we have

$$\left(\binom{3}{0}\binom{5}{2}\binom{7}{2}\binom{4}{0}\binom{1}{2}\binom{6}{1}\binom{2}{0}\right) \mapsto \left(\binom{7}{1}\binom{3}{0}\binom{5}{2}\binom{4}{0}\binom{1}{2}\binom{6}{1}\binom{2}{0}\right).$$

Brown [1] studied Tsetlin library for $p=1$ using LRB, computed the eigenvalues and multiplicities of the transition probability matrix, and showed that they are given by $\lambda_X = \sum_{i\in X} v_i$, $m_X = d_{n-|X|}$, where X ranges over all subsets of $[n]$, and $d_n := \sharp\{\sigma \in \mathfrak{S}_n \,|\, \sigma(k) \neq k,\ k \in [n]\}$ is the number of permutations with no fixed points. Furthermore Chatterjee, Diaconis, and Kim [2] studied some enumerative properties of the Tsetlin library, such as distribution of top (and bottom) k cards. First of all, we extend Brown's result for p-Tsetlin library.

Theorem 1.1
(1) The eigenvalues and corresponding multiplicities of the transition probability matrix of the p-Tsetlin library are given by $\{(\lambda_X, m_X)\}_{X\subset[n]}$ *where* $X(\subset [n])$ *ranges over all subsets of* $[n]$ *and*

$$\lambda_X = \sum_{i\in X} v_i, \quad m_X = (n-|X|)!\, p^{n-|X|} \sum_{k=0}^{n-|X|} \frac{(-1)^k}{k!p^k}, \quad X \subset [n].$$

We have $m_X = d_{n-|X|,p}$ *where* $d_{n,p}$ *is equal to the number of elements of* $G_{n,p}$ *with no fixed points.*

$$\begin{aligned} d_{n,p} &:= \sharp\left\{\tau = \begin{pmatrix} \sigma_1 & \sigma_2 & \cdots & \sigma_n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix} \in G_{n,p} \,\middle|\, \sigma_i \neq i \vee a_i \neq 0\ \forall i \in [n]\right\} \\ &= n!p^n \sum_{j=0}^{n} \frac{(-1)^j}{j!p^j} \end{aligned}$$

(2) The stationary distribution is given by

$$\begin{aligned} &\pi((x_1,b_1),\dots,(x_n,b_n)) \\ &= \frac{h_{b_1}v_{x_1}}{1}\frac{h_{b_2}v_{x_2}}{1-v_{x_1}}\frac{h_{b_3}v_{x_3}}{1-(v_{x_1}+v_{x_2})}\frac{h_{b_{n-1}}v_{x_{n-1}}}{1-(v_{x_1}+\cdots+v_{x_{n-2}})}\frac{h_{b_n}v_{x_n}}{1-(v_{x_1}+\cdots+v_{x_{n-1}})} \end{aligned}$$

which is equal to the probability arranging the cards x_1 with color b_1 , x_2 with color b_2 , and so on. It also has an integral representation as in remarked in [2] :

$\pi((x_1,b_1),\dots,(x_n,b_n))$

$$=v_{x_1}\cdots v_{x_n}h_{b_1}\cdots h_{b_n}\int_{x_1=0}^{\infty}\int_{x_2=x_1}^{\infty}\int_{x_3=x_2}^{\infty}\cdots\int_{x_n=x_{n-1}}^{\infty}\exp(-\sum_{k=1}^{n}v_{x_k}x_k)dx_1\cdots dx_n$$

which is equal to the probability that having colors $b_1, b_2, \ldots, b_n$ and that $X_1 < X_2 < \cdots < X_n$, where X_k has exponential sidtribution with parameter v_{x_k}.

For the top to random shuffle, which is the "inverse" of random to top shuffle of equal weights. Diaconis-Fill-Pitman [4], Garsia [5] ($p = 1$), Nakano-Sadahiro-Sakurai [6] (general p), used some algebraic properties of such shuffles to obtain the primitive orthogonal idempotents, leading to the same results, and moreover also obtained the estimates on mixing time and cut-off. But they have to assume that $\{v_i\}$, $\{h_j\}$ are uniformly distributed. On the other hand, Theorem 1.1 says that the Brown's method of LRB works for general distributions $\{v_i\}$, $\{h_j\}$ for p-Tsetlin library. It also implies that the eigenvalues and multiplicities depend only on "the choice of books" $\{v_i\}$ and not on that of "the choice of colors" $\{h_j\}$.

