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Signatures of surface bundles and scl of a Dehn twist

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A dedication can be included here

Abstract

The first aim of this paper is to give four types of examples of surface bundles over surfaces with non-zero signature. The first example is with base genus 2, a prescribed signature, a 0-section and the fiber genus greater than a certain number which depends on the signature. This provides a new upper bound on the minimal base genus for fixed signature and fiber genus. The second one gives a new asymptotic upper bound for this number in the case that fiber genus is odd. The third one has a small Euler characteristic. The last is a non-holomorphic example.

The second aim is to improve upper bounds for stable commutator lengths of Dehn twists by giving factorizations of powers of Dehn twists as products of commutators. One of the factorizations is used to construct the second examples of surface bundles. As a corollary, we see that there is a gap between the stable commutator length of the Dehn twist along a nonseparating curve in the mapping class group and that in the hyperelliptic mapping class group if the genus of the surface is greater than or equal to 8.

1. *Introduction*

1.1. *Notation*

In here, we introduce notation. Let Σ_g^r be a compact oriented surface of genus g with r boundary components, and let $\mathcal{M}_g^r$ be the mapping class group of Σ_g^r, that is the group of isotopy classes of orientation preserving self-diffeomorphisms of Σ_g^r such that diffeomorphisms and isotopies fix the points of the boundary. For simplicity, we write $\Sigma_g = \Sigma_g^0$ and $\mathcal{M}_g = \mathcal{M}_g^0$. For a subsurface Σ of Σ_g^r, let $\mathcal{M}(\Sigma)$ denote the subgroup of $\mathcal{M}_g^r$ generated by elements whose restrictions on $\Sigma_g^r - \Sigma$ are identity. We denote by $i(a,b)$ the geometric intersection number for two simple closed curves a and b on Σ_g^r.

For $\phi_1, \phi_2 \in \mathcal{M}_g^r$, the notation $\phi_2\phi_1$ means that we first apply ϕ_1 then ϕ_2, the conjugation $\phi_2\phi_1\phi_2^{-1}$ of ϕ_1 by ϕ_2 is denoted by ${}_{\phi_2}(\phi_1)$, and we write $[\phi_1, \phi_2]$ for the commutator of ϕ_1 and ϕ_2. We denote by t_c the right-handed Dehn twist along a simple closed curve c on Σ_g^r. Since $\mathcal{M}_g^r$ is generated by Dehn twists [**13**], every f in $\mathcal{M}_g^r$ can be written as a word in the set of all Dehn twists. If we consider f without explicit word, then we suppose that a certain word of f is given and fixed.

A surface bundle over a surface is a fiber bundle that the fiber and the base are closed oriented surfaces. If the fiber and the base are Σ_g and Σ_h, respectively, then we call this a Σ_g-bundle over Σ_h. For the total space X of this bundle, we denote by $\sigma(X)$ the signature of X. We write it simply σ when no confusion can arise.

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In this paper, we introduce the symbol "$\equiv_P$" in Section 2.3. If the reader is interested only in the results on the (stable) commutator length, then he or she may replace "$\equiv_P$" by "=" and skip Section 2.1, 2.3, 5 and 6.

1.2. *Surface bundles over surfaces with non-zero signature*

Even though to consider surface bundles over surfaces is one simple way to get 4-manifolds, many fundamental problems on such bundles remain open. Problems about surface bundles with non-zero signature are exemplified as one of them.

Euler characteristics multiply for fiber bundles. In contrast, this property does not hold for the signature. Equivalently, there is a surface bundle over a surface with non-zero signature. Such examples were first exhibited by Atiyah [**2**] and, independently, Kodaira [**26**]. Since then, many examples of surface bundles with nonvanishing signature have been constructed (see e.g. [**24**, **14**, **9**, **8**, **42**, **16**, **1**, **34**]).

A Σ_g-bundle over Σ_h gives some restrictions on the signature σ. For example, σ must be divisible by 4, and it vanishes if $h \leq 1$ or $g \leq 2$ using Meyer's signature cocycle and Birman-Hilden's relations [**6**] of $\mathcal{M}_2$ (see [**36**, **37**]). Hence, the case of $g \geq 3$ and $h \geq 2$ is interesting. The existence of an example of $g = 3$ and $\sigma \neq 0$ was shown in [**36**, **37**], and explicit examples were constructed in [**14**, **42**, **16**, **34**]. In particular, for any integer n, there is a Σ_3-bundle over Σ_h with $\sigma = 4n$ if $h \geq 7|n| + 1$ (see [**34**]). An example of $h = 2$ and $\sigma \neq 0$, which solves Problem 2.18 (A) in [**25**], was first given by Bryan-Donagi [**8**]. Precisely, it satisfies $g = 4k^3 - 2k^2 + 1$ and $\sigma = 8(k^3 - k)/3$ for any integer $k \geq 2$. Thus, we notice that g and σ in the example of $h = 2$ take discrete values compared to h and σ in the examples of $g = 3$. If the example of [**8**] has a 0-section (i.e. a section of self-intersection zero), then the genus of a fiber can extend to $g \geq 4k^3 - 2k^2 + 1$ using "section sum operations". However, the author does not know whether it admits a 0-section or not. The motivation for the next result comes from these observations.

THEOREM A. *For any integer n, there is a Σ_g-bundle over Σ_2 with $\sigma = 4n$ if $g \geq 39|n|$. In particular, it admits a 0-section.*

Meyer [**36**, **37**] also proved that for every $g \geq 3$ and n, there is a Σ_g-bundle over Σ_h with $\sigma = 4n$ for some h. Motivated by this result, Problem 1.1 below, which is a refined version of Problem 2.18 (A) in [**25**], was posed by Endo [**14**]. Solving Problem 1.1 is equivalent to computing the minimal genus of the surfaces representing the n times generator of $H_2(\mathcal{M}_g; \mathbb{Z})/\mathrm{Tor}$ for fixed $g \geq 3$ and n (see [**32**]).

PROBLEM 1.1 Endo [**14**]. *Let $h_g(n)$ be the minimal h such that there exists a Σ_g-bundle over Σ_h with $\sigma = 4n$. Determine the value $h_g(n)$.*

Upper bounds on $h_g(n)$ were given in [**14**] after the initial work in [**42**, **16**, **34**]. A sharper bound given by Lee [**34**] is $h_g(n) \leq 5|n| + 1$ for $g \geq 6$. As a corollary of Theorem A, we can compute $h_g(n)$ for the special case and give it's upper bound for $g \geq 39$ by pulling back the bundle to unramified coverings of Σ_2 of degree $|n|$.

COROLLARY 1.2. *For any n, $h_g(n) = 2$ if $g \geq 39|n|$, and $h_g(n) \leq |n| + 1$ if $g \geq 39$.*

Kotschick [**32**] first gave the lower bound on $h_g(n)$. The best known bound was obtained by Hamenstadt [**22**]: $3|n|/(g - 1) + 1 \leq h_g(n)$. Since the upper bound with the same shape as the above lower bound, in which g appears in the denominator, was given in [**16**], we next turn to study the asymptotic behavior of $h_g(n)$. This is natural since the base genus and the signature grow linearly in a sequence of bundles by pulling back by covers of the base of a given bundle. We consider the following problem posed by Mess (see Problem 2.18 (B) in [**25**]).

PROBLEM 1.3 Mess [**25**]. *Let* $H_g := \lim_{n\to\infty} \frac{h_g(n)}{n}$. *Determine the limit* H_g.

The limit exists and is finite and interpreted as the Gromov-Thurston norm of the generator of $H_2(\mathcal{M}_g;\mathbb{Z})/\mathrm{Tor}$ (see [**32**]). The lower bound $3/(g-1) \leq H_g$ is immediately obtained from the result of [**22**]. For any $g \geq 3$, an upper bound on H_g was first given in [**16**]. This bound was improved as follows: $H_g \leq 6/(g-2)$ for even g, $H_g \leq 9/(g-2)$ for $g = 3k \geq 6$ and $H_g \leq 14/(g-1)$ for odd g (see [**8**, **9**, **34**]). Since there is a gap between the even and odd g cases, we fill it.

THEOREM B. *If g is odd, then, for any integer n, there is an Σ_g-bundle over $\Sigma_{6|n|+5}$ with $\sigma = 4(g-1)n$. Therefore, $H_g \leq 6/(g-1)$ for odd g.*

We next focus on surface bundles over surfaces with non-zero signatures and small Euler characteristics. The Euler characteristic of a Σ_g-bundle over Σ_h is $4(g-1)(h-1)$. The smallest known example is that of [**34**] ($g = 3$, $h = 8$ and $\sigma = 4$). We slightly improve it.

THEOREM C. *There exists a Σ_3-bundle over Σ_7 with $\sigma = 4$ and a 0-section.*

Finally, we give non-holomorphic examples with non-zero signature. Thurston [**43**] showed that the total space of a Σ_g-bundle over Σ_h is symplectic for $g \geq 2$. Then, the following question arises: *For which pairs of g and h does there exist a Σ_g-bundle over Σ_h with $\sigma \neq 0$, whose total space does not admit a complex structure?* If a holomorphic surface bundle is isotrivial, then $\sigma = 0$ (see [**8**]), and there are simple examples with $\sigma = 0$ that is non-isotrivial and whose total space can not be complex (see [**4**]). From this, we require the assumption that $\sigma \neq 0$. Baykur [**4**] showed that for any positive integer N and for any $h \geq 3$, there exists $g > N$ such that there are infinite families of (pairwise non-homotopic) 4-manifolds with $\sigma \neq 0$ admitting a Σ_g-bundle over Σ_h and not admitting any complex structure with either orientation (The same holds for any $g \geq 4$ if $h \geq 9$). Using Theorem 4 (2) of [**4**] and Theorem A, we see that the same is true for $h = 2$ (i.e. the smallest h satisfying $\sigma \neq 0$).

COROLLARY 1.4. *For any integer n and for any $g \geq 39|n| + 1$, there is an infinite family of (pairwise non-homotopic) 4-manifolds with $\sigma = 4n$ admitting a Σ_g-bundle over Σ_2 and not admitting any complex structure with either orientation.*

1.3. *Stable commutator lengths of Dehn twists*

Since the monodromy factorization of a Σ_g-bundle over Σ_h is a factorization of the identity as a product of h commutators in $\mathcal{M}_g$, techniques constructing commutators and reducing the number of them are required to prove Theorem A, B and C. We apply the techniques of (stable) commutator lengths on $\mathcal{M}_g$ to obtain the results on surface bundles. Especially, Theorem D (1) below will be used to show Theorem B.

Let $[G,G]$ be the commutator subgroup of a group G. For $x \in [G,G]$, the *commutator length* $\mathrm{cl}_G(x)$ of x is defined to be the smallest number of commutators whose product is equal to x. The *stable commutator length* $\mathrm{scl}_G(x)$ of x is the limit

$$\mathrm{scl}_G(x) = \lim_{n\to\infty} \frac{\mathrm{cl}_G(x^n)}{n}.$$

Note that the limit exists. We define $\mathrm{cl}_G(x) := \infty$ if $x \notin [G,G]$, $\mathrm{scl}_G(x) := \mathrm{scl}_G(x^k)/|k|$ if $x \notin [G,G]$ but $x^k \in [G,G]$ for some k and $\mathrm{scl}_G(x) := \infty$ if $x^k \notin [G,G]$ for any k. From the results of [**6**] and [**41**], $\mathrm{scl}_{\mathcal{M}_g}(x) < \infty$ for any $x \in \mathcal{M}_g$ and any $g \geq 1$. Since Dehn twists are the most fundamental generators of $\mathcal{M}_g$, computing $\mathrm{cl}_{\mathcal{M}_g}(t_c)$ and $\mathrm{scl}_{\mathcal{M}_g}(t_c)$ is the natural problem. Korkmaz and Ozbagci [**29**] showed that $\mathrm{cl}_{\mathcal{M}_g}(t_c) = 2$ for any non-trivial (separating or nonseparating) Dehn twist t_c if $g \geq 3$. Therefore, our next problem is to calculate $\mathrm{cl}_{\mathcal{M}_g}(t_c^n)$ for any n and $\mathrm{scl}_{\mathcal{M}_g}(t_c)$. However, since it is difficult to compute cl_G and scl_G in general, it makes sense to give estimates on $\mathrm{cl}_{\mathcal{M}_g}(t_c^n)$ and $\mathrm{scl}_{\mathcal{M}_g}(t_c)$.

A lower bound on $\mathrm{scl}_{\mathcal{M}_g}(t_c)$ was given by Endo-Kotschick [**17**]. Consequently, $\mathcal{M}_g$ is not uniformly perfect, and the natural homomorphism from the second bounded cohomology of $\mathcal{M}_g$ to its ordinary cohomology is not injective, which were conjectured by Morita [**40**]. For technical reasons, they showed that $|n|/(18g-6)+1 \leq \mathrm{cl}_{\mathcal{M}_g}(t_c^n)$ for any n if c is a separating curve. This gives $1/(18g-6) \leq \mathrm{scl}_{\mathcal{M}_g}(t_c)$ for a separating curve c. This assumption that c is separating was removed by Korkmaz [**27**], and the above results were extended to positive multi twists in [**7**]. In [**27**], an upper bound on $\mathrm{scl}_{\mathcal{M}_g}(t_c)$ was also given. He showed that $\mathrm{scl}_{\mathcal{M}_g}(t_c) < 2/30$ for a nonseparating curve c if $g \geq 2$. On the other hand, there is an estimate $\mathrm{scl}_{\mathcal{M}_g}(t_c) = O(1/g)$ for any simple closed curve c, so $\lim_{g\to\infty} \mathrm{scl}_{\mathcal{M}_g}(t_c) = 0$ (see [**33**] and also [**10**]). Explicit upper bounds that realize such an estimate were given in [**11**] if c is nonseparating, and in [**39**] if c is separating. However, they do not give an explicit factorization of t_c^n as a product of commutators realizing $\lim_{g\to\infty} \mathrm{scl}_{\mathcal{M}_g}(t_c) = 0$ explicitly.

The purpose is to give sharper upper bounds for stable commutator lengths of Dehn twists giving explicit factorizations of powers of Dehn twists as products of commutators. We call a simple closed curve s on Σ_g the *separating curve of type h* if s separates Σ_g into two components with genera h and $g-h$ for $h = 1, 2, \ldots, [\frac{g}{2}]$. To state our results, let s_0 be a nonseparating curve on Σ_g and let s_h a separating curve of type h on Σ_g. Our main results are following.

THEOREM D. *Let $g \geq 2$ and $h \geq 2$. For any integer n, we have the following.*
(1) $\mathrm{cl}_{\mathcal{M}_g}(t_{s_0}^{10(g-1)n}) \leq |n|+3$, *and therefore* $\mathrm{scl}_{\mathcal{M}_g}(t_{s_0}) \leq 1/(10g-10)$,
(2) $\mathrm{cl}_{\mathcal{M}_g}(t_{s_1}^{5(g-1)n}) \leq [7|n|/2]+5$, *and therefore* $\mathrm{scl}_{\mathcal{M}_g}(t_{s_1}) \leq 7/(10g-10)$,
(3) $\mathrm{cl}_{\mathcal{M}_g}(t_{s_h}^{[g/h]n}) \leq [(|n|+3)/2]$, *and therefore* $\mathrm{scl}_{\mathcal{M}_g}(t_{s_h}) \leq 1/(2[g/h])$.
In particular, there are factorizations of powers of Dehn twists as products of commutators realizing the above upper bounds for the commutator lengths.

Sharper upper and lower bounds were given in [**38**, **11**, **39**] if $g = 2$.

Let $\mathcal{H}_g$ be the hyperelliptic mapping class group of Σ_g, that is the subgroup of $\mathcal{M}_g$ consisting of all mapping classes that commute with isotopy class of some fixed hyperelliptic involution. Since $\mathcal{M}_g = \mathcal{H}_g$ if $g = 1, 2$, we have $\mathrm{scl}_{\mathcal{M}_g} \equiv \mathrm{scl}_{\mathcal{H}_g}$. In general, for a subgroup H of a group G, we have $\mathrm{scl}_G(x) \leq \mathrm{scl}_H(x)$. By $1/(8g+4) \leq \mathrm{scl}_{\mathcal{H}_g}(t_{s_0})$ (see [**38**]) and Theorem D (1), we obtain the following corollary.