We next turn to the q-analogue. Let $\mathbf{F}_q$ be the finite field of q elements and set $V_{n,q} = \mathbf{F}_q^n \setminus \{\vec{0}\}$. We identify $GL_n(V_{n,q})$ with the set of $C_{n,q}$, where $C_{n,q} := \{X = (x_1, \ldots, x_n) \ | x_i \in V_{n,q},\, x_1, \cdots, x_n$: linearly independent$\}$. We define the action of $v \in V_{n,q}$ on $A = (x_1, \ldots, x_n) \in C_{n,q}$ from the left as follows. Let k be the minimum integer satisfying the condition : " x_k is equal to a linear combination of $v, x_0, x_1, \cdots, x_{k-1}$". Then we eliminate x_k from A and set $vA := (v, x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n) \in C_{n,q}$.

Example $(q = 3,\ n = 3)$:

$$v = \begin{pmatrix}1\\1\\0\end{pmatrix},\quad A = \left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\},\quad vA = \left\{\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$$

Let $\{w_v\}_{v\in V_{n,q}}$ be a distribution on $V_{n,q}$. Given $A = (x_1, \ldots, x_n) \in C_{n,q}$, we pick up $v \in V_{n,q}$ with probability w_v and set vA. The corresponding Markov chain on $GL_n(V_{n,q})$ is called **the q-analogue I of the Tsetlin library** (we will discuss q-analogue II later).

Theorem 1.2
(1) The eigenvalues and corresponding multiplicities of the transition probability matrix of the q-analogue I of the Tsetlin library are given by $\{(\lambda_X, m_X)\}_{X\subset V_{n,q}}$ where X ranges over all subspaces of $V_{n,q}$ and

$$\lambda_X = \sum_{v\in X} w_v, \quad X \subset V_{n,q},\ X : \text{subspace}$$

$$m_X = \{n-\dim X\}_q! \sum_{j=0}^{n-\dim X} \frac{(-1)^j q^{\binom{j}{2}} q^{\dim X (n-\dim X - j)}}{\{j\}_q!}$$

where

$$\{0\}_q := 0 \qquad \{n\}_q := q^{n-1}(q^n-1) \quad (n \geq 1)$$
$$\{0\}_q! := 1 \qquad \{n\}_q! := \prod_{k=1}^{n} \{k\}_q = \prod_{k=1}^{n} q^{k-1}(q^k-1) \quad (n \geq 1)$$

(2) The stationary distribution is given by

$$\pi(a_1, \ldots, a_n) = \frac{w_{a_1}}{1} \frac{w_{a_2}}{1-(\sum_{d\in \mathrm{span}\{a_1\}} w_d)} \frac{w_{a_3}}{1-(\sum_{d\in \mathrm{span}\{a_1,a_2\}} w_d)} \cdots \frac{w_{a_i}}{1-(\sum_{d\in \mathrm{span}\{a_1,\ldots,a_{i-1}\}} w_d)} \cdots \frac{w_{a_n}}{1-(\sum_{d\in \mathrm{span}\{a_1,\ldots,a_{n-1}\}} w_d)}$$

Remark
m_X is related to a q-analogue of derangement number discussed in [3] which we briefly recall here. Let V be an n-dimensional linear space over the finite field $\mathbb{F}_q$ $G_n(q)$ (resp. $F_n(k)$) be the number of automorphisms on V with no fixed points (resp. the number of automorphisms on V fixing points in a k-dimensional subspace and does not fix any other points) :

$$G_n(q) := \sharp \{A \in auto\,(V) \mid Ax \neq x, \ \text{for any } x \neq 0\}$$
$$F_n(k) := \sharp \Big\{ f \in auto\,(V) \ \Big| \ f(x) = x, \ f(y) \neq y, \ \text{for any } x \in X, \ y \notin X \ \text{for some } k\text{-dimensional subspace } X \Big\}$$

Then Chen and Rota [3] showed

$$G_n(q) = \{n\}_q! \sum_{j=0}^{n} (-1)^j \frac{q^{\binom{j}{2}}}{\{j\}_q!}$$
$$F_n(k) = \begin{bmatrix} n \\ k \end{bmatrix}_q \{n-k\}_q! \sum_{j=0}^{n-k} (-1)^j \frac{q^{\binom{j}{2}} q^{k(n-k-j)}}{\{j\}_q!}$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[n-k]_q![k]_q!}$ is the q-binomial coefficient and

$$[0]_q := 0, \quad [n]_q := \frac{q^n-1}{q-1} \ (n \geq 1), \quad [0]_q! := 1, \quad [n]_q! := \prod_{k=1}^{n} [k]_q \ (n \geq 1).$$

These formulas imply that we cannot have the analogy of the relations which hold in the usual set situation.

$$F_n(k) \neq \begin{bmatrix} n \\ k \end{bmatrix}_q G_{n-k}(q).$$

And the formula for m_X in Theorem 1.2 implies that m_X is equal to "the derangement number per a subspace". That is,

$$m_X = \frac{F_n(\dim X)}{\begin{bmatrix} n \\ \dim X \end{bmatrix}_q}.$$

On the other hand, Brown [1] considererd another Markov chain, which we call **the q-analogue II of the Tsetlin library**, such that the state space is the set of the chains of subspaces of $V_{n,q}$ and the random operation corresponding to the map $(v, A) \mapsto vA$ mentioned in the definition of q-analogue I. The eigenvalues of the transition probability matrix are similar as those in Theorem 1.2, but the multiplicities are equal to $d_{n-\dim X}(q)$ where $d_n(q)$ is the q-derangement number introduced by [7]. $d_n(q)$ satisfies $[n]_q! = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q d_k(q)$ as the usual derangement number does, but $F_n(k)$ has explicit meaning as "the derangement as a map". In Sections 2, 3, we prove Theorems 1.1 and 1.2 respectively, which are done by choosing appropriate semigroups and apply Brown's theory of LRB. In Appendix I, we briefly recall Brown's result on LRB [1]. In Appendix II, we recall the definition of the q-analogue I and the statement corresponding to Theorem 1.2. In Appendix III, we show that the random walk on the hypercube $\{0, 1\}^n$ with any weights can be analyzed using LRB.

2. Proof of Theorem 1.1

In this section, we introduce a LRB by which p-Tsetling library is described, and apply Brown's result [1] to prove Theorem 1.1. Let S'_n be the set of arrangement of distinct numbers in $[n]$ of length $l(= 1, 2, \cdots, n)$. And let S_n be the corresponding set with colors.

$$S'_n := \left\{ x = (x_1, \ldots, x_l) \in [n]^l \,\middle|\, x_i \neq x_j \text{ for } i \neq j,\ l = 1, 2, \cdots, n \right\} \cup \{e\}$$
$$S_n := \left\{ x = \big((x_1, a_1), \ldots, (x_l, a_l)\big) \,\middle|\, (x_1, \ldots, x_l) \in S'_n,\ a_i \in C_p,\ l = 1, 2, \cdots, n \right\} \cup \{e\}$$

where e is the identity element. For $x = \big((x_1, a_1), \ldots, (x_l, a_l)\big) \in S_n$, let $\dot{x} = \big(x_1 \ldots x_l\big) \in S'_n$ be the one obtained by eliminating colors from x. Also, for $x = \big((x_1, a_1), \ldots, (x_i, a_i)\big), y = \big((y_1, b_1), \cdots, (y_j, b_j)\big) \in S_n$, we define product $x * y$ of x, y to be

$$x * y = \big((x_1, a_1), \ldots, (x_l, a_l), (y_1, b_1), \ldots, (y_m, b_m)\big)^{\wedge}$$

where $\wedge$ means to eliminate any y_i's which already appears in x_j's : that is, on the formal concatenation of $\dot{x}$ and $\dot{y}$

$$(x_1, \ldots, x_l, y_1, \ldots, y_m),$$

if we find a number which appears twice, then we delete what is on the right.

Example : $p = 3, n = 7$

$$\left(\binom{2}{2}\binom{1}{1}\binom{7}{0}\right) * \left(\binom{3}{0}\binom{5}{2}\binom{7}{2}\binom{4}{0}\binom{1}{2}\binom{6}{1}\right)$$
$$= \left(\binom{2}{2}\overline{\binom{1}{1}}\underline{\binom{7}{0}}\binom{3}{0}\binom{5}{2}\underline{\binom{7}{2}}\binom{4}{0}\overline{\binom{1}{2}}\binom{6}{1}\right)^{\wedge}$$
$$= \left(\binom{2}{2}\overline{\binom{1}{1}}\underline{\binom{7}{0}}\binom{3}{0}\binom{5}{2}\binom{4}{0}\binom{6}{1}\right)$$

Then $(S_n, *)$ is LRB. Moreover if we take $l = 1$ for x and $l = n$ for y, say $x = (i, q) \in [n] \times C_p$ and $y = \big((y_1, b_1), \cdots, (y_n, b_n)\big) \in S_n$, $x * y$ is equal to an element in S_n obtained by first picking up $(y_k, b_k) = (i, b_k)$ whose number is equal to i, changing color to q and putting it in the leftmost position :

$$x * y = \big((i, q), (y_1, b_1), \cdots, (y_n, b_n)\big),$$

and this operation coincides with that in p-Tsetlin library. In what follows, we write xy instead of $x * y$.

Remark 1
When $p = 1$, Tsetlin library can also be represented by a suitable choice of hyperplane arrangement. It is also the case for $p = 2$ which will be discussed in our forthcoming paper.

Remark 2
Another (natural) definition of p-Tsetlin library would be to pick up $(i, q) \in [n] \times C_p$, $r \in C_p$ with probability v_i, h_r respectively, to change color to $(i, q + r) \in [n] \times C_p$ and to put in the leftmost position. However, this Markov chain cannot be represented by LRB unless $\{h_r\}_{r \in C_p}$ is uniformly distributed as in [6].