COROLLARY 1.5. *If $g \geq 8$, then $\mathrm{scl}_{\mathcal{M}_g}(t_{s_0}) < \mathrm{scl}_{\mathcal{H}_g}(t_{s_0})$.*

From [**41**], we have $\mathrm{cl}_{\mathcal{M}_g}(t_c^n) < \infty$ for any n if $g \geq 3$. In contrast, $\mathrm{cl}_{\mathcal{M}_2}(t_{s_0}^n) < \infty$ if and only if $n \equiv 0 \pmod{10}$, $\mathrm{cl}_{\mathcal{M}_2}(t_{s_1}^n) < \infty$ if and only if $n \equiv 0 \pmod 5$, and $\mathrm{cl}_{\mathcal{M}_1}(t_{s_0}^n) < \infty$ if and only if $n \equiv 0 \pmod{12}$. Even though $\mathrm{scl}_{\mathcal{M}_1}(t_{s_0}) = 1/12$ (see Remark 4.5 in [**11**]), to my knowledge, $\mathrm{cl}_{\mathcal{M}_1}(t_{s_0}^{12})$ is still unknown. We determine $\mathrm{cl}_{\mathcal{M}_1}(t_{s_0}^{12n})$. It was shown in [**29**] (resp. [**31**]) that $t_{s_0}^{10}$ (resp. $t_{s_1}^5$) in $\mathcal{M}_2$ is written as products of 2 commutators (resp. 6 commutators). Hence, $\mathrm{cl}_{\mathcal{M}_2}(t_{s_0}^{10}) \leq 2$ and $\mathrm{cl}_{\mathcal{M}_2}(t_{s_1}^5) \leq 6$. We generalize the results to $10n$ and $5n$ and improve the result of [**31**] slightly.

Theorem E. *For any integer n, we have the following.*
(1) $\mathrm{cl}_{\mathcal{M}_1}(t_{s_0}^{12n}) = |n| + 1$,
(2) $\mathrm{cl}_{\mathcal{M}_2}(t_{s_0}^{10n}) \leq |n| + 1$,
(3) $\mathrm{cl}_{\mathcal{M}_2}(t_{s_1}^{5n}) \leq [7|n|/2] + 2$.
In particular, there are factorizations of powers of Dehn twists as products of commutators realizing the above upper bounds.

1.4. *Outline*

The outline of the paper is as follows. In Section 2, we introduce some relators in $\mathcal{M}_g^r$ and a signature formula for achiral Lefschetz fibrations given by Endo-Hasegawa-Kamada-Tanaka [**15**]. They will be used to compute the signatures of surface bundles over surfaces. Section 3 exhibits techniques to write certain words as products of commutators and to reduce the number of commutators. In Section 4–8, we give the proofs of the main results. Throughout the paper, we only give proofs for $n \geq 0$ since the case of $n < 0$ is immediately follows from the case of $n \geq 0$.

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2. *Relators in mapping class groups and a signature formula*

In this section, we present the signature formula for achiral Lefschetz fibrations given in [**15**]. When we consider an achiral Lefschetz fibration, we obtain its global monodromy in the mapping class group of the fiber. The result in [**15**] says that we can compute the signature of the total space of the fibration by "counting the numbers of certain relators" included in the global monodromy.

The outline of this section is as follows. We give a brief summary of the global monodromy of an achiral Lefschetz fibration in Subsection 2.1. In Subsection 2.2, we describe four fundamental relators and the infinite presentation of $\mathcal{M}_g$ given by Luo [**35**]. In Subsection 2.3, we review the result of [**15**].

2.1. *The global monodromy of an achiral Lefschetz fibration*

We briefly describe the global monodromy and the section of an achiral Lefschetz fibration.

Let $g \geq 2$. Roughly speaking, a genus-g *achiral Lefschetz fibration* $\pi : X \to \Sigma_h$ is a smooth fibration of a 4-manifold X over Σ_h with regular fiber Σ_g and finitely many singular fibers.

The singular fibers are classified two types: *of type* $+1$, and *of type* -1. Each singular fiber is obtained by collapsing a simple closed curve v on Σ_g, called the vanishing cycle. Note that if π has no singular fibers, then it is an Σ_g-bundle over Σ_h. When we give a genus-g achiral Lefschetz fibration $X \to \Sigma_h$ with n singular fibers of type $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ whose vanishing cycles are $v_1, v_2, \ldots, v_n$, where $\epsilon_i = \pm 1$, we obtain the following relator (up to cyclic permutations), called the *global monodromy* of π, in $\mathcal{M}_g$:

$$t_{v_1}^{\epsilon_1} t_{v_2}^{\epsilon_2} \cdots t_{v_n}^{\epsilon_n} [\mathcal{X}_1, \mathcal{Y}_1][\mathcal{X}_2, \mathcal{Y}_2] \cdots [\mathcal{X}_h, \mathcal{Y}_h] = \mathrm{id}, \tag{2.1}$$

where $\mathcal{X}_1, \mathcal{Y}_1$ are some words in $\mathcal{M}_g$. Conversely, if we give a relator of the above form, then we get a genus-g achiral Lefschetz fibration $X \to \Sigma_h$ with n singular fibers of type $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ whose vanishing cycles are $v_1, v_2, \ldots, v_n$.

A genus-g achiral Lefschetz fibration $\pi : X \to \Sigma_h$ with the global monodromy (2.1) admits a $(-k)$-section (that is, $s : \Sigma_h \to X$ such that $\pi \circ s = \mathrm{id}_{\Sigma_h}$ and $[s(\Sigma_h)]^2 = -k$) if and only if there exists a lift of (2.1) from $\mathcal{M}_g$ to $\mathcal{M}_g^1$ in the form

$$t_{\widetilde{v}_1}^{\epsilon_1} t_{\widetilde{v}_2}^{\epsilon_2} \cdots t_{\widetilde{v}_n}^{\epsilon_n} [\widetilde{\mathcal{X}}_1, \widetilde{\mathcal{Y}}_1][\widetilde{\mathcal{X}}_2, \widetilde{\mathcal{Y}}_2] \cdots [\widetilde{\mathcal{X}}_h, \widetilde{\mathcal{Y}}_h] = t_\partial^k,$$

where ∂ is the boundary curve on Σ_g^1, $t_{\widetilde{v}_i}$ is a Dehn twist mapped to t_{v_i} under the map $\mathcal{M}_g^1 \to \mathcal{M}_g$ induced by the inclusion $\Sigma_g^1 \to \Sigma_g$, and similarly, $\widetilde{\mathcal{X}}_j$ and $\widetilde{\mathcal{Y}}_j$ are mapped to $\mathcal{X}_j$ and $\mathcal{Y}_j$, respectively.

By the result of [**15**], the signature of X is determined by "the numbers of certain relators" of $\mathcal{M}_g$ included in (2.1). In the next subsection, we introduce the relators.

2.2. *Infinite presentations of mapping class groups*

In [**15**], the authors employ an infinite presentation of $\mathcal{M}_g^r$ given by Luo [**35**] building on earlier work of Gervais [**21**]. To state it, we introduce four fundamental relators in $\mathcal{M}_g^r$.

DEFINITION 2.1. Let a, b be simple closed curve on Σ_g^r.

- If a is homotopically trivial, then $t_a = \mathrm{id}$, so we call it the *trivial relator* and write
$$T := t_a.$$
- Let $c = t_b(a)$. Then, we have the relation $t_c = t_b t_a t_b^{-1}$, called the *primitive braid relation*. Therefore, we obtain the *primitive braid relator*
$$P := t_c^{-1} t_b t_a t_b^{-1}.$$
- Let a, b be simple closed curves on the subsurface Σ_1^1 bounded by d with $i(a, b) = 1$ as in Figure 1. Then, the *2-chain relation* $t_d = (t_a t_b)^6$ holds in $\mathcal{M}_1^1 \subset \mathcal{M}_g^r$. This gives the *2-chain relator*
$$C_2 := t_d^{-1} (t_a t_b)^6.$$
- Let x, y, z be the interior curves on a subsurface Σ_0^4 in Σ_g^r as in Figure 2, and let a, b, c, d be the boundary curves on Σ_0^4 as in the figure. Then, the *lantern relation* $t_a t_b t_c t_d = t_x t_y t_z$ holds in $\mathcal{M}_0^4 \subset \mathcal{M}_g^r$. Then, we have the *lantern relator*
$$L := t_d^{-1} t_c^{-1} t_b^{-1} t_a^{-1} t_x t_y t_z.$$

Luo [**35**] gave the following infinite presentation of the mapping class group $\mathcal{M}_g^r$.

THEOREM 2.2 ([**35**]). *$\mathcal{M}_g^r$ has an infinite presentation whose generators are the set of all Dehn twists and whose relators are T, P, C_2 and L.*

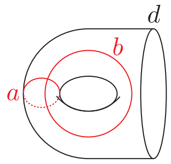

FIGURE 1. *The curves a, b, d on Σ_1^1.*

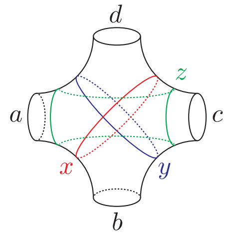

FIGURE 2. *The curves a, b, c, d, x, y, z on Σ_0^4.*

In the rest of this subsection, we present variations of the primitive braid relator P. They are used throughout this paper. Before it, we give the following lemma.

LEMMA 2.3. *Let f be a word in $\mathcal{M}_g^r$. For a simple closed curve a on Σ_g^r, $t_{f(a)}^{-1} f t_a f^{-1}$ is a product of conjugates of primitive braid relators and their inverses.*

Proof. Let $f = t_{b_k}^{\epsilon_k} \cdots t_{b_2}^{\epsilon_2} t_{b_1}^{\epsilon_1}$, where $\epsilon_i = \pm 1$ and each b_i is a simple closed curve on Σ_g^r. For simplicity, we set $c_0 = a$, $c_i = t_{b_i}^{\epsilon_i}(c_{i-1})$ for $i = 1, 2, \ldots, k$, so $c_k = f(a)$. Then, $P_{i,\epsilon_i} = t_{c_i}^{-1} t_{b_i}^{\epsilon_i} t_{c_{i-1}} t_{b_i}^{-\epsilon_i}$ is a primitive braid relator if $\epsilon_i = 1$, and P_{i,ϵ_i} is the conjugation of the inverse of the primitive relator $t_{c_{i-1}}^{-1} t_{b_i} t_{c_i} t_{b_i}^{-1}$ by $t_{b_i}^{-1}$ if $\epsilon_i = -1$ since $t_{b_i}(c_i) = c_{i-1}$ from $c_i = t_{b_i}^{-1}(c_{i-1})$. Here, let us consider the following conjugation Q_{i,ϵ_i} of P_{i,ϵ_i}:

$$\begin{aligned} Q_{k,\epsilon_k} &= P_{k,\epsilon_k}, \\ Q_{k-1,\epsilon_{k-1}} &= t_{b_k}^{\epsilon_k} P_{k-1,\epsilon_{k-1}} t_{b_k}^{-\epsilon_k}, \end{aligned}$$

and in general

$$Q_{i,\epsilon_i} = t_{b_k}^{\epsilon_k} \cdots t_{b_{i+1}}^{\epsilon_{i+1}} P_{i,\epsilon_i} t_{b_{i+1}}^{-\epsilon_{i+1}} \cdots t_{b_k}^{-\epsilon_k}.$$

Then, we have

$$\begin{aligned} Q_{k,\epsilon_k} Q_{k-1,\epsilon_{k-1}} \cdots Q_{1,\epsilon_1} &= P_{k,\epsilon_k} t_{b_k}^{\epsilon_k} \cdot P_{k-1,\epsilon_{k-1}} t_{b_{k-1}}^{\epsilon_{k-1}} \cdots P_{2,\epsilon_2} t_{b_2}^{\epsilon_2} \cdot P_{1,\epsilon_1} \cdot t_{b_2}^{-\epsilon_2} t_{b_3}^{-\epsilon_3} \cdots t_{b_k}^{-\epsilon_k} \\ &= t_{c_k}^{-1} t_{b_k}^{\epsilon_k} t_{c_{k-1}} \cdot t_{c_{k-1}}^{-1} t_{b_{k-1}}^{\epsilon_{k-1}} t_{c_{k-2}} \cdots t_{c_2}^{-1} t_{b_2}^{\epsilon_2} t_{c_1} \cdot t_{c_1}^{-1} t_{b_1}^{\epsilon_1} t_{c_0} t_{b_1}^{-\epsilon_1} \cdot t_{b_2}^{-\epsilon_2} t_{b_3}^{-\epsilon_3} \cdots t_{b_k}^{-\epsilon_k} \\ &= t_{c_k}^{-1} \cdot t_{b_k}^{\epsilon_k} t_{b_{k-1}}^{\epsilon_{k-1}} \cdots t_{b_2}^{\epsilon_2} t_{b_1}^{\epsilon_1} \cdot t_{c_0} \cdot t_{b_1}^{-\epsilon_1} t_{b_2}^{-\epsilon_2} \cdots t_{b_k}^{-\epsilon_k} \\ &= t_{f(a)}^{-1} f t_a f^{-1}. \end{aligned}$$

This finishes the proof. □

From Lemma 2.3, we can regard the word $t_{f(a)}^{-1} f t_a f^{-1}$ as a primitive relator, so we use the same letter P for $t_{f(a)}^{-1} f t_a f^{-1}$, and we call the relation $f t_a f^{-1} = t_{f(a)}$ the *primitive braid relation* again. Moreover, the two well-known relations, called the commutative and the braid relations, are also the primitive braid relations.

DEFINITION 2.4. Let a, b be two simple closed curves on Σ_g^r.

- Let f be a word in $\mathcal{M}_g^r$. Then, we have the *primitive braid relation* $f t_a f^{-1} = t_{f(a)}$ and the *primitive relator*

$$P := t_{f(a)}^{-1} f t_a f^{-1}.$$

– If $i(a,b)=0$, then $t_b(a)=a$. Therefore, we have the *commutative relation* $t_a t_b = t_b t_a$ in $\mathcal{M}_g^r$ and the *commutative relator*

$$P := t_a^{-1} t_b t_a t_b^{-1},$$

– If $i(a,b)=1$, then $t_a t_b(a)=b$. Then, the *braid relation* $t_a t_b t_a = t_b t_a t_b$ holds in $\mathcal{M}_g^r$. This gives the *braid relator*

$$P := t_b^{-1} t_a t_b t_a t_b^{-1} t_a^{-1}.$$

2.3. *A signature formula*

We now present the work of [**15**]. This was essentially derived in the earlier work of Endo and Nagami [**18**], which gives a signature formula for Lefschetz fibrations over S^2. Since (2.1) is normally generated by T, P, C_2, L from Theorem 2.2, we can count the number of these four relators included in (2.1). This fact is the key to state the result in [**15**].

THEOREM 2.5 ([**15**], Proposition 5.1). *Let $n^{\pm}(R)$ be the number of a relator $R^{\pm 1}$ included in the global monodromy of a genus-g achiral Lefschetz fibration $\pi : X \to \Sigma_h$, where $R = T, P, C_2, L$. We set $n(R) = n^{+}(R) - n^{-}(R)$. Then, we have*

$$\sigma(X) = -n(T) - 7n(C_2) + n(L).$$

REMARK 2.6. *Originally, Proposition 5.1 in [**15**] is stated in terms of a graphical method, called the "chart" description.*

From Theorem 2.5, we notice that primitive braid relators are not needed for the computation of $\sigma(X)$. Equivalently, if we have an achiral Lefschetz fibration $\pi' : X' \to \Sigma_h$ with the monodromy obtained by applying primitive braid relations to that of $\pi : X \to \Sigma_h$, then $\sigma(X) = \sigma(X')$ holds. For this reason, we introduce the following notation.

DEFINITION 2.7. Let Q be a conjugate of primitive braid relator in $\mathcal{M}_g^r$.

– Let V and V' be words in $\mathcal{M}_g^r$ with $V'V^{-1} = Q^{\epsilon}$, where $\epsilon = \pm 1$. Set

$$W := U_1 V U_2,$$
$$W' := U_1 V' U_2,$$

where U_1 and U_2 are words in $\mathcal{M}_g^r$. Then, we can construct W' from W using Q as follows:

$$(U_1 Q^{\epsilon} U_1^{-1}) W = (U_1 Q^{\epsilon} U_1^{-1}) U_1 V U_2 = U_1 V' U_2 = W'.$$

When W' is obtained from W by applying a sequence of the above operations (i.e. by using the primitive braid relations), we denote it by

$$W \equiv_P W'.$$

– We say that W *commutes with* W' *modulo* P if the next relation holds:

$$W \cdot W' \equiv_P W' \cdot W.$$

– Let $W_1, W_2, \ldots, W_n$ be words in $\mathcal{M}_g^r$. If the relation

$$W_1 W_2 \cdots W_{n-1} W_n \equiv_P W_n W_1 W_2 \cdots W_{n-1}$$

holds, then we call it a *cyclic permutation*.