Having expressed p-Tselin library by LRB, we then apply Brown's theory [1] in Appendix I. First of all, we set an equivalence relation $x \sim y \overset{def}{\Longleftrightarrow} xy =$

$x, yx = y$, define $\mathcal{L} := S_n/\sim$, and let supp : $S_n \to \mathcal{L}$ be the natural surjective map. $\mathcal{L}$ becomes a lattice under the ordering : $\operatorname{supp} x \leq \operatorname{supp} y \overset{def}{\Longleftrightarrow} yx = y$. For given $X \in \mathcal{L}$, then let $x \in \operatorname{supp}^{-1}(X)$ and set $c_X := \sharp\{c \in C_n \mid xc = c\}$. This number is independent of the choice of $x \in S_n$ such that $\operatorname{supp} x = X$ so that c_X is well defined.

Lemma 2.1.
(1) $\mathcal{L} = 2^{[n]}$ with $X \leq Y \Leftrightarrow X \subset Y$ $(X, Y \in \mathcal{L})$ and support map is given as follows.

$$\begin{array}{ccc} S_n & \twoheadrightarrow & \mathcal{L}(= 2^{[n]}) \\ x = \big((x_1, j_1), \cdots, (x_l, j_l)\big) & \mapsto & \{x_1 \cdots x_l\} \end{array}$$

(2) When $X = \{x_1, \cdots, x_k\} \in \mathcal{L}$, we have $c_X = p^{n-k}(n-k)!$.

Proof.
For given $x = ((x_1, j_1), \cdots, (x_l, j_l)) \in S_n$, $s(x) := \{x_1, \cdots, x_l\}$. Then $x \sim y$ iff $yx = y$ and $xy = x$. Since $yx = y$ iff $s(x) \subset s(y)$, we have $x \sim y$ iff $s(x) = s(y)$. Therefore we have proved $\mathcal{L} = S/\sim = 2^{[n]}$ and $yx = y$ iff $s(x) \subset s(y)$.
(2) For given $X = \{x_1, \cdots, x_k\} \in \mathcal{L}$, let $x = ((x_1, q_1), \cdots, (x_l, q_k)) \in \operatorname{supp}^{-1}(X)$. Since $xc = c$ iff the first k words in c coincides with x, c_X is equal to the number of choices of remaining $(n-k)$ words in c which is given by $c_X = p^{n-k}(n-k)!$. □ □

Proof of Theorem 1.1
(1) We apply Theorem A.1.1. To compute the eigenvalues of the transition probability matrix P, we take a subset X of $[n]$, and

$$\lambda_X = \sum_{\operatorname{supp} y \leq X} w_y = \sum_{\substack{y=((i,q)) \\ \operatorname{supp} y \leq X}} v_i h_q = \sum_{i \in X} \sum_{j=0}^{p-1} v_i h_j = \sum_{i \in X} v_i.$$

Since the Möbius function of $\mathcal{L} = 2^{[n]}$ is given by $\mu(X, Y) = (-1)^{|Y|-|X|}$ $(X \subset Y)$, we have

$$\begin{aligned} m_X &= \sum_{Y \geq X} \mu(X, Y) c_Y = \sum_{Y \supseteq X} (-1)^{|Y|-|X|} p^{n-|Y|}(n - |Y|)! \\ &= \sum_{k=0}^{n-|X|} (-1)^k \binom{n-|X|}{k} p^{n-|X|-k}(n-k-|X|)! \\ &= (n-|X|)! p^{n-|X|} \sum_{k=0}^{n-|X|} \frac{(-1)^k}{k! p^k} = d_{n-|X|, p}. \end{aligned}$$

(2) The stationary distribution π can be computed directly by using its definition :

$$\begin{aligned}
&\pi((x_1, b_1), \ldots, (x_n, b_n)) \\
&= \lim_{k\to\infty} \mathbb{P}(X_k = ((x_1, b_1), \ldots, (x_n, b_n))) \\
&= \lim_{m\to\infty} \sum_{a_1+\cdots a_n=m} h_{b_1} v_{x_1} v_{x_1}^{a_1} h_{b_2} v_{x_2} (v_{x_1} + v_{x_2})^{a_2} + \cdots + (v_{x_1} + \cdots + v_{x_{n-1}})^{a_{n-1}} h_{b_n} v_{x_n} 1^{a_n} \\
&= \frac{h_{b_1} v_{x_1}}{1} \frac{h_{b_2} v_{x_2}}{1 - v_{x_1}} \frac{h_{b_3} v_{x_3}}{1 - (v_{x_1} + v_{x_2})} \frac{h_{b_{n-1}} v_{x_{n-1}}}{1 - (v_{x_1} + \cdots + v_{x_{n-2}})} \frac{h_{b_n} v_{x_n}}{1 - (v_{x_1} + \cdots + v_{x_{n-1}})}
\end{aligned}$$

□

One can also derive it by computing the primitive idempotents as in Section 8.5 in [1].

3. Proof of Theorem 1.2

In this section, we consider the q-analogue I of Tsetlin library. Set $V_{n,q} = \mathbf{F}_q^n$ and we consider the set $S_{n,q}$ of l-tuples of independent vectors in $V_{n,q}$ with $l = 0, 1, \cdots, n$.

$$S_{n,q} := \{(x_1, \ldots, x_l) \,|\, x_i \in V_{n,q},\ 1 \le l \le n,\ x_1, \cdots, x_l : \text{linearly independent}\ \} \cup \{e\}$$

where e is the identity element. For $x = (x_1, \cdots, x_l) \in S_{n,q}$ and $y = (y_1, \cdots, y_m) \in S_{n,q}$, we define $x * y \in S_{n,q}$ as follows.

$$x * y = (x_1, \ldots, x_l) * (y_1, \ldots, y_m) := (x_1, \ldots, x_l, y_1, \ldots, y_m)^{\wedge}$$

where $\wedge$ means that "delete any vector that is linearly dependent on the earlier vectors". As in the proof of Theorem 1.1, $(S_{n,q}, *)$ is a finite LRB with identity. Since $x \sim y$ iff $xy = x$, $yx = y$ iff span x = span y, the lattice $\mathcal{L}$ corresponding to $S_{n,q}$ is given as follows.

$$\mathcal{L} := S_{n,q}/\sim = \{W \subset V_{n,q} \,|\, W \text{ is a subspace of } V_{n,q}\} \text{ (isomorphism as a poset).}$$

The set $C_{n,q} := \{c \in S_{n,q} \,|\, cx = c,\ \forall x \in S_{n,q}\}$ of chambers is given by setting $l = n$ in the definition of $S_{n,q}$.

$$C_{n,q} = \{(x_1, \cdots, x_n) \,|\, x_i \in V_{n,q},\ x_1, \cdots, x_n : \text{independent}\ \}.$$

Moreover, for given $X = \text{span}\{x_1, \cdots, x_k\} \in \mathcal{L}$, we take $x = (x_1, \cdots, x_k) \in \text{supp}^{-1}(X)$ so that

$$\begin{aligned}
c_X &= \sharp\{c \in S_{n,q} \,|\, xc = c\} \\
&= \sharp\big\{c = (x_1, \cdots, x_k, y_1, \cdots, y_{n-k}) \,|\, y_1, \cdots, y_{n-k} \in V_{n,q}, \\
&\qquad\qquad (x_1, \cdots, x_k, y_1, \cdots, y_{n-k}) : \text{independent}\ \big\}
\end{aligned}$$

Therefore the number of chambers and c_X are given as follows.

$$|C_{n,q}| = \prod_{j=0}^{n-1} (q^n - q^j), \quad c_X = \prod_{j=k}^{k+l-1} (q^n - q^j), \quad k := \dim X.$$

For completeness, we shall derive the Möbius function on $\mathcal{L}$.

Lemma 3.1. The Möbius function μ on $\mathcal{L}$ is given as follows.

$$\mu(X,Y) = (-1)^{\dim Y - \dim X} q^{\binom{\dim Y - \dim X}{2}}, \quad X, Y \in \mathcal{L},\ X \subset Y$$

Proof. Let $X, Y \in \mathcal{L}$, $X \subset Y$, and let $\nu(X,Y)$ be the quantity in RHS in the statement of Lemma 3.1. Then by definition, it suffices to show the following equality.

$$\sum_{Z\,:\,X\subset Z\subset Y} \nu(X,Z) = 1(X=Y) \tag{3.1}$$

Let $k = \dim X$, $m = \dim Y$. Since (3.1) is clear for $k = m$, we suppose $k < m$. Then

$$\begin{aligned}
\sum_{Z\,:\,X\subset Z\subset Y} \nu(X,Z) &= \sum_{l=0}^{m-k} \sum_{\substack{Z\,:\,X\subset Z\subset Y,\\ \dim Z = k+l}} \nu(X,Z) \\
&= \sum_{l=0}^{m-k} \sharp\{Z \mid X \subset Z \subset Y, \dim Z = k+l\}(-1)^l q^{\binom{l}{2}} \\
&= \sum_{l=0}^{m-k} \begin{bmatrix} m-k \\ l \end{bmatrix}_q (-1)^l q^{\binom{l}{2}} = 0
\end{aligned}$$

At the last step, we used q-binomial theorem. □ □

Proof of Theorem 1.2

We apply Theorem A.1.1 again. To compute the eigenvalue of the transition probability matrix, we take a subspace X of $V_{n,q}$ and use the fact that $\operatorname{supp} v \leq X$ iff $v \in X$.

$$\lambda_X = \sum_{v\,:\,\operatorname{supp} v \leq X} w_v = \sum_{v\,:\,v\in X} w_v, \quad X \in \mathcal{L}.$$