Remark 2.8 below collects fundamental properties of the equivalence relation $\equiv_P$. We will use it (without specifying) repeatedly.

Remark 2.8. *Let f, X_1, X_2 be words in $\mathcal{M}_g^r$, and let $a, a_1, a_2, \ldots, a_k$ be simple closed curves on Σ_g^r. We follow the notation of Definition 2.7.*

(1) *For a primitive braid relator $Q = t_{f(a)}^{-1} f t_a f^{-1}$, we set $V = t_{f(a)}$, $V' = f t_a f^{-1}$, $U_1 = X_1$, $U_2 = X_2$. Then, we have*
$$X_1 \cdot t_{f(a)} \cdot X_2 \equiv_P X_1 \cdot f t_a f^{-1} \cdot X_2.$$
(2) *For a primitive braid relator $Q = t_{f(a)}^{-1} f t_a f^{-1}$, we set $V = f$, $V' = t_{f(a)}^{-1} f t_a$, $U_1 = X_1 t_{f(a)}$, $U_2 = X_2$. Then, we have*
$$X_1 \cdot t_{f(a)} \cdot f \cdot X_2 \equiv_P X_1 \cdot f \cdot t_a \cdot X_2,$$
In particular, for any f, the Dehn twist along a boundary curve ∂ of Σ_g^r commutes with f modulo P from $f(\partial) = \partial$.

(3) *When dealing with a relator R one can always perform any cyclic permutation for the following reason: we set $R = t_{a_1}^{\epsilon_1} t_{a_2}^{\epsilon_2} \cdots t_{a_k}^{\epsilon_k}$ and $Q_{\epsilon_k} = t_{a_k}^{\epsilon_k} R t_{a_k}^{-\epsilon_k} R^{-1}$, where $\epsilon_i = \pm 1$. Then, Q_{-1} is a primitive braid relator from $R(a_k) = a_k$, and $Q_1 = (t_{a_k} Q_{-1} t_{a_k}^{-1})^{-1}$. Therefore, when we set $Q = Q_{\epsilon_k}$, $V = R$, $V' = t_{a_k}^{-\epsilon_k} R t_{a_k}^{\epsilon_k}$, $U_1 = U_2 = \mathrm{id}$, we have*
$$t_{a_1}^{\epsilon_1} t_{a_2}^{\epsilon_2} \cdots t_{a_{k-1}}^{\epsilon_{k-1}} t_{a_k}^{\epsilon_k} \equiv_P t_{a_k}^{\epsilon_k} \cdot t_{a_1}^{\epsilon_1} t_{a_2}^{\epsilon_2} \cdots t_{a_{k-1}}^{\epsilon_{k-1}}$$
(4) *It is clear that $WW' = W'W$ as elements in $\mathcal{M}_g^r$ if $WW' \equiv_P W'W$. Conversely, we see that $WW' \equiv_P W'W$ if $WW' = W'W$ as follows: we set $W' = t_{a_1}^{\epsilon_1} t_{a_2}^{\epsilon_2} \cdots t_{a_k}^{\epsilon_k}$, where $\epsilon_i = \pm 1$. From $W' = WW'W^{-1}$ and the primitive braid relation, we obtain $W' = t_{a_1}^{\epsilon_1} \cdots t_{a_k}^{\epsilon_k} \equiv_P t_{W(a_1)}^{\epsilon_1} \cdots t_{W(a_k)}^{\epsilon_k}$. This gives $WW' = W t_{a_1}^{\epsilon_1} \cdots t_{a_k}^{\epsilon_k} \equiv_P t_{W(a_1)}^{\epsilon_1} \cdots t_{W(a_k)}^{\epsilon_k} W \equiv_P W'W$.*

3. *Lemmas*

This section exhibits techniques to prove the main results.

From Section 2, we see that we need to write relators as a product of commutators. The next lemma will be useful for constructing commutators. This technique was used for example in [**23**], [**29**] and [**5**].

Lemma 3.1. *Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be simple closed curves on Σ_g^r. If there is a word f in $\mathcal{M}_g^r$ mapping $(a_1, a_2, \ldots, a_n)$ to $(b_1, b_2, \ldots, b_n)$, then for any integers $k_1, k_2, \ldots, k_n$, the following holds:*
$$t_{a_1}^{k_1} t_{a_2}^{k_2} \cdots t_{a_n}^{k_n}\, t_{b_n}^{-k_n} \cdots t_{b_2}^{-k_2} t_{b_1}^{-k_1} \equiv_P [t_{a_1}^{k_1} t_{a_2}^{k_2} \cdots t_{a_n}^{k_n}, f].$$

Proof. By the primitive braid relations and $(f t_{a_i} f^{-1})^{-k_i} = f t_{a_i}^{-k_i} f^{-1}$, we have
$$\begin{aligned} t_{a_1}^{k_1} t_{a_2}^{k_2} \cdots t_{a_n}^{k_n}\, t_{b_n}^{-k_n} \cdots t_{b_2}^{-k_2} t_{b_1}^{-k_1} &= t_{a_1}^{k_1} t_{a_2}^{k_2} \cdots t_{a_n}^{k_n} \cdot t_{f(a_n)}^{-k_n} \cdots t_{f(a_2)}^{-k_2} t_{f(a_1)}^{-k_1} \\ &\equiv_P t_{a_1}^{k_1} t_{a_2}^{k_2} \cdots t_{a_n}^{k_n} f t_{a_n}^{-k_n} \cdots t_{a_2}^{-k_2} t_{a_1}^{-k_1} f^{-1}. \end{aligned}$$
By $t_{a_n}^{-k_n} \cdots t_{a_2}^{-k_2} t_{a_1}^{-k_1} = (t_{a_1}^{k_1} t_{a_2}^{k_2} \cdots t_{a_n}^{k_n})^{-1}$, we obtain the required formula. □

The next three lemmas are used to construct a word f in Lemma 3.1.

Lemma 3.2. *Let a, b, c be nonseparating curves on Σ_g^r such that $i(a, b) = i(b, c) = 1$. Then the following holds.*

(1) $t_b t_c t_a t_b$ *maps* a *to* c. *It maps* (a, c) *to* (c, a) *if* $i(a, c) = 0$,
(2) $t_a t_b t_c$ *maps* (a, b) *to* (b, c) *if* $i(a, c) = 0$.

Proof. Since $t_a t_b(a) = b$, $t_b t_c(b) = c$, $t_c t_b(c) = b$ and $t_b t_a(b) = a$, and $t_a(c) = c$, $t_c(a) = a$ and $t_a t_c = t_c t_a$ if $i(a, c) = 0$ (see Definition 2.4), (1) follows from

$$t_b t_c t_a t_b(a) = t_b t_c(b) = c,$$
$$t_b t_c t_a t_b(c) = t_b t_a t_c t_b(c) = t_b t_a(b) = a,$$

and (2) is obtained as follows:

$$t_a t_b t_c(a) = t_a t_b(a) = b,$$
$$t_a t_b t_c(b) = t_a(c) = c.$$

□

LEMMA 3.3. *Let* $a, b, c, \alpha, \beta, \gamma$ *be nonseparating curves on* Σ_g^r *such that* $i(a, b) = i(b, c) = i(\alpha, \beta) = i(\beta, \gamma) = 1$. *Suppose that* a *is disjoint from* α, β, γ *and that* γ *is disjoint from* a, b, c. *Then,* $t_b t_c t_a t_b \cdot t_\beta t_\gamma t_\alpha t_\beta$ *maps* (a, α) *to* (c, γ). *It maps* (a, c, α, γ) *to* (c, a, γ, α) *if* c *is disjoint from* a, α, β, γ, *and if* α *is disjoint from* γ, a, b, c.

Proof. Since a (resp. γ) is disjoint from α, β, γ (resp. a, b, c), we have

$$t_b t_c t_a t_b \cdot t_\beta t_\gamma t_\alpha t_\beta(a) = t_b t_c t_a t_b(a) = c,$$
$$t_b t_c t_a t_b \cdot t_\beta t_\gamma t_\alpha t_\beta(\alpha) = t_b t_c t_a t_b(\gamma) = \gamma$$

by the farmer part of Lemma 3.2 (1). By a similar argument, the latter part of Lemma 3.3 follows from that of Lemma 3.2 (1). This finishes the proof. □

LEMMA 3.4. *Let* $a, b, c, \alpha, \beta, \gamma$ *be nonseparating curves on* Σ_g^r *such that* $i(a, b) = i(b, c) = i(\alpha, \beta) = i(\beta, \gamma) = 1$. *Suppose that* a, c *are disjoint from* α, β, γ *and that* β *is disjoint from* a, b, c. *Then,* $t_\beta t_\gamma \cdot t_b t_c t_a t_b \cdot t_\alpha t_\beta$ *maps* (a, α) *to* (c, γ).

Proof. Since a, c (resp. β) are disjoint from α, β, γ (resp. a, b, c), by $t_\alpha t_\beta(\alpha) = \beta$, $t_\beta t_\gamma(\beta) = \gamma$ (see Definition 2.4) and the farmer part of Lemma 3.2 (1), we have

$$t_\beta t_\gamma \cdot t_b t_c t_a t_b \cdot t_\alpha t_\beta(a) = t_\beta t_\gamma \cdot t_b t_c t_a t_b(a) = t_\beta t_\gamma(c) = c,$$
$$t_\beta t_\gamma \cdot t_b t_c t_a t_b \cdot t_\alpha t_\beta(\alpha) = t_\beta t_\gamma \cdot t_b t_c t_a t_b(\beta) = t_\beta t_\gamma(\beta) = \gamma,$$

and this finishes the proof. □

The key lemma of this paper is following.

LEMMA 3.5. *Let* $a_1, a_2, \ldots, a_{m+1}$ *be disjoint simple closed curves on* Σ_g^r. *If there is a word* f *in* $\mathcal{M}_g^r$ *such that* $f(a_i) = a_{i+1}$ *for* $i = 1, 2, \ldots, m$, *then we have the following relations in* $\mathcal{M}_g^r$ *for any integers* $k_1, k_2, \ldots, k_{m+1}$:
(1) $t_{a_1}^{k_1} t_{a_2}^{k_2} \cdots t_{a_{m+1}}^{k_{m+1}} \equiv_P [t_{a_1}^{k_1} t_{a_2}^{k_1+k_2} \cdots t_{a_m}^{k_1+k_2+\cdots+k_m}, f] \cdot t_{a_{m+1}}^{k_1+k_2+\cdots+k_{m+1}}$,
(2) $t_{a_1}^{k_1} t_{a_2}^{k_2} \cdots t_{a_{m+1}}^{k_{m+1}} \equiv_P t_{a_{m+1}}^{k_1+k_2+\cdots+k_{m+1}} \cdot [t_{a_1}^{k_1} t_{a_2}^{k_1+k_2} \cdots t_{a_m}^{k_1+k_2+\cdots+k_m}, f]$,
(3) $t_{a_1}^{k_1} t_{a_2}^{k_2} \cdots t_{a_{m+1}}^{k_{m+1}} \equiv_P t_{a_{m+1}}^{k_1+k_2+\cdots+k_{m+1}} \cdot [f, t_{a_1}^{-k_1} t_{a_2}^{-k_1-k_2} \cdots t_{a_m}^{-k_1-k_2-\cdots-k_m}]$.

Proof. For abbreviation, set $K_i := k_1 + k_2 + \cdots + k_i$. Then, we have

$$t_{a_1}^{k_1} t_{a_2}^{k_2} \cdots t_{a_{m+1}}^{k_{m+1}} = t_{a_1}^{K_1} t_{a_2}^{-K_1} \cdot t_{a_2}^{K_2} t_{a_3}^{-K_2} \cdot t_{a_3}^{K_3} t_{a_4}^{-K_3} \cdots t_{a_m}^{K_m} t_{a_{m+1}}^{-K_m} \cdot t_{a_{m+1}}^{K_{m+1}}.$$

This relation and the commutative relations give the following three relations:

$$\begin{aligned} t_{a_1}^{k_1} t_{a_2}^{k_2} \cdots t_{a_{m+1}}^{k_{m+1}} &\equiv_P t_{a_1}^{K_1} t_{a_2}^{K_2} \cdots t_{a_m}^{K_m} \cdot t_{a_{m+1}}^{-K_m} \cdots t_{a_3}^{-K_2} t_{a_2}^{-K_1} \cdot t_{a_{m+1}}^{K_{m+1}}, \\ t_{a_1}^{k_1} t_{a_2}^{k_2} \cdots t_{a_{m+1}}^{k_{m+1}} &\equiv_P t_{a_{m+1}}^{K_{m+1}} \cdot t_{a_1}^{K_1} t_{a_2}^{K_2} \cdots t_{a_m}^{K_m} \cdot t_{a_{m+1}}^{-K_m} \cdots t_{a_3}^{-K_2} t_{a_2}^{-K_1}, \\ t_{a_1}^{k_1} t_{a_2}^{k_2} \cdots t_{a_{m+1}}^{k_{m+1}} &\equiv_P t_{a_{m+1}}^{K_{m+1}} \cdot t_{a_2}^{-K_1} t_{a_3}^{-K_2} \cdots t_{a_{m+1}}^{-K_m} \cdot t_{a_m}^{K_m} \cdots t_{a_2}^{K_2} t_{a_1}^{K_1}. \end{aligned}$$

Here, by the primitive braid relation $t_{a_{i+1}} \equiv_P f t_{a_i} f^{-1}$ and $(f t_{a_i} f^{-1})^{-K_i} = f t_{a_i}^{-K_i} f^{-1}$ for $i = 1, 2, \ldots, m$, we obtain

$$\begin{aligned} t_{a_1}^{K_1} t_{a_2}^{K_2} \cdots t_{a_m}^{K_m} \cdot t_{a_{m+1}}^{-K_m} \cdots t_{a_3}^{-K_2} t_{a_2}^{-K_1} &\equiv_P t_{a_1}^{K_1} t_{a_2}^{K_2} \cdots t_{a_m}^{K_m} \cdot f t_{a_m}^{-K_m} \cdots t_{a_2}^{-K_2} t_{a_1}^{-K_1} f^{-1}, \\ t_{a_2}^{-K_1} t_{a_3}^{-K_2} \cdots t_{a_{m+1}}^{-K_m} \cdot t_{a_m}^{K_m} \cdots t_{a_2}^{K_2} t_{a_1}^{K_1} &\equiv_P f t_{a_1}^{-K_1} t_{a_2}^{-K_2} \cdots t_{a_m}^{-K_m} f^{-1} \cdot t_{a_m}^{K_m} \cdots t_{a_2}^{K_2} t_{a_1}^{K_1}. \end{aligned}$$

Hence, the relations (1)–(3) follow from $t_{a_m}^{-K_m} \cdots t_{a_2}^{-K_2} t_{a_1}^{-K_1} = (t_{a_1}^{K_1} t_{a_2}^{K_2} \cdots t_{a_m}^{K_m})^{-1}$ and $t_{a_m}^{K_m} \cdots t_{a_2}^{K_2} t_{a_1}^{K_1} = (t_{a_1}^{-K_1} t_{a_2}^{-K_2} \cdots t_{a_m}^{-K_m})^{-1}$. □

The next four lemmas are used to reduce the number of commutators.

LEMMA 3.6. *For words X_1, X_2, Y_1, Y_2 in $\mathcal{M}_g^r$ with $X_i Y_j \equiv_P Y_j X_i$ $(i, j = 1, 2)$, we have*

$$[X_1, X_2][Y_1, Y_2] \equiv_P [X_1 Y_1, X_2 Y_2].$$

Proof. It follows from

$$X_1 X_2 X_1^{-1} X_2^{-1} \cdot Y_1 Y_2 Y_1^{-1} Y_2^{-1} \equiv_P X_1 Y_1 X_2 Y_2 Y_1^{-1} X_1^{-1} Y_2^{-1} X_2^{-1}.$$

□

LEMMA 3.7. *For any three words X, Y, Z in a group G, we have*

$$[X, Y][Y, Z] = [XZ^{-1}, ZYZ^{-1}].$$

Proof. The equation immediately follows from the following computations:

$$\begin{aligned} [X, Y][Y, Z] &= XYX^{-1}Y^{-1} \cdot YZY^{-1}Z^{-1} = XYX^{-1}ZY^{-1}Z^{-1}, \\ [XZ^{-1}, ZYZ^{-1}] &= (XZ^{-1})(ZYZ^{-1})(ZX^{-1})(ZY^{-1}Z^{-1}) = XYX^{-1}ZY^{-1}Z^{-1}. \end{aligned}$$

□

LEMMA 3.8. *Let X, Y be words in $\mathcal{M}_g^r$. For any integer n, we have*
(1) $(XY)^n = {}_X(Y)_{X^2}(Y) \cdots {}_{X^n}(Y) \cdot X^n$,
(2) $(XY)^n = X^n \cdot {}_{X^{-n+1}}(Y) \cdots {}_{X^{-2}}(Y)_{X^{-1}}(Y) Y$.

Proof. The equations immediately follow from

$$\begin{aligned} (XY)^n &= (XYX^{-1})(X^2YX^{-2}) \cdots (X^nYX^{-n})X^n, \\ (XY)^n &= X^n(X^{-n+1}YX^{n-1}) \cdots (X^{-2}YX^2)(X^{-1}YX)Y. \end{aligned}$$

□

Lemma 3.9. *Let X and f be words in $\mathcal{M}_g^r$ such that X is the product $X = X_1X_2\cdots X_n$ whose factors X_i satisfy that $X_i \cdot X_j \equiv_P X_j \cdot X_i$ for $i \neq j$, $X_{i+1} \cdot f \equiv_P f \cdot X_i$ and $X_1 \cdot f \equiv_P f \cdot X_n$. Then, we have*

$$X \cdot f \equiv_P f \cdot X.$$

Proof. We obtain the claim as follows:

$$X_1X_2\cdots X_{n-1}X_n \cdot f \equiv_P X_2X_3\cdots X_nX_1 \cdot f \equiv_P f \cdot X_1X_2\cdots X_{n-1}X_n.$$

□

4. *Scl of the Dehn twist along a nonseparating curve*

We first give the proof of Theorem D (1) since some of the results that will be obtained in the course of this proof will also be used in the proofs of Theorems A, B and E. Note that since Dehn twists along two nonseparating curves s_0, s_0' (resp. two separating curves s_h, s_h' of type h and a separating curve s_{g-h} of type $g-h$) are conjugate, and a conjugate of a commutator is again a commutator, it suffices to prove Theorem D and E for some nonseparating curve (resp. separating curve of type h).

In order to prove Theorem D (1), we present the 3-chain relator and factorize its n-the power as a product of commutators and Dehn twists. The factorization will be used to show Theorem A, C, D (1) and E (1) and (2).

Definition 4.1. Let a, b, c be simple closed curves on Σ_1^2 bounded by d, d' with $i(a,b) = i(b,c) = 1$ and $i(c,a) = 0$ as in Figure 3. Then, we have the *3-chain relation* $t_{d'}t_d = (t_at_bt_c)^4$ in $\mathcal{M}_1^2$ and the *3-chain relator*

$$C_3 := t_d^{-1}t_{d'}^{-1}(t_at_bt_c)^4.$$

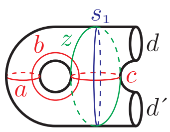

Figure 3. *The curves a, b, c, d, d', s_1, z on Σ_1^2.*

The next proposition is the key result in this section. We will use some equations in the proof to show Theorems A, C, D (1) and E (1) and (2).

Proposition 4.2. *In the notation of Definition 4.1, for any integer n, there are words $V_1, W_1, V_2, W_2, \ldots, V_{|n|+1}, W_{|n|+1}$ in $\mathcal{M}_1^2$ such that the following holds in $\mathcal{M}_1^2$:*

$$C_3^n \equiv_P t_b^{12n}[V_1, W_1][V_2, W_2]\cdots[V_{|n|+1}, W_{|n|+1}] \cdot t_d^{-n}t_{d'}^{-n}.$$

Proof. Let $v = t_at_c(b)$. Since a is disjoint from c, $t_c^{-1}t_a^{-1}(c) = c$ and $t_c^{-1}t_a^{-1}(a) = a$ (see Definition 2.4). By the primitive braid relation and Lemma 3.2 (2), we have

$$t_bt_v(a) = t_bt_at_ct_bt_c^{-1}t_a^{-1}(a) = t_bt_at_ct_b(a) = c,$$
$$t_bt_v(c) = t_bt_at_ct_bt_c^{-1}t_a^{-1}(c) = t_bt_at_ct_b(c) = a.$$

This gives the following two relations:

$$t_b t_v \cdot t_a \equiv_P t_c \cdot t_b t_v, \tag{4.1}$$

$$t_b t_v \cdot t_c \equiv_P t_a \cdot t_b t_v. \tag{4.2}$$

Note that using the primitive braid relation, we have

$$\begin{aligned} t_a t_b t_c t_a t_b t_c &\equiv_P t_a t_b t_a t_c t_b t_c \\ &\equiv_P t_a \cdot t_b (t_a t_c t_b t_c^{-1} t_a^{-1}) \cdot t_a t_c t_c \\ &\equiv_P t_a \cdot t_b t_v \cdot t_a t_c t_c. \end{aligned}$$

This equation, together with the relations (4.1) and (4.2), the commutative relation $t_a t_c = t_c t_a$ and a cyclic permutation, gives

$$C_3 \equiv_P t_a^4 t_c^4 (t_b t_v)^2 t_d^{-1} t_{d'}^{-1}.$$

When we take n-th power of this relation, by the property of boundary curves d, d', the relations (4.1) and (4.2) and the commutative relation $t_a t_c = t_c t_a$, we have

$$C_3^n \equiv_P t_a^{4n} t_c^{4n} (t_b t_v)^{2n} t_d^{-n} t_{d'}^{-n}. \tag{4.3}$$

By this equation and the primitive braid relations, we have

$$\begin{aligned} C_3^n &\equiv_P t_a^{4n} t_c^{4n} (t_b^4 \cdot t_b^{-1} (t_b^{-2} t_v t_b^2) t_b^{-1} t_v)^n t_d^{-n} t_{d'}^{-n} \\ &\equiv_P t_a^{4n} t_c^{4n} (t_b^4 \cdot t_b^{-1} t_{t_b^{-2}(v)} t_b^{-1} t_v)^n t_d^{-n} t_{d'}^{-n}. \end{aligned}$$

Here, when we set $\phi_3 := t_a t_c t_b^3$ in $\mathcal{M}_1^2$, $\phi_3(b) = t_a t_c(b) = v$ and $\phi_3(t_b^{-2}(v)) = t_a t_c t_b(v) = t_a t_c t_b t_a t_c(b)$. From the commutative and the braid relations, we have

$$t_a t_c t_b t_a t_c = t_a t_c t_b t_c t_a = t_a t_b t_c t_b t_a.$$

By Lemma 3.2 (2), we see that

$$t_a t_b t_c t_b t_a(b) = t_a t_b t_c(a) = b,$$

so $\phi_3(t_b^{-2}(v)) = b$. Therefore, ϕ_3 maps $(b, t_b^{-2}(v))$ to (v, b). This gives

$$C_3^n \equiv_P t_a^{4n} t_c^{4n} (t_b^4 \cdot [t_b^{-1} t_{t_b^{-2}(v)}, \phi_3])^n t_d^{-n} t_{d'}^{-n}$$

from Lemma 3.1. Therefore, by Lemma 3.8 (2), we obtain the following relation:

$$C_3^n \equiv_P t_a^{4n} t_c^{4n} t_b^{4n} \cdot \prod_{i=1}^{n} {}_{t_b^{-4(n-i)}}([t_b^{-1} t_{t_b^{-2}(v)}, \phi_3]) \cdot t_d^{-n} t_{d'}^{-n}. \tag{4.4}$$

Note that the conjugation of a commutator is also a commutator, and that we have

$$\begin{aligned} t_a^{4n} t_c^{4n} t_b^{4n} &\equiv_P t_b^{12n} \cdot t_b^{-4n} (t_b^{-8n} t_a^{4n} t_b^{8n}) t_b^{-4n} (t_b^{-4n} t_c^{4n} t_b^{4n}) \\ &\equiv_P t_b^{12n} \cdot t_b^{-4n} t_{t_b^{-8n}(a)}^{4n} t_b^{-4n} t_{t_b^{-4n}(c)}^{4n}. \end{aligned}$$

Since $t_a t_b t_c$ maps (a, b) to (b, c) by Lemma 3.2 (2), we find that $t_b^{-4n} t_a t_b t_c t_b^{8n}$, denoted by ϕ_4, maps $(b, t_b^{-8n}(a))$ to $(t_b^{-4n}(c), b)$, so Lemma 3.1 gives

$$t_a^{4n} t_c^{4n} t_b^{4n} \equiv_P t_b^{12n} \cdot [t_b^{-4n} t_{t_b^{-8n}(a)}^{4n}, \phi_4],$$

and this establishes the formula. □

Theorem D (1) directly follows from Theorem 4.3 below, which will also be used to prove Theorem B, since the left hand side of the equation in it is a relator.

Theorem 4.3. *Let s_0 be a nonseparating curve on Σ_g for $g \geq 2$. Then, there exist 3-chain relators $C_{3,j}$ $(j = 1, \dots, g-1)$ such that for any integer n there are words $\mathcal{V}_1, \mathcal{W}_1, \mathcal{V}_2, \mathcal{W}_2, \dots, \mathcal{V}_{|n|+3}, \mathcal{W}_{|n|+3}$ in $\mathcal{M}_g$ that satisfy*

$$\prod_{j=1}^{g-1} C_{3,j}^n \equiv_P t_{s_0}^{10(g-1)n}[\mathcal{V}_1, \mathcal{W}_1][\mathcal{V}_2, \mathcal{W}_2] \cdots [\mathcal{V}_{|n|+3}, \mathcal{W}_{|n|+3}].$$

Proof. Let us consider the simple closed curves a_1, b_1, c_1 on the genus-1 subsurface S_1^2 of Σ_g bounded by d_1, d_{g-1} as in Figure 4. Then, we obtain the 3-chain relator $C_{3,1} := t_{d_{g-1}}^{-1} t_{d_1}^{-1} (t_{a_1} t_{b_1} t_{c_1})^4$. By Proposition 4.2, the relation

$$C_{3,1}^n \equiv_P t_{b_1}^{12n}[V_{1,1}, W_{1,1}][V_{2,1}, W_{2,1}] \cdots [V_{|n|+1,1}, W_{|n|+1}, 1] t_{d_{g-1}}^{-n} t_{d_1}^{-n},$$

holds in $\mathcal{M}(S_1^2)$ for any integer n, where $V_{i,1}, W_{i,1}$ are some words in $\mathcal{M}(S_1^2)$.

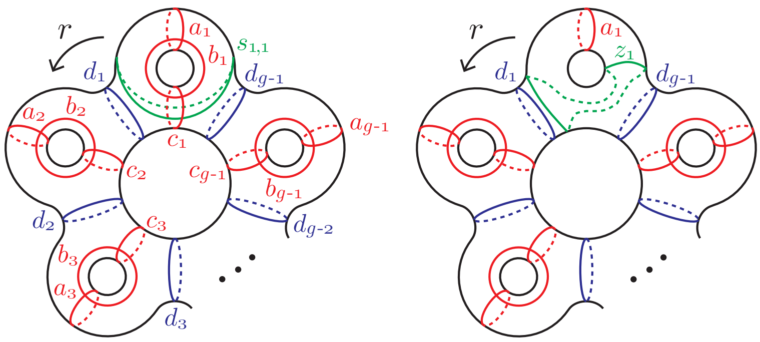

Figure 4. *The rotation r of Σ_g and the curves $a_1, b_1, c_1, d_1, d_{g-1}, s_{1,1}, z_1$.*

Let r be the rotation of Σ_g by $2\pi/(g-1)$ as in Figure 4. We set

$$\begin{aligned} C_{3,j} &:= {}_{r^{j-1}}(C_{3,1}), & & \\ b_j &:= r^{j-1}(b_1), & d_j &:= r^{j-1}(d_1), \\ V_{i,j} &:= {}_{r^{j-1}}(V_{i,1}) & W_{i,j} &:= {}_{r^{j-1}}(W_{i,1}) \end{aligned}$$

for $j = 1, 2, \dots, g-1$. Also set $d_0 = d_{g-1}$. Then, using the primitive braid relations, the relation holds in $\mathcal{M}(r^{j-1}(S_1^2))$:

$$C_{3,j}^n \equiv_P t_{b_j}^{12n}[V_{1,j}, W_{1,j}][V_{2,j}, W_{2,j}] \cdots [V_{|n|+1,j}, W_{|n|+1}, j] t_{d_j}^{-n} t_{d_{j-1}}^{-n}$$

for $j = 1, 2, \dots, g-1$. Here, any simple closed curves on $\mathrm{Int}(r^{j-1}(S_1^2))$ are disjoint from any simple closed curves on $\mathrm{Int}(r^{j'-1}(S_1^2))$ if $j \neq j'$, and d_j, d_{j-1} are boundary curves of $r^{j-1}(S_1^2)$. Hence, for any words e_j in $\mathcal{M}(r^{j-1}(S_1^2))$ and any words $f_{j'}$ in $\mathcal{M}(r^{j'-1}(S_1^2))$, we have $e_j f_{j'} = f_{j'} e_j$ by the commutative relations and the property of boundary curves if $j \neq j'$. From Lemma 3.6 and $d_g = d_1$, we have

$$\begin{aligned} \prod_{j=1}^{g-1} C_{3,j}^n &\equiv_P \prod_{j=1}^{g-1} t_{b_j}^{12n} \cdot \prod_{i=1}^{|n|+1} [\mathcal{V}_i, \mathcal{W}_i] \cdot \prod_{j=1}^{g-1} t_{d_j}^{-2n} \\ &\equiv_P \prod_{j=1}^{g-1} t_{b_j}^{12n} \cdot \prod_{j=1}^{g-1} t_{d_j}^{-2n} \cdot \prod_{i=1}^{|n|+1} [\mathcal{V}_i, \mathcal{W}_i] \end{aligned}$$

where $\mathcal{V}_i = V_{i,1}V_{i,2}\cdots V_{i,g-1}$ and $\mathcal{W}_i = W_{i,1}W_{i,2}\cdots W_{i,g-1}$. Using Lemma 3.5 (2) and (3), we see that

$$\prod_{j=1}^{g-1} t_{b_j}^{12n} \equiv_P t_{b_{g-1}}^{12(g-1)n}[B,r],$$
$$\prod_{j=1}^{g-1} t_{d_j}^{-2n} \equiv_P t_{d_{g-1}}^{-2(g-1)n}[r,D],$$

where $B := t_{b_1}^{12n}t_{b_2}^{24n}\cdots t_{b_{g-2}}^{12(g-2)n}$ and $D := t_{d_1}^{2n}t_{d_2}^{4n}\cdots t_{d_{g-2}}^{2(g-2)n}$. This gives

$$\prod_{j=1}^{g-1} C_{3,j}^n \equiv_P t_{b_{g-1}}^{12(g-1)n}[B,r]\cdot t_{d_{g-1}}^{-2(g-1)n}[r,D]\cdot \prod_{i=1}^{|n|+1}[\mathcal{V}_i,\mathcal{W}_i],$$

Since b_j, a_j are disjoint from d_k for any j and k, we have $B(d_k) = d_k$. This gives $[B,r](d_{g-1}) = BrB^{-1}r^{-1}(d_{g-1}) = d_{g-1}$, so we have $[B,r]t_{d_{g-1}}^{12(g-1)n} \equiv_P t_{d_{g-1}}^{12(g-1)n}[B,r]$. From this and Lemma 3.7, we obtain

$$\prod_{j=1}^{g-1} C_{3,j}^n \equiv_P t_{b_{g-1}}^{12(g-1)n}t_{d_{g-1}}^{-2(g-1)n}\cdot[BD^{-1},DrD^{-1}]\cdot \prod_{i=1}^{|n|+1}[\mathcal{V}_i,\mathcal{W}_i].$$

Since b_{g-1} and d_{g-1} are nonseparating, there exists a diffeomorphism f satisfying $f(b_{g-1}) = d_{g-1}$. Therefore, by Lemma 3.1 we have

$$\begin{aligned} t_{b_{g-1}}^{12(g-1)n}t_{d_{g-1}}^{-2(g-1)n} &= t_{b_{g-1}}^{10(g-1)n}\cdot t_{b_{g-1}}^{2(g-1)n}t_{d_{g-1}}^{-2(g-1)n} \\ &\equiv_P t_{b_{g-1}}^{10(g-1)n}[t_{b_{g-1}}^{2(g-1)n}, f], \end{aligned}$$

and this proves Theorem 4.3 and therefore Theorem D (1). □

Remark 4.4. *M. Korkmaz gave interesting proof of an upper bound on* $\mathrm{scl}_{\mathcal{M}_g}(t_{s_0})$ *in his talk at Max Plank, 2013 (see [**28**]). The main idea is to use his result of [**27**] and quasi-morphisms and to consider* $[\frac{g}{2}]$ *disjoint subsurfaces of* Σ_g *each of which has genus-2 and one boundary component. The proof of Theorem D is much inspired by his idea.*

5. *Surface bundles with base genus two*

In this section, we prove Theorem A.