For the multiplicity m_X of the eigenvalue λ_X, we set $k = \dim X$ and compute

$$\begin{aligned}
m_X &= \sum_{Y \geq X} \mu(X,Y)\, c_Y \\
&= \sum_{Y\,:\,Y\supseteq X} (-1)^{\dim Y - \dim X} q^{\binom{\dim Y - \dim X}{2}} \prod_{l=\dim Y}^{n-1} (q^n - q^l) \\
&= \sum_{j=0}^{n-k} \sum_{Y\,:\,\dim Y = k+j} (-1)^j q^{\binom{j}{2}} \prod_{l=k+j}^{n-1} (q^n - q^l)
\end{aligned}$$

$$= \sum_{j=0}^{n-k} \left[\begin{array}{c} n-k \\ j \end{array} \right]_q (-1)^j q^{\binom{j}{2}} \prod_{l=k+j}^{n-1} (q^n - q^l). \tag{3.2}$$

Here we note that

$$\prod_{l=k+j}^{n-1} (q^n - q^l) = \prod_{l=k+j}^{n-1} q^l (q^{n-l} - 1) = \frac{q^{\binom{n}{2}}}{q^{\binom{k+j}{2}}} [n-k-j]_q! (q-1)^{n-k-j}$$

On the other hand, using

$$[n]_q! = \frac{\{n\}_q!}{q^{\binom{n}{2}} (q-1)^n}$$

we have

$$\left[\begin{array}{c} n-k \\ j \end{array} \right]_q = \frac{[n-k]_q!}{[j]_q! [n-k-j]_q!} = \frac{\dfrac{\{n-k\}_q!}{q^{\binom{n-k}{2}} (q-1)^{n-k}}}{[n-k-j]_q! \dfrac{\{j\}_q!}{q^{\binom{j}{2}} (q-1)^j}}$$

$$= \frac{\{n-k\}_q!}{\{j\}_q!} \cdot \frac{q^{\binom{j}{2}}}{q^{\binom{n-k}{2}}} \cdot \frac{1}{[n-k-j]_q!} \cdot \frac{1}{(q-1)^{n-k-j}}.$$

Substituting them into (3.2) yields

$$m_X = \sum_{j=0}^{n-k} (-1)^j q^{\binom{j}{2}} \frac{\{n-k\}_q!}{\{j\}_q!} \frac{q^{\binom{j}{2}}}{q^{\binom{n-k}{2}}} \frac{q^{\binom{n}{2}}}{q^{\binom{k+j}{2}}} = \{n-k\}_q! \sum_{j=0}^{n-k} \frac{(-1)^j q^{\binom{j}{2}}}{\{j\}_q!} q^{k(n-k-j)}.$$

□

A. Appendix 1 : LRB

In this section, we recall the Brown's theory [1] on LRB. Let $(S, *)$ be a finite semigroup with identity. We say $(S, *)$ is a left regular band(LRB) if it satisfies the following condition.

$$x^2 = x, \quad xyx = xy, \quad x, y \in S$$

We introduce an order $\leq$, an binary relationship $\preceq$, and an equivalence relation $\sim$ corresponding to $\preceq$ as follows.

(1) $x \leq y \overset{def}{\iff} xy = y$

(2) $x \preceq y \overset{def}{\iff} yx = y$

(3) $x \sim y \overset{def}{\iff} x \preceq y, \ y \preceq x$

Let $\mathcal{L} := S/\sim$ and let supp : $S \to \mathcal{L}$ be the natural surjection. $\mathcal{L}$ is an ordered set under

$$X \leq Y \overset{def}{\Longleftrightarrow} x \preceq y, \quad \text{where } X, Y \in \mathcal{L}, \quad x \in \operatorname{supp}^{-1}(X),\ y \in \operatorname{supp}^{-1}(Y).$$

We note that the relation $x \preceq y$ does not depend on the choice of x, y in $\operatorname{supp}^{-1}(X)$, $\operatorname{supp}^{-1}(Y)$ respectively, so the definition of $X \leq Y$ is well-defined. Moreover $\mathcal{L}$ is a lattice with minimal $\widehat{0}(= \operatorname{supp} e)$ and maximal element $\widehat{1}$. The set C of elements in $\operatorname{supp}^{-1}(\widehat{1})$

$$C := \{c \in S \,|\, \operatorname{supp} c = \widehat{1}\} = \{c \in C \,|\, cx = c,\ \forall x \in S\}$$

is called the **chamber** and we denote by $C_{\geq x}$ the set of chambers c satisfying $x \leq c$.

$$C_{\geq x} := \{c \in C \,|\, x \leq c\} = \{c \in C \,|\, xc = c\}.$$

For $X \in \mathcal{L}$, we take $x \in \operatorname{supp}^{-1}(X)$ and set

$$c_X := \sharp C_{\geq x} = \#\{c \in C \,|\, xc = c\}.$$

Since $\operatorname{supp} x = \operatorname{supp} x'$ implies $\#C_{\geq x} = \#C_{\geq x'}$, c_X is well-defined.