Throughout this section, we suppose that $g \geq 39$. Let us consider Σ_g^1 with one boundary component ∂ as in Figure 5. Then, we can take 13 disjoint subsurfaces $S_1, S_2, \ldots, S_{12}$ and S of genus 3 with one boundary component and a word Φ in $\mathcal{M}_g^1$ such that $\Phi(S_i) = S_{i+1}$, $\Phi(S_{12}) = S_1$ and $\Phi|_S = \mathrm{id}|_S$ as in Figure 5.

Let $\alpha_1, \beta_1, \gamma_1, \delta_1, \epsilon_1, \zeta_1, x_1, y_1, z_1$ be the simple closed curves on S_1 as Figure 6, and let $a_1, b_1, s_{1,1}, d_1, d_2, \gamma, \delta, y, z, \epsilon, \zeta$ be simple closed curves on S as in the figure. We consider the following two lantern relators L_1 and L:

$$L_1 := t_{\alpha_1}^{-1}t_{\delta_1}^{-1}t_{\gamma_1}^{-1}t_{\beta_1}^{-1}t_{x_1}t_{y_1}t_{z_1},$$
$$L := t_{d_1}t_yt_zt_\delta^{-1}t_\gamma^{-1}t_{d_2}^{-1}t_{s_{1,1}}^{-1}.$$

The next lemma was proved in [**29**].

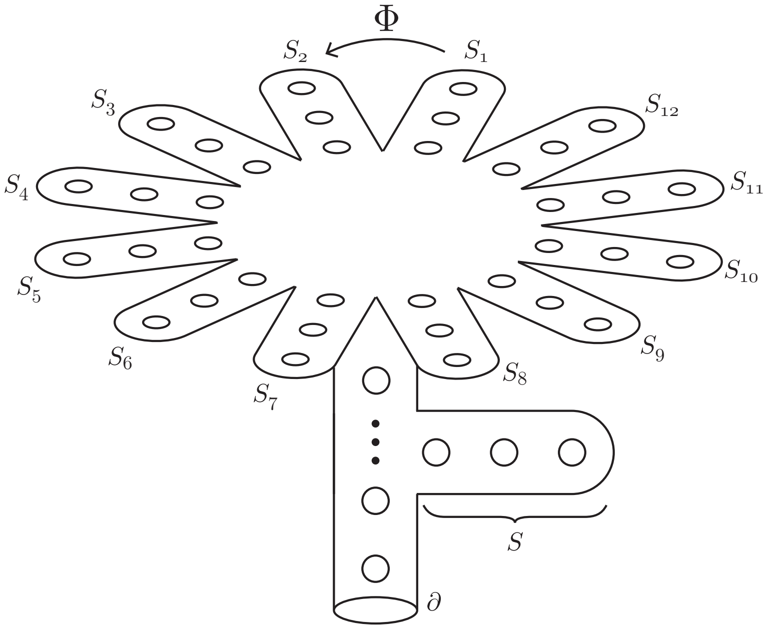

FIGURE 5. *The subsurfaces* $S, S_1, S_2, \dots, S_{12}$ *of* Σ_g^1.

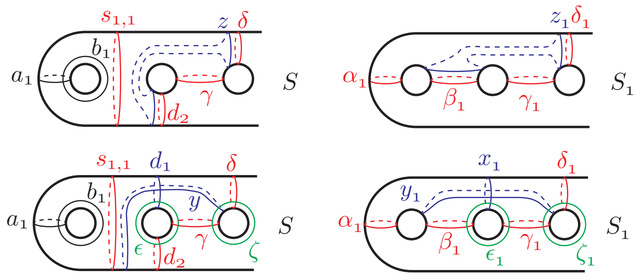

FIGURE 6. *The curves* $a_1, b_1, s_{1,1}, d_1, d_2, \gamma, \delta, \epsilon, \zeta, y, z$ *on* S *and the curves* $\alpha_1, \beta_1, \gamma_1, \delta_1, \epsilon_1, \zeta_1, x_1, y_1, z_1$ *on* S_1.

LEMMA 5.1 ([**29**]). *Set* $\psi_1 = t_{\epsilon_1} t_{\gamma_1} t_{x_1} t_{\epsilon_1}$, $\omega_1 = t_{\epsilon_1} t_{z_1} t_{\beta_1} t_{\epsilon_1} t_{\zeta_1} t_{\delta_1} t_{y_1} t_{\zeta_1}$, $\psi = t_\epsilon t_\gamma t_{d_1} t_\epsilon$ *and* $\phi = t_\zeta t_y t_\delta t_\zeta t_\epsilon t_{d_2} t_z t_\epsilon$. *The followings hold in* $\mathcal{M}(S_1)$ *and* $\mathcal{M}(S)$, *respectively:*

$$L_1 \equiv_P [t_{x_1}, \psi_1] \cdot [t_{y_1} t_{\beta_1}^{-1}, \omega_1] \cdot t_{\alpha_1}^{-1},$$
$$L^{-1} \equiv_P t_{s_{1,1}} \cdot [t_\delta t_z^{-1}, \phi] \cdot [t_\gamma, \psi],$$

Proof. Since $\alpha_1, \beta_1, \gamma_1, \delta_1$ (resp. $\delta, \gamma, d_2, s_{1,1}$) are disjoint from x_1, y_1, z_1 (resp. d_1, y, z) and disjoint from each other, the commutative relations give

$$L_1 \equiv_P t_{x_1} t_{\gamma_1}^{-1} \cdot t_{y_1} t_{\beta_1}^{-1} t_{z_1} t_{\delta_1}^{-1} \cdot t_{\alpha_1}^{-1}$$
$$L^{-1} \equiv_P t_{s_{1,1}} \cdot t_\delta t_z^{-1} t_{d_2} t_y^{-1} \cdot t_\gamma t_{d_1}^{-1}.$$

By Lemma 3.2 (2) and 3.3, ψ_1 maps x_1 to γ_1, ω_1 maps (y_1, β_1) to (δ_1, z_1), ϕ maps (δ, z) to (y, d_2), and ψ maps d_1 to γ. Lemma 3.1 gives the required formulas. □

For a simple closed curve on S_1 appeared in the above, say α_1, we set $\alpha_i := \Phi^{i-1}(\alpha_1)$ which is a simple closed curve on S_i, curves β_i, γ_i, etc. are defined accordingly, and we write the lantern relation $L_i := {}_{\Phi^{i-1}}(L_1)$. From Lemma 5.1 and the primitive braid relation, we obtain

$$\psi_i = t_{\epsilon_i} t_{\gamma_i} t_{x_i} t_{\epsilon_i},$$
$$\omega_i = t_{\epsilon_i} t_{z_i} t_{\beta_i} t_{\epsilon_i} t_{\zeta_i} t_{\delta_i} t_{y_i} t_{\zeta_i},$$
$$L_i \equiv_P [t_{x_i}, \psi_i][t_{y_i} t_{\beta_i}^{-1}, \omega_i] t_{\alpha_i}^{-1}$$

for $i = 1, 2, \ldots, 12$. Moreover, we define the 2-chain relator $C_{2,1}$, which holds in $\mathcal{M}(S)$, to be

$$C_{2,1} := (t_{a_1} t_{b_1})^6 t_{s_{1,1}}^{-1}.$$

The following proposition is the key result to prove Theorem A.

Proposition 5.2. *For $g \geq 39$, there are words $\widetilde{\mathcal{A}}_1, \widetilde{\mathcal{B}}_1, \widetilde{\mathcal{C}}_1, \widetilde{\mathcal{D}}_1$ in $\mathcal{M}_g^1$ such that*

$$L_1 L_2 \cdots L_{12} C_{2,1} L^{-1} \equiv_P [\widetilde{\mathcal{A}}_1, \widetilde{\mathcal{B}}_1][\widetilde{\mathcal{C}}_1, \widetilde{\mathcal{D}}_1],$$

To prove Proposition 5.2, we prepare two lemmas (Lemma 5.3 and 5.4).

Lemma 5.3. *For $g \geq 39$, the following relation holds in $\mathcal{M}_g^1$:*

$$L_1 L_2 \cdots L_{12} \equiv_P t_{\alpha_{12}}^{-12} [X, \Psi][YA, \Omega\Phi],$$

where $X := t_{x_1} t_{x_2} \cdots t_{x_{12}}$, $\Psi := \psi_1 \psi_2 \cdots \psi_{12}$, $Y := t_{y_1} t_{\beta_1}^{-1} t_{y_2} t_{\beta_2}^{-1} \cdots t_{y_{12}} t_{\beta_{12}}^{-1}$, $\Omega := \omega_1 \omega_2 \cdots \omega_{12}$, and $A := t_{\alpha_1}^{-1} t_{\alpha_2}^{-2} \cdots t_{\alpha_{11}}^{-11}$.

Proof. Since S_i is disjoint from $S_{i'}$ for $i \neq i'$, any words in $\mathcal{M}(S_i)$ commute with any words in $\mathcal{M}(S_{i'})$ modulo P from the commutative relations. Therefore, by $L_i \equiv_P [t_{x_i}, \psi_i][t_{y_i} t_{\beta_i}^{-1}, \omega_i] t_{\alpha_i}^{-1} \in \mathcal{M}(S_i)$ and Lemma 3.6, we have

$$L_1 L_2 \cdots L_{12} \equiv_P [X, \Psi][Y, \Omega] t_{\alpha_1}^{-1} t_{\alpha_2}^{-1} \cdots t_{\alpha_{12}}^{-1}.$$

By Lemma 3.5 (1) and the definition of the curve α_i, we obtain

$$L_1 L_2 \cdots L_{12} \equiv_P [X, \Psi][Y, \Omega][A, \Phi] t_{\alpha_{12}}^{-12}.$$

Since α_i is disjoint from $\beta_i, \delta_i, \epsilon_i, \zeta_i, y_i, z_i$ for $i = 1, 2, \ldots, 12$ and S_i is disjoint from $S_{i'}$ for $i \neq i'$, A commutes with Y, Ω modulo P by the commutative relations. Besides, ω_i, Ω (resp. $t_{y_i} t_{\beta_i}^{-1}$, Y) and Φ satisfy the condition of Lemma 3.9 from the commutative and the primitive braid relations, so Φ commutes with Ω (resp. Y) modulo P. Lemma 3.6 and a cyclic permutation give the required formula. □

The next lemma will be also used to prove Theorem C.

Lemma 5.4. *There are words V', W' in $\mathcal{M}(S)$ such that the following relation holds in $\mathcal{M}(S)$:*

$$C_{2,1} L^{-1} \equiv_P [V', W'][t_\gamma, \psi] t_{a_1}^8 t_{b_1}^4.$$

Proof. Let C_3 be the 3-chain relator in Definition 4.1. By the inclusion $\iota : \Sigma_1^2 \to \Sigma_1^1$ obtained by gluing a disk along d', ι maps c on Σ_1^2 to a on Σ_1^1. Then, from the map $\iota_* : \mathcal{M}_1^2 \to \mathcal{M}_1^1$ induced by ι, the trivial relation $t_{d'} = \mathrm{id}$ and the braid relation $t_a t_b t_a = t_b t_a t_b$ give the 2-chain relator C_2 from C_3. From the equation (4.4) in the case of $n = 1$ and ι_*, the equation

$$C_2 \equiv_P t_a^8 t_b^4 [V, W] t_d^{-1}$$

holds in $\mathcal{M}_1^1$, where V, W are some words in $\mathcal{M}_1^1$. Therefore, when we denote by S_1^1 the genus-1 subsurface bounded by $s_{1,1}$ as in Figure 6, Lemma 5.1 gives

$$C_{2,1}L^{-1} \equiv_P t_{a_1}^8 t_{b_1}^4 [V_1, W_1][t_\delta t_z^{-1}, \phi][t_\gamma, \psi],$$

where V_1, W_1 are in $\mathcal{M}(S_1^1)$. Since S_1^1 is disjoint from $\delta, \zeta, \epsilon, y, z, d_2$, and V_1, W_1 are in $\mathcal{M}(S_1^1)$, V_1, W_1 commute with $t_\delta t_z^{-1}, \phi$ modulo P by the commutative relations. This gives

$$C_{2,1}L^{-1} \equiv_P t_{a_1}^8 t_{b_1}^4 [V_1 t_\delta t_z^{-1}, W_1\phi][t_\gamma, \psi]. \tag{5.1}$$

Lemma 3.6 and a cyclic permutation give the required formula. □

We are now ready to prove Proposition 5.2.

Proof of Proposition 5.2. In the notation of Lemma 5.3 and 5.4, each of V' and W' commutes with both of YA and $\Omega\Phi$ modulo P since V' and W' are supported in S_1^1 while YA and $\Omega\Phi$ are supported in the complement of S_1^1. Hence, by Lemma 5.3, 5.4 and 3.6 and a cyclic permutation, we have

$$\begin{aligned} L_1 L_2 \cdots L_{12} \cdot C_{2,1}L^{-1} &\equiv_P t_{\alpha_{12}}^{-12}[X, \Psi][YA, \Omega\Phi] \cdot [V', W'][t_\gamma, \psi] t_{a_1}^8 t_{b_1}^4 \\ &\equiv_P [t_\gamma, \psi] t_{a_1}^8 t_{b_1}^4 t_{\alpha_{12}}^{-12}[X, \Psi][YAV', \Omega\Phi W']. \end{aligned}$$

Note that t_γ and ψ are supported in S while $t_{a_1}, t_{b_1}, t_{\alpha_{12}}, X$ and Ψ are supported in the complement of S. Hence t_γ and ψ commute with $t_{a_1}, t_{b_1}, t_{\alpha_{12}}, X$ and Ψ modulo P. Therefore, with Lemma 3.6 we have

$$\begin{aligned} L_1 L_2 \cdots L_{12} \cdot C_{2,1}L^{-1} &\equiv_P t_{a_1}^8 t_{b_1}^4 t_{\alpha_{12}}^{-12}[t_\gamma, \psi][X, \Psi][YAV', \Omega\Phi W'] \\ &\equiv_P t_{a_1}^8 t_{b_1}^4 t_{\alpha_{12}}^{-12}[t_\gamma X, \psi\Psi][YAV', \Omega\Phi W']. \end{aligned} \tag{5.2}$$

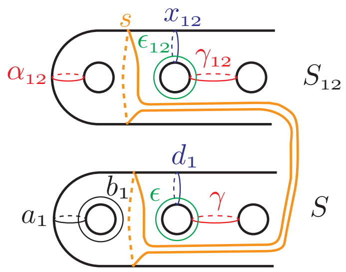

FIGURE 7. *The separating curve s on Σ_g^1.*

We take a separating curve s such that it bounds a genus-2 subsurface S_2^1 of Σ_g^1 that contains simple closed curves a_1, b_1, α_{12} (see Figure 7) and s is disjoint from d_1, γ, ϵ and $x_i, \gamma_i, \epsilon_i$ for any $i = 1, 2, \ldots, 12$. Then, we can consider a half twist H_s along s such that $H_s|_{\Sigma_g^1 - S_2^1} = \mathrm{id}|_{\Sigma_g^1 - S_2^1}$, $H_s(a_1) = \alpha_{12}$ and $H_s(\alpha_{12}) = a_1$. Here we set

$$H := t_{a_1} t_{b_1} H_s.$$

We see that $H|_{\Sigma_g^1 - S_2^1} = \mathrm{id}|_{\Sigma_g^1 - S_2^1}$ and that $H(\alpha_{12}) = b_1$ and $H(a_1) = \alpha_{12}$ since $t_{a_1} t_{b_1}(a_1) = b_1$ and α_{12} is disjoint from a_1, b_1. Therefore, Lemma 3.1 gives

$$\begin{aligned} t_{\alpha_{12}}^{-12} t_{a_1}^8 t_{b_1}^4 &\equiv_P t_{a_1}^8 t_{\alpha_{12}}^{-4} t_{b_1}^4 t_{\alpha_{12}}^{-8} \\ &\equiv_P [t_{a_1}^8 t_{\alpha_{12}}^{-4}, H]. \end{aligned}$$

By this equation and the equation (5.2), we obtain

$$L_1L_2\cdots L_{12}C_{2,1}L^{-1} \equiv_P [t_{a_1}^8 t_{\alpha_{12}}^{-4}, H][t_\gamma X, \psi\Psi][YAV', \Omega\Phi W'].$$

Note that a_1, b_1, α_{12}, s are disjoint from d_1, γ, ϵ and $\gamma_i, \epsilon_i, x_i$ for any $i = 1, 2, \ldots, 12$. Hence, by $H|_{\Sigma_g^1 - S_2^1} = \mathrm{id}|_{\Sigma_g^1 - S_2^1}$, the definitions of X, Ψ, ψ and the commutative relations, we see that $t_{a_1}^8 t_{\alpha_{12}}^{-4}$ and H (supported in S_2^1) commute with $t_\gamma X$ and $\psi\Psi$ (supported outside S_2^1) modulo P. Lemma 3.6 gives

$$L_1L_2\cdots L_{12}C_{2,1}L^{-1} \equiv_P [t_{a_1}^8 t_{\alpha_{12}}^{-4} t_\gamma X, \psi\Psi H][YAV', \Omega\Phi W'],$$

and the proof is complete. □

We show Theorem A.