For given probability distribution $\{w_x\}_{x \in S}$ on S, we define the random walk on C by picking up $x \in S$ with probability w_x and multiply x to $c \in S$ from the left : $c \mapsto xc$. The transition probability matrix P is given by

$$P(c, d) := \sum_{x \,:\, xc = d} w_x, \quad c, d \in C.$$

Theorem A.1.1
P is diagonalizable and its eigenvalues and corresponding multiplicities are given by $\{(\lambda_X, m_X)\}_{X \in \mathcal{L}}$ where

$$\lambda_X := \sum_{y \,:\, \operatorname{supp} y \leq X} w_y, \quad m_X := \sum_{Y \,:\, X \leq Y} \mu(X, Y)\, c_Y, \quad X \in \mathcal{L}$$

and $\mu(X, Y)$ is the Möbius function on $\mathcal{L}$.

B. Appendix 2 : q-analogue II

In this section, we shall discuss the q-analogue II of the Tsetlin library which is studied in [1]. The state space $C_{n,q}$ is the set of chains of subspaces of $V_{n,q}(= \mathbf{F}_q^n)$ instead of taking the basis of them :

$$C_{n,q} := \Big\{ \{X_i\}_{i=0}^n \,\Big|\, 0 = X_0 < X_1 < \cdots < X_n = V_{n,q},\ \dim X_i = i,\ i = 0, 1, \cdots, n \Big\}.$$

Given $\{X_i\}_{i=0}^n \in C_{n,q}$, pick up one-dimensional subspace(line) ℓ of $V_{n,q}$ with given probability weight $\{w_\ell\}_\ell$ and add ℓ to each components of $\{X_i\}_{i=0}^n$:

$$0 < \ell \leq X_1 + \ell \leq X_2 + \ell \leq \cdots \leq X_{n-1} + \ell \leq V_{n,q}$$

Eliminating the repetition, we obtain another element of $C_{n,q}$. We call this Markov chain the q-analogue II of the Tsetling library.

Theorem A.2.1
The eigenvalues and corresponding multiplicities of the q-analogue II of the Tsetling library is given as follows.

$$\lambda_X = \sum_{v\in X} w_v, \quad X : \text{subspace of } V_{n,q},\ \dim X \neq n-1$$
$$m_X = [n-\dim X]_q! \sum_{j=0}^{n^{\dim X}} \frac{(-1)^j}{[j]_q!} q^{\binom{j}{2}}.$$

where $[n]_q, [n]_q!$ are q-analogue of natural number and factorial defined by

$$[0]_q := 0, \quad [n]_q := \frac{q^n-1}{q-1}\ (n\geq 1), \quad [0]_q! := 1, \quad [n]_q! := \prod_{k=1}^{n}[k]_q\ (n\geq 1).$$

To prove Theorem A.2.1, we consider a LRB and the corresponding lattice $\mathcal{L}$.

$$\overline{S}_{n,q} := \Big\{ \{X_i\}_{i=0}^{\ell} \,\Big|\, X_i\text{: subspaces, } 0 = X_0 < X_1 < \cdots < X_{\ell-1} < X_\ell = V,$$
$$\dim X_i = i,\ 0\leq i\leq \ell-1,\ i=0,\cdots,\ell,\ \ell = 0,1,\cdots,n \Big\}$$
$$\overline{\mathcal{L}} := \{W \mid W \text{ subspace of } V_{n,q},\ \dim W \neq n-1\}$$

Then support map is given by

$$\mathrm{supp}(X_0,\cdots,X_\ell) := \begin{cases} X_{\ell-1} & \ell < n \\ V_{n,q} & \ell = n \end{cases}$$

Remark The multiplicity m_X can be written as $m_X = d_{n-\dim X}(q)$ where

$$d_n(q) := [n]_q! \sum_{k=0}^{n} \frac{(-1)^k}{[k]_q!} q^{\binom{k}{2}}$$

is called the q-derangement number [7]. To compare with $F_n(k) / \begin{bmatrix} n \\ \dim X \end{bmatrix}_q$, which appeared in Theorem 2.1, we recall its definition. We write the element $\sigma \in \mathfrak{S}_n$ of $\mathfrak{S}_n$ as $\sigma = (\sigma_1,\cdots,\sigma_n)$. We define the descent set des(σ) and major index maj(σ) as

$$des(\sigma) := \{i \in [n-1] \mid \sigma_i > \sigma_{i+1}\}$$
$$maj(\sigma) := \sum_{i\in des(\sigma)} i.$$

The following formula due to MacMahon is well-known.

$$\sum_{\sigma\in\mathfrak{S}_n} q^{maj(\sigma)} = [n]_q!$$

which motivates to define the q-derangement number.

$$d_n(q) := \sum_{\sigma\in D_n} q^{maj(\sigma)} \quad \text{where} \quad D_n := \{\sigma\in\mathfrak{S}_n \mid \sigma_i \neq i \text{ for any } i\in[n]\}\,.$$

Then being different from $F_n(k)$, it satisfies

$$[n]_q! = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q d_k(q)$$

and the Möbius inversion formula yields

$$d_n(q) = [n]_q! \sum_{k=0}^{n} \frac{(-1)^k}{[k]_q!} q^{\binom{k}{2}}.$$

C. Apendix 3 : RW on hypercube

In this section we show that the random walk on the hypercube $\{0,1\}^n$ can be analyzed by LRB. Let $\{v_j\}_{j=1}^n$ be a probability distribution on $[n]$ and let $(X_k)_{k=1}^{\infty}$ be be a lazy random walk such that for given $X_k = (X_k^{(1)},\dots,X_k^{(n)}) \in \{0,1\}^n$, choose $j\in[n]$ and flip $X_k^{(j)}$ with probability $v_j/2$. X_k does not move with probability 1/2. In this appendix, we show that $\{X_k\}$ can be regarded as a random walk on a hyperplane arrangement which is a typical example of LRB.

Theorem A.3.1
The eigenvalues of the transition probability matrix of this random walk are given as follows.

$$\lambda_X := \sum_{i\notin X} v_i \ ,\ X\subset[n]\ ,\ m_X = 1.$$

In particular, if $v_i = \dfrac{1}{n}$for all i eigenvalues and corresponding multiplicities are given by

$$\lambda_i = \frac{i}{n}\ ,\ m_i = \binom{n}{i}\ ,\ i = 0,1,\dots,n$$

We begin by recalling briefly the random walk on hyperplane arrangement. First of all, we consider a set of hyperplanes

$$\mathcal{A} := \{H_i\}_{i=1}^n, H_i = \{x = (x_1,\dots,x_n)\in\mathbb{R}^n \mid x_i = 0\}.$$

Let $H_i^+ := \{x \in \mathbb{R}^n \mid x_i > 0\}, H_i^- := \{x \in \mathbb{R}^n \mid x_i < 0\}$,and $H_i^0 := H_i$. Then $\mathcal{A}$ induces a partitation of $\mathbb{R}^n$ into convex sets called faces :

$$\mathcal{F} := \{F = \bigcap_{i=1}^n H_i^{\sigma_i} \mid \sigma_i = +, -, 0,\ i = 1, 2, \ldots, n\}.$$

Since each face $F = \cap_{i=1}^n H_i^{\sigma_i} \in \mathcal{F}$ is characterized by $\{\sigma_1, \ldots, \sigma_n\}$, we write $\sigma(F) := \{\sigma_i(F)\}_{i=1}^n$, with $\sigma_i(F) := \sigma_i$ and call it the sign sequence of F. Given $F, G \in \mathcal{F}$, the product FG is defined to be the face whose sign sequence is given by

$$\sigma_i(F * G) = \begin{cases} \sigma_i(F) & , \quad \sigma_i(F) \neq 0 \\ \sigma_i(G) & , \quad \sigma_i(F) = 0 \end{cases}.$$

Then $\mathcal{F}$ becomes a LRB under this product, the corresponding lattice $\mathcal{L}$ is the set of affine subspaces in $\mathbb{R}^n$, and the support map supp : $\mathcal{F} \to \mathcal{L}$ is given by

$$\operatorname{supp} F = \bigcap_{\sigma_i(F)=0} H_i$$

We consider a set $\{F_j^\pm\}_{j=1}^n$ of faces as follows.

$$\begin{aligned} F_j^+ &= \{x \in \mathbb{R}^n \mid x_j > 0, x_i = 0, i \neq j\} \\ F_j^- &= \{x \in \mathbb{R}^n \mid x_j < 0, x_i = 0, i \neq j\}, j = 1, 2, \ldots, n \end{aligned}$$

and set a distribution $\{w_F\}_{F \in \mathcal{F}}$ as follows.

$$w_F := \begin{cases} \frac{w_j}{2} & F = F_j^+ \\ \frac{w_j}{2} & F = F_j^- \\ 0 & \text{otherwise} \end{cases}$$

Then the corresponding Markov chain on $\mathcal{A}$ is nothing but the lazy random walk on $\{0, 1\}^n$ and Theorem A.3.1 follows from Theorem A.1.1.

Remark

Since the transition probability matrix P of the random walk and that P_{lazy} of its lazy version satisfies $P_{lazy} = \frac{1}{2}(P + I)$, the eigenvalues and its multiplicities are given by

$$\lambda_X = 1 - 2 \sum_{i \in X} v_i\ ,\ X \subset [n]\ ,\ m_X = 1.$$

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