Proof of Theorem A. Assume that $g \geq 39n$ and $n \geq 1$. Then, we can take n disjoint subsurfaces $S'_1, S'_2, \ldots, S'_n$ of Σ_g^1 of genus 39 with one boundary component and find a diffeomorphism Φ' on Σ_g^1 such that $\Phi'(S'_i) = S'_{i+1}$. Identify the subsurface S'_1 with the entire surface for Proposition 5.2 (with genus 39) and let

$$R_1 := L_1L_2\cdots L_{12}C_{2,1}L^{-1},$$
$$R_{i+1} := {}_{\Phi'}(R_i).$$

Since S'_i is disjoint from S'_j, by the commutative relations, Lemma 3.6 and Proposition 5.2, we have

$$R_1R_2\cdots R_n \equiv_P [\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}][\widetilde{\mathcal{C}}, \widetilde{\mathcal{D}}],$$

where $\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\mathcal{C}}, \widetilde{\mathcal{D}}$ are some words in $\mathcal{M}_g^1$. In particular, we see that this relation also holds in $\mathcal{M}_g$. This gives a Σ_g-bundle $X \to \Sigma_2$ with a 0-section for $g \geq 39n$. From the above argument, in the notation of Proposition 2.5, we have

$$n(T) = n^+(T) - n^-(T) = 0 - 0,$$
$$n(C_2) = n^+(C_2) - n^-(C_2) = n - 0,$$
$$n(L) = n^+(L) - n^-(L) = 12n - n.$$

This gives

$$\sigma(X) = -1 \cdot 0 - 7 \cdot n + 1 \cdot 11n = 4n$$

for $g \geq 39n$, and this finishes the proof. □

6. *Surface bundles with odd fiber genera*

This section shows Theorem B and C. To prove them, we prepare some results (Proposition 6.1 and 6.2 and Lemma 6.3).

Let α_1, β_1, γ_1, x_1, y_1, z_1, x'_1, y'_1, z'_1 be the nonseparating curves on the genus-2 subsurface S_2^2 of Σ_g bounded by δ_1, δ'_1 as in Figure 8. We consider the following two lantern relators:

$$L_1 := t_{\alpha_1}^{-1} t_{\delta_1}^{-1} t_{\gamma_1}^{-1} t_{\beta_1}^{-1} t_{x_1} t_{y_1} t_{z_1},$$
$$L'_1 := t_{\beta_1}^{-1} t_{\gamma_1}^{-1} t_{\delta'_1}^{-1} t_{\alpha_1}^{-1} t_{x'_1} t_{y'_1} t_{z'_1}.$$

Proposition 6.1. *For any integer n, there are words $X_1, Y_1, X_2, Y_2, \ldots, X_{|n|+2}, Y_{|n|+2}$ in $\mathcal{M}(S_2^2)$ such that the following holds in $\mathcal{M}(S_2^2)$:*

$$(L_1)^{2n}(L'_1)^{2n} \equiv_P [X_1, Y_1][X_2, Y_2]\cdots[X_{|n|+2}, Y_{|n|+2}] \cdot t_{\delta_1}^{-2n} t_{\delta'_1}^{-2n}.$$

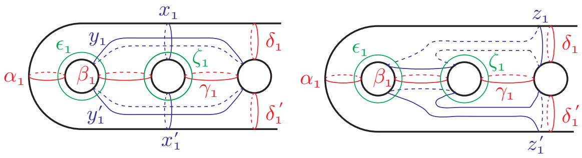

FIGURE 8. *The curves* $\alpha_1, \beta_1, \gamma_1, \delta_1, \epsilon_1, \zeta_1, x_1, y_1, z_1, x_1', y_1', z_1'$ *on* S_2^2.

Proof. Note that $\alpha_1, \beta_1, \gamma_1, \delta_1$ (resp. $\alpha_1, \beta_1, \gamma_1, \delta_1'$) are disjoint from each other and z_1 (resp. x_1'). Therefore, by the lantern relations $t_{x_1}t_{y_1}t_{z_1} = t_{\beta_1}t_{\gamma_1}t_{\delta_1}t_{\alpha_1}$ and $t_{x_1'}t_{y_1'}t_{z_1'} = t_{\alpha_1}t_{\gamma_1}t_{\beta_1}t_{\delta_1'}$ and the commutative relations, we have

$$t_{x_1}t_{y_1}t_{\alpha_1}^{-1}(z_1) = t_{\beta_1}t_{\gamma_1}t_{\delta_1}t_{z_1}^{-1}(z_1) = z_1,$$
$$t_{y_1'}t_{z_1'}t_{\gamma_1}^{-1}(x_1') = t_{x_1'}^{-1}t_{\alpha_1}t_{\beta_1}t_{\delta_1'}(x_1') = x_1'.$$

Using these facts and the primitive braid relations, we obtain

$$t_{x_1}t_{y_1}t_{\alpha_1}^{-1} \cdot t_{z_1} \equiv_P t_{z_1} \cdot t_{x_1}t_{y_1}t_{\alpha_1}^{-1},$$
$$t_{x_1'} \cdot t_{y_1'}t_{z_1'}t_{\gamma_1}^{-1} \equiv_P t_{y_1'}t_{z_1'}t_{\gamma_1}^{-1} \cdot t_{x_1'}.$$

These two relations and the commutative relations give

$$(L_1)^{2n} \equiv_P (t_{y_1}t_{z_1}t_{\alpha_1}^{-1})^{2n}t_{x_1}^{2n}t_{\beta_1}^{-2n}t_{\gamma_1}^{-2n}t_{\delta_1}^{-2n},$$
$$(L_1')^{2n} \equiv_P (t_{x_1'}t_{y_1'}t_{\gamma_1}^{-1})^{2n}t_{z_1'}^{2n}t_{\alpha_1}^{-2n}t_{\beta_1}^{-2n}t_{\delta_1'}^{-2n}.$$

Since $x_1, y_1, z_1, x_1', y_1', z_1'$ are disjoint from $\alpha_1, \beta_1, \gamma_1, \delta_1$, and x_1, y_1, z_1 are disjoint from x_1', y_1', z_1', by the commutative relations, we have

$$(L_1)^{2n}(L_1')^{2n} \equiv_P (t_{y_1}t_{z_1}t_{\gamma_1}^{-1}t_{x_1'}t_{y_1'}t_{\alpha_1}^{-1})^{2n} \cdot t_{x_1}^{2n}t_{z_1'}^{2n} \cdot t_{\alpha_1}^{-2n}t_{\beta_1}^{-4n}t_{\gamma_1}^{-2n} \cdot t_{\delta_1}^{-2n}t_{\delta_1'}^{-2n}.$$

Since y_1, z_1 are disjoint from x_1', y_1', and α_1, γ_1 are disjoint from y_1, z_1, x_1', y_1', by the commutative and the primitive braid relations, we obtain

$$(t_{y_1}t_{z_1}t_{\gamma_1}^{-1}t_{x_1'}t_{y_1'}t_{\alpha_1}^{-1})^2 \equiv_P t_{y_1}t_{\alpha_1}^{-1}t_{x_1'}t_{\gamma_1}^{-1} \cdot (t_{z_1}t_{y_1}t_{z_1}^{-1})t_{\gamma_1}^{-1}(t_{y_1'}t_{x_1'}t_{y_1'}^{-1})t_{\alpha_1}^{-1} \cdot t_{z_1}^2t_{y_1'}^2$$
$$\equiv_P t_{y_1}t_{\alpha_1}^{-1}t_{x_1'}t_{\gamma_1}^{-1} \cdot t_{t_{y_1'}(x_1')}t_{\gamma_1}^{-1}t_{t_{z_1}(y_1)}t_{\alpha_1}^{-1} \cdot t_{z_1}^2t_{y_1'}^2.$$

Here, let $f_1 := t_{z_1}t_{y_1'} \cdot t_{\epsilon_1}t_{y_1}t_{\alpha_1}t_{\epsilon_1} \cdot t_{\zeta_1}t_{x_1'}t_{\gamma_1}t_{\zeta_1}$ in $\mathcal{M}(S_2^2)$. By the latter part of Lemma 3.3, $t_{\epsilon_1}t_{y_1}t_{\alpha_1}t_{\epsilon_1} \cdot t_{\zeta_1}t_{x_1'}t_{\gamma_1}t_{\zeta_1}$ maps $(y_1, \alpha_1, x_1', \gamma_1)$ to $(\alpha_1, y_1, \gamma_1, x_1')$. From that α_1, γ_1, y_1 are disjoint from y_1', $t_{y_1'}$ maps $(\alpha_1, y_1, \gamma_1, x_1')$ to $(\alpha_1, y_1, \gamma_1, t_{y_1'}(x_1'))$. Note that y_1' and x_1' are disjoint from z_1, so $t_{y_1'}(x_1')$ is disjoint from z_1. From this, t_{z_1} maps $(\alpha_1, y_1, \gamma_1, t_{y_1'}(x_1'))$ to $(\alpha_1, t_{z_1}(y_1), \gamma_1, t_{y_1'}(x_1'))$ since $\alpha_1, \gamma_1, t_{y_1'}(x_1')$ are disjoint from z_1. Therefore, we see that f_1 maps $(y_1, \alpha_1, x_1', \gamma_1)$ to $(\alpha_1, t_{z_1}(y_1), \gamma_1, t_{y_1'}(x_1'))$. From Lemma 3.1, we obtain

$$t_{y_1}t_{\alpha_1}^{-1}t_{x_1'}t_{\gamma_1}^{-1} \cdot t_{t_{y_1'}(x_1')}t_{\gamma_1}^{-1}t_{t_{z_1}(y_1)}t_{\alpha_1}^{-1} \equiv_P [t_{y_1}t_{\alpha_1}^{-1}t_{x_1'}t_{\gamma_1}^{-1}, f_1].$$

When we write $[X, Y] = [t_{y_1}t_{\alpha_1}^{-1}t_{x_1'}t_{\gamma_1}^{-1}, f_1]$, we have

$$(t_{y_1}t_{z_1}t_{\gamma_1}^{-1}t_{x_1'}t_{y_1'}t_{\alpha_1}^{-1})^2 \equiv_P [X, Y]t_{z_1}^2t_{y_1'}^2.$$

Since z_1 is disjoint from y_1', the commutative relations and Lemma 3.8 (1) give

$$([X, Y] \cdot t_{z_1}^2t_{y_1'}^2)^n = \prod_{i=1}^{n}[X_i, Y_i] \cdot (t_{z_1}^2t_{y_1'}^2)^n \equiv_P \prod_{i=1}^{n}[X_i, Y_i] \cdot t_{z_1}^{2n}t_{y_1'}^{2n},$$

where $[X_i, Y_i] = {}_{(t_{z_1}^2t_{y_1'}^2)^{i-1}}([X, Y])$, which is a commutator since the conjugation of a commutator is also a commutator. Note that $\alpha_1, \beta_1, \gamma_1$ (resp. z_1) are disjoint from x_1, z_1, y_1', z_1' (resp.

y'_1). From the above arguments and the commutative relations give

$$\begin{aligned}(L_1)^{2n}(L'_1)^{2n} &\equiv_P \prod_{i=1}^{n}[X_i,Y_i]\cdot t_{z_1}^{2n}t_{y'_1}^{2n}\cdot t_{x_1}^{2n}t_{z'_1}^{2n}\cdot t_{\alpha_1}^{-2n}t_{\beta_1}^{-4n}t_{\gamma_1}^{-2n}\cdot t_{\delta_1}^{-2n}t_{\delta'_1}^{-2n}\\ &\equiv_P \prod_{i=1}^{n}[X_i,Y_i]\cdot t_{z_1}^{2n}t_{\alpha_1}^{-2n}t_{y'_1}^{2n}t_{\beta_1}^{-2n}\cdot t_{x_1}^{2n}t_{\beta_1}^{-2n}t_{z'_1}^{2n}t_{\gamma_1}^{-2n}\cdot t_{\delta_1}^{-2n}t_{\delta'_1}^{-2n}.\end{aligned}$$

We set $f_2 = t_{\zeta_1}t_{\beta_1}t_{\epsilon_1}t_{y'_1}t_{\alpha_1}t_{\epsilon_1}t_{z_1}t_{\zeta_1}$ and $f_3 = t_{\epsilon_1}t_{z'_1}t_{\zeta_1}t_{\gamma_1}t_{x_1}t_{\zeta_1}t_{\beta_1}t_{\epsilon_1}$ in $\mathcal{M}(S_2^2)$. By Lemma 3.3, f_2 in $\mathcal{M}(S_2^2)$ maps (z_1,α_1) to (β_1,y'_1), and f_3 in $\mathcal{M}(S_2^2)$ maps (x_1,β_1) to (γ_1,z'_1). Therefore, by Lemma 3.1, we have

$$\begin{aligned}t_{z_1}^{2n}t_{\alpha_1}^{-2n}t_{y'_1}^{2n}t_{\beta_1}^{-2n} &= [t_{z_1}^{2n}t_{\alpha_1}^{-2n}, f_2],\\ t_{x_1}^{2n}t_{\beta_1}^{-2n}t_{z'_1}^{2n}t_{\gamma_1}^{-2n} &= [t_{x_1}^{2n}t_{\beta_1}^{-2n}, f_3],\end{aligned}$$

and the proposition follows. □

PROPOSITION 6.2. *Suppose that g is odd. Let s_0 be a nonseparating curve on Σ_g. Then, for any integer n, there are lantern relators $\mathcal{L}_1,\mathcal{L}_2,\ldots,\mathcal{L}_{2(g-1)|n|}$ and words $\mathcal{X}_1,\mathcal{Y}_1,\mathcal{X}_2,\mathcal{Y}_2,\ldots,\mathcal{X}_{|n|+2},\mathcal{Y}_{|n|+2}$ in $\mathcal{M}_g$ such that*

$$\left(\prod_{i=1}^{2(g-1)|n|}\mathcal{L}_i\right)^{\epsilon} \equiv_P \prod_{j=1}^{|n|+2}[\mathcal{X}_j,\mathcal{Y}_j]\cdot t_{s_0}^{-2(g-1)n},$$

where $\epsilon = 1$ when $n > 0$ and $\epsilon = -1$ when $n < 0$.

Proof of Proposition 6.2. If $g = 3$, Proposition 6.2 immediately follows from Proposition 6.1 for $g = 3$ by setting $s_0 = \delta_1 = \delta'_1$.

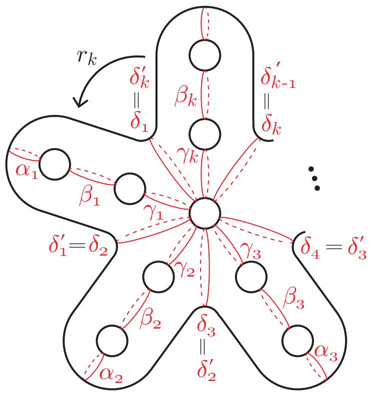

FIGURE 9. *The rotation r_k of Σ_g for $g = 2k+1$.*

If $g = 2k+1$ and $k \geq 2$, then there is a rotation r_k of Σ_g by $2\pi/k$ as in Figure 9. We identify the genus-2 subsurface bounded by δ_1 and δ'_1 with S_2^2 in Proposition 6.1 and then we write

$$\begin{aligned}L_j &:= r_k^{j-1}(L_1), & L'_j &:= r_k^{j-1}(L'_1),\\ \delta_j &:= r_k^{j-1}(\delta_1), & \delta'_j &:= r_k^{j-1}(\delta'_1),\\ X_{i,j} &:= r_k^{j-1}(X_i), & Y_{i,j} &:= r_k^{j-1}(Y_i),\end{aligned}$$

for $j = 1, 2, \ldots, k$. Note that $\delta_1 = \delta'_k$. For $j = 1, 2, \ldots, k$, Proposition 6.1 and the primitive braid relations give

$$(L_j)^{2n}(L'_j)^{2n} \equiv_P [X_{1,j}, Y_{1,j}][X_{2,j}, Y_{2,j}] \cdots [X_{|n|+2,j}, Y_{|n|+2,j}] \cdot t_{\delta_j}^{-2n} t_{\delta'_j}^{-2n}.$$

Any simple closed curves on $\mathrm{Int}(r_k^{j-1}(S_2^2))$ are disjoint from any simple closed curves on $\mathrm{Int}(r_k^{j'-1}(S_2^2))$ if $j \neq j'$, and δ_j, δ'_j are boundary curves of $r_k^{j-1}(S_2^2)$. Hence, for any words e_j in $\mathcal{M}(r_k^{j-1}(S_2^2))$ and any words $f_{j'}$ in $\mathcal{M}(r_k^{j'-1}(S_2^2))$, we have $e_j f_{j'} \equiv_P f_{j'} e_j$ for $j \neq j'$ by the commutative relations and the property of boundary curves. By $\delta_{j+1} = \delta'_j$, $\delta_1 = \delta'_k$, the commutative relations and Lemma 3.6 we have

$$\prod_{j=1}^{k} (L_j)^{2n}(L'_j)^{2n} \equiv_P [\mathcal{X}_1, \mathcal{Y}_1][\mathcal{X}_2, \mathcal{Y}_2] \cdots [\mathcal{X}_{|n|+2}, \mathcal{Y}_{|n|+2}]\ t_{\delta_1}^{-4n} t_{\delta_2}^{-4n} \cdots t_{\delta_k}^{-4n},$$

where $\mathcal{X}_i = X_{i,1}X_{i,2} \cdots X_{i,k}$ and $\mathcal{Y}_i = Y_{i,1}Y_{i,2} \cdots Y_{i,k}$, and Lemma 3.5 (1) gives

$$t_{\delta_1}^{-4n} t_{\delta_2}^{-4n} \cdots t_{\delta_k}^{-4n} \equiv_P [t_{\delta_1}^{-4n} t_{\delta_2}^{-8n} \cdots t_{\delta_{k-1}}^{-4(k-1)n}, r_k] \cdot t_{\delta_k}^{-4kn}.$$

Since from their definition, $\mathcal{X}_{|n|+2}, X_{|n|+2,j}$ (resp. $\mathcal{Y}_{|n|+2}, Y_{|n|+2,j}$) and r_k satisfy the condition of Lemma 3.9, by the primitive braid relations, we obtain $\mathcal{X}_{|n|+2} r_k \equiv_P r_k \mathcal{X}_{|n|+2}$ (resp. $\mathcal{Y}_{|n|+2} r_k \equiv_P r_k \mathcal{Y}_{|n|+2}$). Moreover, since δ_j is a boundary curve of $r_k^{j-1}(S_2^2)$ and disjoint from $r_k^{j'-1}(S_2^2)$ if $j \neq j'$, $\mathcal{X}_{|n|+2}$ and $\mathcal{Y}_{|n|+2}$ commute with t_{δ_j} modulo P for any j by the commutative relations and the property of boundary curves. From the above argument, Lemma 3.6 gives

$$[\mathcal{X}_{|n|+2}, \mathcal{Y}_{|n|+2}][t_{\delta_1}^{-4n} t_{\delta_2}^{-8n} \cdots t_{\delta_{k-1}}^{-4(k-1)n}, r_k] = [\mathcal{X}_{|n|+2} t_{\delta_1}^{-4n} t_{\delta_2}^{-8n} \cdots t_{\delta_{k-1}}^{-4(k-1)n}, \mathcal{Y}_{|n|+2} r_k],$$

and we obtain the desired conclusion. □

Lemma 6.3 ([**30**]). *Let us consider the lantern relator* $L := t_a^{-2} t_d^{-1} t_{d'}^{-1} t_c t_{s_1} t_z$, *the 2-chain relator* $C_2 := t_{s_1}^{-1}(t_a t_b)^6$ *and the 3-chain relator* $C_3 := t_d^{-1} t_{d'}^{-1}(t_a t_b t_c)^4$ *in* $\mathcal{M}_1^2$, *where the curves are as in Figure 3. Then we have*

$$C_3 \equiv_P L \cdot C_2.$$

Proof. Since a, d, d' are disjoint from c, z and each other, $t_a, t_d, t_{d'}$ commute with t_c, t_z modulo P by the commutative relations. Combining this with a cyclic permutation give $L \equiv_P t_z t_c t_d^{-1} t_{d'}^{-1} t_a^{-2} t_{s_1}$. Here, by the braid relation, we have $t_a t_b t_a t_b t_a t_b \equiv_P t_a t_a t_b t_a t_a t_b$. Therefore, using a cyclic permutation we have

$$L \cdot C_2 \equiv_P t_d^{-1} t_{d'}^{-1} \cdot t_b t_a t_a t_b \cdot t_a t_a t_b t_a t_a t_b \cdot t_z t_c.$$

By drawing corresponding curves and applying the corresponding Dehn twist, we find that $t_b t_a t_a t_b(z) = c$. This gives $t_b t_a t_a t_b \cdot t_z \equiv_P t_c \cdot t_b t_a t_a t_b$ by the primitive braid relation. Using this equation, we have

$$L \cdot C_2 \equiv_P t_d^{-1} t_{d'}^{-1} \cdot \underline{t_b t_a t_a t_b \cdot t_a t_a \cdot t_c \cdot t_b t_a} t_a t_b \cdot t_c.$$

We focus on the underlined part. By Lemma 3.2, we have $t_b t_a t_a t_b(a) = a$, $t_a t_b t_c(b) = c$ and $t_a t_b t_c(a) = b$. This gives $t_b t_a t_a t_b \cdot t_a \equiv_P t_a \cdot t_b t_a t_a t_b$, $t_a t_b t_c \cdot t_b \equiv_P t_c \cdot t_a t_b t_c$ and $t_a t_b t_c \cdot t_a \equiv_P t_b \cdot t_a t_b t_c$. Applying them on the underlined parts, we obtain

$$\begin{aligned} \underline{t_b t_a t_a t_b t_a t_a} t_c t_b t_a &\equiv_P t_a t_a t_b t_a \underline{t_a t_b t_c t_b t_a} \\ &\equiv_P t_a t_a t_b t_a t_c t_b t_a t_b t_c. \end{aligned}$$

By the braid relation and $t_at_bt_c \cdot t_b \equiv_P t_c \cdot t_at_bt_c$ on the underlined parts, we get

$$\begin{aligned} t_a\underline{t_at_bt_a}t_ct_bt_at_bt_c &\equiv_P t_at_b\underline{t_at_bt_ct_b}t_at_bt_c \\ &\equiv_P t_at_bt_ct_at_bt_ct_at_bt_c. \end{aligned}$$

This finishes the proof. □

We now prove Theorem B.

Proof of Theorem B. We may assume that the two simple closed curves in Proposition 6.2 and Theorem 4.3 both of which were denoted by s_0 are the same since for any two nonseparating curves c and c' there is a word f in $\mathcal{M}_g$ such that $f(c) = c'$ and the desired relations are preserved after such an identification.

Replacing n in the equation in Proposition 6.2 by $5n$ for non-negative n and applying a cyclic permutation, by Theorem 4.3 we get

$$\prod_{i=1}^{10(g-1)n} \mathcal{L}_i \cdot \prod_{j=1}^{g-1} C_{3,j}^n \equiv_P \prod_{j=1}^{5n+2} [\mathcal{X}_j, \mathcal{Y}_j] \cdot \prod_{j=1}^{n+3} [\mathcal{V}_j, \mathcal{W}_j],$$

This gives a Σ_g-bundle $X \to \Sigma_{6n+5}$ for odd g (This construction is called "subtraction of Lefschetz fibration" in [**16**]). By Lemma 6.3, we see that

$$\begin{aligned} n(T) &= n^+(T) - n^-(T) = 0 - 0, \\ n(C_2) &= n^+(C_2) - n^-(C_2) = (g-1)n - 0, \\ n(L) &= n^+(L) - n^-(L) = 11(g-1)n - 0 \end{aligned}$$

in the notation of Proposition 2.5. Therefore, we have

$$\sigma(X) = -1 \cdot 0 - 7 \cdot (g-1)n + 1 \cdot 11(g-1)n = 4(g-1)n.$$

This completes the proof. □

Remark 6.4. *We do not know whether the surface bundles constructed in Theorem B admit sections or not.*

In the rest of this section, we prove Theorem C.

Proof of Theorem C. Let us consider the two (sub)surfaces of genus 3 with one boundary component as in Figure 8 and the left side of Figure 6. Since a_1, b_1 are disjoint from γ, ϵ, d_1, t_{a_1}, t_{b_1} commute with $t_\gamma, \psi(= t_\epsilon t_\gamma t_{d_1} t_\epsilon)$ modulo P by the commutative relations. Therefore, the equation (5.1) (writing V_1 and W_1 simply as V and W) and a cyclic permutation give

$$C_{2,1}L^{-1} \equiv_P t_{a_1}^8 t_{b_1}^4 [t_\gamma, \psi][V t_\delta t_z^{-1}, W\phi].$$

Here, there is a word f in $\mathcal{M}_3^1$ such that $f(\delta_1) = a_1$ and $f(\delta_1') = a_1'$, where a_1' is the simple

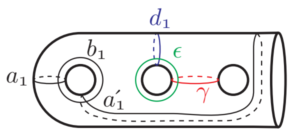

Figure 10. *The curves $a_1, a_1', b_1, d_1, \gamma, \epsilon$ on S.*

closed curve as in Figure 10 since δ_1, δ_1' are boundary curves of the genus-2 subsurface S_2^2 of

Σ_g^1 and a_1, a_1' are also boundary curves of the genus-2 subsurface of S. By Proposition 6.1 with $n = 3$ and the primitive braid and the commutative relations, we have

$$_f\left((L_1)^6(L_1')^6\right) C_{2,1}L^{-1} \equiv_P [X_1,Y_1][X_2,Y_2]\cdots[X_5,Y_5]t_{a_1'}^{-6}t_{a_1}^2t_{b_1}^4[t_\gamma,\psi][Vt_\delta t_z^{-1}, W\phi].$$

in $\mathcal{M}_3^1$, where we write X_i, Y_i again to mean $_f(X_{i,1}), {}_f(Y_{i,1})$, respectively. Let $f_5 := t_{b_1}^2 \cdot t_{a_1'}t_{b_1}t_{a_1}$ in $\mathcal{M}_3^1$. Since then $t_{a_1'}t_{b_1}t_{a_1}$ maps (a_1', b_1) to (b_1, a_1) by Lemma 3.2 (2), we see that f_5 maps (a_1', b_1) to $(b_1, t_{b_1}^2(a_1))$. Therefore, the primitive braid relation and Lemma 3.1 give

$$\begin{aligned} t_{a_1'}^{-6}t_{a_1}^2t_{b_1}^4 &= t_{a_1'}^{-6}t_{b_1}^{-2}(t_{b_1}^2t_{a_1}^2t_{b_1}^{-2})t_{b_1}^6 \\ &\equiv_P t_{a_1'}^{-6}t_{b_1}^{-2}t_{t_{b_1}^2(a_1)}^2t_{b_1}^6 \\ &\equiv_P [t_{a_1'}^{-6}t_{b_1}^{-2}, f_5]. \end{aligned}$$

Note that $t_{a_1'}, t_{b_1}, f_5$ commute with t_γ, ψ by the commutative relations since a_1, a_1', b_1 are disjoint from γ, ϵ, d_1. By the above argument, Lemma 3.6 gives

$$_f\left((L_1)^6(L_1')^6\right) C_{2,1}L^{-1} \equiv_P [X_1,Y_1][X_2,Y_2]\cdots[X_5,Y_5][t_{a_1'}^{-6}t_{b_1}^{-2}t_\gamma, f_5\psi][Vt_\delta t_z^{-1}, W\phi].$$

in $\mathcal{M}_3^1$. In particular, this equation holds in $\mathcal{M}_3$, so we get an Σ_3-bundle over Σ_7 with a 0-section. To find the signature, we compute

$$\begin{aligned} n(T) &= n^+(T) - n^-(T) = 0 - 0, \\ n(C_2) &= n^+(C_2) - n^-(C_2) = 1 - 0, \\ n(L) &= n^+(L) - n^-(L) = 12 - 1 \end{aligned}$$

in the notation of Proposition 2.5, and

$$\sigma(X) = -0 - 7\cdot 1 + 1\cdot 11 = 4.$$

The proof is complete. □

7. *Proofs of Theorem E Parts (1) and (2)*

We show E(1): $\mathrm{cl}_{\mathcal{M}_1}(t_{s_0}^{12n}) = |n| + 1$ and E(2): $\mathrm{cl}_{\mathcal{M}_2}(t_{s_0}^{10n}) \le |n| + 1$. Since we no longer need to compute signatures of surface bundles, replacing "$\equiv_P$" by "$=$" and ignoring the numbers of the relators L, T, C_2 pose no problem. From now on, we do not write $\equiv_P$ and relators explicitly.

We use the next result to prove Theorem E (1).

THEOREM 7.1 ([**3**]). *Let $h_1, g_1, h_2, g_2, \ldots, h_k, g_k$ be words in a group G. Then, for any integer n, $([h_1,g_1][h_2,g_2]\cdots[h_k,g_k])^n$ is written as a product of $|n|(k-1) + \left[\frac{|n|}{2}\right] + 1$ commutators.*

Proof of Thoerem E (1). We apply Proposition 4.2 to a closed torus where d and d' bound disks and hence t_d and $t_{d'}$ become trivial. Therefore, we see that t_b^{12n} can be written as a product of $|n| + 1$ commutators in $\mathcal{M}_1$. This gives $\mathrm{cl}_{\mathcal{M}_1}(t_b^{12n}) \le |n| + 1$ for any integer n.

We now show that $\mathrm{cl}_{\mathcal{M}_1}(t_b^{12n}) \ge |n| + 1$. Assume that for some integer $k \ge 1$, t_b^{12k} can be written as a product of k commutators. We will show that this assumption leads to a contradiction with $\mathrm{scl}_{\mathcal{M}_1}(t_{s_0}) = 1/12$. Theorem 7.1 gives

$$\mathrm{cl}_{\mathcal{M}_1}(t_b^{12kn}) \le n(k-1) + \left[\frac{n}{2}\right] + 1.$$

for any positive integer n. Therefore, we have

$$\mathrm{scl}_{\mathcal{M}_1}(t_b^{12k}) \leq (k-1) + \frac{1}{2} = k - \frac{1}{2}.$$

Since $\mathrm{scl}_{\mathcal{M}_1}(t_b) = \mathrm{scl}_{\mathcal{M}_1}(t_b^{12k})/12k$ (see Section 1.3), we obtain

$$\mathrm{scl}_{\mathcal{M}_1}(t_b) \leq \frac{1}{12} - \frac{1}{24k} < \frac{1}{12}.$$

This contradicts our assumption, which proves the theorem. □

Next, we give a proof of Theorem E (2).

Proof of Theorem E (2). We embed Σ_1^2 into Σ_2 so that $d = d'$ (see Figure 11), and consider

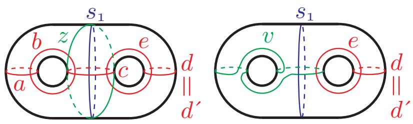

FIGURE 11. *Σ_2 and the curves a, b, c, d, e, v on Σ_1^2.*

the relation (4.3): $1 = t_a^{4n} t_c^{4n} (t_b t_v)^{2n} t_d^{-2n}$. Lemma 3.8 (2) and the primitive braid relation ${}_{t_b^{-2(n-i)}}(t_v) = {}_{t_b^{-2(n-i)-1}}(t_{t_b(v)})$ give

$$(t_b t_v)^{2n} = t_b^{2n} \prod_{i=1}^{n} {}_{t_b^{-2(n-i)-1}}(t_v t_{t_b(v)}).$$

In addition, the primitive braid relations give

$$\begin{aligned} t_a^{4n} t_c^{4n} t_b^{2n} &= t_b^{10n} \cdot t_b^{-4n} (t_b^{-6n} t_a^{4n} t_b^{6n}) t_b^{-4n} (t_b^{-2n} t_c^{4n} t_b^{2n}) \\ &= t_b^{10n} \cdot t_b^{-4n} t_{t_b^{-6n}(a)}^{4n} t_b^{-4n} t_{t_b^{-2n}(c)}^{4n}. \end{aligned}$$

By combining the above two relations with the relation (4.3) and using the commutative relations, we obtain

$$C_3^n = t_b^{10n} \cdot t_b^{-4n} t_{t_b^{-6n}(a)}^{4n} t_b^{-4n} t_{t_b^{-2n}(c)}^{4n} \cdot \prod_{i=1}^{n} {}_{t_b^{-2(n-i)-1}}(t_v t_d^{-1} t_{t_b(v)} t_d^{-1}).$$

Since $t_a t_b t_c$ maps (a, b) to (b, c) by Lemma 3.2 (2), we find that $t_b^{-2n} t_a t_b t_c t_b^{6n}$, denoted by f_3, maps $(t_b^{-6n}(a), b)$ to $(b, t_b^{-2n}(c))$. Let e be a nonseparating curve as in Figure 11. Since $t_e t_d t_v t_e$ maps (v, d) to (d, v) by Lemma 3.2 (1), $t_b t_e t_d t_v t_e$, denoted by f_4, maps (v, d) to $(d, t_b(v))$ by $i(b, d) = 0$. By Lemma 3.1 we see that

$$\begin{aligned} t_b^{-4n} t_{t_b^{-6n}(a)}^{4n} t_b^{-4n} t_{t_b^{-2n}(c)}^{4n} &= [t_b^{-4n} t_{t_b^{-6n}(a)}^{4n}, f_3], \\ t_v t_d^{-1} t_{t_b(v)} t_d^{-1} &= [t_v t_d^{-1}, f_4]. \end{aligned}$$

Since the conjugation of a commutator is also a commutator, Theorem E (2) follows. □

8. *Scl of the Dehn twist along a separating curve*

8.1. *A separating curve of type 1*

We show Theorem D (2): $\mathrm{cl}_{\mathcal{M}_g}(t_{s_1}^{5(g-1)n}) \leq [7|n|/2] + 5$ and E (3): $\mathrm{cl}_{\mathcal{M}_2}(t_{s_1}^{5n}) \leq [7|n|/2] + 2$.

We consider the subsurface S_1^2 in the proof of Theorem 4.3 and the curves $a_1, b_1, c_1, s_{1,1}, z_1, d_1, d_{g-1}$ as in Figure 4. The separating curve $s_{1,1}$ is of type 1.

PROPOSITION 8.1. *For any integer n, there are words $V_1', W_1', V_2', W_2', \dots, V'_{[\frac{|n|}{2}]+1}, W'_{[\frac{|n|}{2}]+1}$ in $\mathcal{M}(S_1^2)$ such that the following holds in $\mathcal{M}(S_1^2)$:*

$$t_{s_{1,1}}^n = [V_1', W_1'][V_2', W_2'] \cdots [V'_{[\frac{|n|}{2}]+1}, W'_{[\frac{|n|}{2}]+1}] t_{d_{g-1}}^n t_{d_1}^n.$$

Proof. From the lantern relation $t_{c_1} t_{s_{1,1}} t_{z_1} = t_{d_1} t_{d_{g-1}} t_{a_1}^2$, we get $t_{s_{1,1}} = t_{c_1}^{-1} t_{d_1} t_{d_{g-1}} t_{a_1}^2 t_{z_1}^{-1}$. Since a_1, d_1, d_{g-1} are disjoint from each other and c_1, z_1, using the commutative relation and Lemma 3.8 (1), we have

$$\begin{aligned} t_{s_{1,1}}^n &= (t_{c_1}^{-1} t_{z_1}^{-1})^n t_{a_1}^n t_{a_1}^n t_{d_{g-1}}^n t_{d_1}^n \\ &= {}_{t_{c_1}^{-1}}(t_{z_1}^{-1})\, {}_{t_{c_1}^{-2}}(t_{z_1}^{-1}) \cdots {}_{t_{c_1}^{-n}}(t_{z_1}^{-1}) \cdot t_{c_1}^{-n} t_{a_1}^n t_{a_1}^n t_{d_{g-1}}^n t_{d_1}^n. \end{aligned}$$

From the commutative relations and the primitive braid relations ${}_{t_{c_1}^{-2i+1}}(t_{z_1}^{-1}) = {}_{t_{c_1}^{-2i}}(t_{t_{c_1}(z_1)}^{-1})$ and ${}_{t_{c_1}^{-2m-1}}(t_{z_1}^{-1}) = t_{t_{c_1}^{-2m-1}(z_1)}^{-1}$, we have

$$t_{s_{1,1}}^{2m} = \prod_{i=1}^{m} {}_{t_{c_1}^{-2i}}(t_{t_{c_1}(z_1)}^{-1} t_{a_1} t_{z_1}^{-1} t_{a_1}) \cdot t_{c_1}^{-2m} t_{a_1}^{2m} \cdot t_{d_{g-1}}^{2m} t_{d_1}^{2m}$$

for even $n = 2m$ and

$$t_{s_{1,1}}^{2m+1} = \prod_{i=1}^{m} {}_{t_{c_1}^{-2i}}(t_{t_{c_1}(z_1)}^{-1} t_{a_1} t_{z_1}^{-1} t_{a_1}) \cdot t_{t_{c_1}^{-2m-1}(z_1)}^{-1} t_{a_1}^{2m+1} t_{c_1}^{-2m-1} t_{a_1} \cdot t_{d_{g-1}}^{2m+1} t_{d_1}^{2m+1}$$

for odd $n = 2m+1$. Note that in either case $m = [\frac{n}{2}]$. Since $t_{b_1} t_{a_1} t_{z_1} t_{b_1}$ maps (z_1, a_1) to (a_1, z_1) by Lemma 3.2 (1), $t_{b_1} t_{a_1} t_{z_1} t_{b_1} t_{c_1}^i$ maps $(t_{c_1}^{-i}(z_1), a_1)$ to (a_1, z_1) by $i(a_1, c_1) = 0$. From the proof of Lemma 6.3, $t_{b_1} t_{a_1} t_{a_1} t_{b_1}$ maps (a_1, z_1) to (a_1, c_1). Therefore, when we set $\phi_1 := t_{b_1} t_{a_1} t_{z_1} t_{b_1} t_{c_1}^{-1}$ and $\phi_2 := t_{b_1} t_{a_1} t_{a_1} t_{b_1} \cdot t_{b_1} t_{a_1} t_{z_1} t_{b_1} t_{c_1}^{2m+1}$, ϕ_1 maps $(t_{c_1}(z_1), a_1)$ to (a_1, z_1), and ϕ_2 maps $(t_{c_1}^{-2m-1}(z_1), a_1)$ to (a_1, c_1). Moreover, $t_{b_1} t_{c_1} t_{a_1} t_{b_1}$, denoted by ϕ_3, maps a_1 to c_1 by Lemma 3.2 (1). Lemma 3.1 gives

$$\begin{aligned} t_{t_{c_1}(z_1)}^{-1} t_{a_1} t_{z_1}^{-1} t_{a_1} &= [t_{t_{c_1}(z_1)}^{-1} t_{a_1}, \phi_1], \\ t_{t_{c_1}^{-2m-1}(z_1)}^{-1} t_{a_1}^{2m+1} t_{c_1}^{-2m-1} t_{a_1} &= [t_{t_{c_1}^{-2m-1}(z_1)}^{-1} t_{a_1}^{2m+1}, \phi_2], \\ t_{c_1}^{-2m} t_{a_1}^{2m} &= [t_{c_1}^{-2m}, \phi_3]. \end{aligned}$$

Since the conjugation of a commutator is also a commutator, the proof is complete. □

Proof of Theorem E (3). In Proposition 8.1, if we consider $g = 2$ then $d_1 = d_{g-1}$ and we have

$$t_{s_{1,1}}^{5n} = [V_1', W_1'][V_2', W_2'] \cdots [V'_{[\frac{|5n|}{2}]+1}, W'_{[\frac{|5n|}{2}]+1}] \cdot t_{d_1}^{10n}.$$

By applying Theorem E (2) to the nonseparating Dehn twist t_{d_1}, Theorem E (3) is proved. □

Proof of Theorem D (2). In the notation of proofs of Theorem 4.3 and Proposition 8.1, we write

$$\begin{aligned} s_{1,j} &:= r^{j-1}(s_{1,1}), & d_j &:= r^{j-1}(d_1), \\ V'_{i,j} &:= {}_{r^{j-1}}(V_i'), & W'_{i,j} &:= {}_{r^{j-1}}(W_i') \end{aligned}$$

for $j = 1, 2, \ldots, g-1$. Also write $d_0 = d_{g-1}$. Then, for $j = 1, 2, \ldots, g-1$, Proposition 8.1 and the primitive braid relations give

$$t_{s_{1,j}}^{5n} = [V'_{1,j}, W'_{1,j}][V'_{2,j}, W'_{2,j}] \cdots [V'_{[\frac{|5n|}{2}]+1,j}, W'_{[\frac{|5n|}{2}]+1,j}] t_{d_j}^{5n} t_{d_{j-1}}^{5n}$$

in $\mathcal{M}(r^{j^1}(S_1^2))$. Here, any simple closed curves on $\mathrm{Int}(r^{j-1}(S_1^2))$ are disjoint from any simple closed curves on $\mathrm{Int}(r^{j'-1}(S_1^2))$ if $j \neq j'$, and d_j, d_{j-1} are boundary curves of $r^{j-1}(S_1^2)$. Hence, for any words e_j in $\mathcal{M}(r^{j-1}(S_1^2))$ and any words $f_{j'}$ in $\mathcal{M}(r^{j'-1}(S_1^2))$ where $j \neq j'$, we have $e_j f_{j'} = f_{j'} e_j$ by the commutative relations and the property of boundary curves. When we set $\mathcal{V}'_i = V'_{i,1} V'_{i,2} \cdots V'_{i,g-1}$ and $\mathcal{W}'_i = W'_{i,1} W'_{i,2} \cdots W'_{i,g-1}$, from Lemma 3.6 and $d_g = d_1$, we have

$$t_{s_{1,1}}^{5n} t_{s_{1,2}}^{5n} \cdots t_{s_{1,g-1}}^{5n} = [\mathcal{V}'_1, \mathcal{W}'_1][\mathcal{V}'_2, \mathcal{W}'_2] \cdots [\mathcal{V}'_{[\frac{|5n|}{2}]+1}, \mathcal{W}'_{[\frac{|5n|}{2}]+1}] \cdot t_{d_1}^{10n} t_{d_2}^{10n} \cdots t_{d_{g-1}}^{10n}.$$

Moreover, Lemma 3.5 gives

$$t_{s_{1,g-1}}^{5(g-1)n}[T_{s_{1,1}}, r] = [\mathcal{V}'_1, \mathcal{W}'_1][\mathcal{V}'_2, \mathcal{W}'_2] \cdots [\mathcal{V}'_{[\frac{|5n|}{2}]+1}, \mathcal{W}'_{[\frac{|5n|}{2}]+1}] \cdot t_{d_{g-1}}^{10(g-1)n}[T_d, r],$$

where $T_{s_{1,1}} = t_{s_{1,1}}^{5n} t_{s_{1,2}}^{10n} \cdots t_{s_{1,g-2}}^{5(g-2)n}$ and $T_d = t_{d_1}^{10n} t_{d_2}^{20n} \cdots t_{d_{g-2}}^{10(g-2)n}$. We obtain Theorem D (2) by $[T_d, r][T_{s_{1,1}}, r]^{-1} = [T_d, r][r, T_{s_{1,1}}]$, Lemma 3.7 and Theorem 4.3. □

8.2. *A separating curve of type h*

We give the proof of Theorem D (3): $\mathrm{cl}_{\mathcal{M}_g}(t_{s_h}^{[g/h]n}) \leq [(|n|+3)/2]$ for $g \geq 3$ and $h \geq 2$. Note that the small letter h in this subsection differs from that of the base genus of surface bundles.

Let a, b, c, d, e, x, y, z be the nonseparating curves on the genus-h subsurface S_h^1 of Σ_g bounded by the separating curve s_h of type h as in Figure 12.

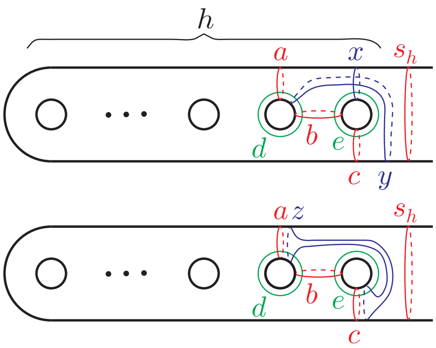

FIGURE 12. *The curves $s_h, a, b, c, d, e, x, y, z$.*

PROPOSITION 8.2 ([**5**]). *For any integer n, there are $X_1, Y_1, X_2, Y_2, \ldots, X_{[\frac{|n|+3}{2}]}, Y_{[\frac{|n|+3}{2}]}$ in $\mathcal{M}(S_h^1)$ such that the following holds in $\mathcal{M}(S_h^1)$:*

$$t_{s_h}^n = [X_1, Y_1][X_2, Y_2] \cdots [X_{[\frac{|n|+3}{2}]}, Y_{[\frac{|n|+3}{2}]}].$$

Proof. Let us consider the lantern relation $t_x t_y t_z = t_b t_c t_{s_h} t_a$. Since a, b, c, s_h, z are disjoint from each other, the commutative relations give $t_x t_y \cdot t_z = t_z \cdot t_x t_y$ and therefore $t_{s_h}^n = (t_x t_y)^n t_z^n t_a^{-n} t_b^{-n} t_c^{-n}$. By Lemma 3.8 (1), we have

$$t_{s_h}^n = {}_{t_x}(t_y)_{t_x^2}(t_y) \cdots {}_{t_x^n}(t_y) t_x^n t_z^n t_a^{-n} t_b^{-n} t_c^{-n}.$$

From the primitive braid relations $t_{x}^{2i}(t_y) = t_{x}^{2i-1}(t_{t_x(y)})$ and $t_{x}^{2m+1}(t_y) = t_{t_x^{2m+1}(y)}$ and the commutative relations, we obtain

$$t_{s_h}^{2m} = \prod_{i=1}^{m} t_x^{2i-1}(t_y t_a^{-1} t_{t_x(y)} t_a^{-1}) \cdot t_x^{2m} t_b^{-2m} t_z^{2m} t_c^{-2m}$$

for even $n = 2m$ and

$$t_{s_h}^{2m+1} = \prod_{i=1}^{m} t_x^{2i-1}(t_y t_a^{-1} t_{t_x(y)} t_a^{-1}) \cdot t_{t_x^{2m+1}(y)} t_a^{-1} \cdot t_x^{2m+1} t_b^{-2m-1} t_z^{2m+1} t_c^{-2m-1}$$

for odd $n = 2m+1$. Note that $m = [\frac{n}{2}]$. Since $t_d t_a t_y t_d$ maps (y, a) to (a, y) by Lemma 3.2 (1), $t_x t_d t_a t_y t_d$, denoted by ϕ', maps (y, a) to $(a, t_x(y))$ by $i(a, x) = 0$, and $t_d t_a t_y t_d t_x^{-2m-1}$, denoted by ψ', maps $t_x^{2m+1}(y)$ to a. Moreover, by Lemma 3.4, $t_d t_z \cdot t_e t_c t_x t_e \cdot t_b t_d$, denoted by τ', maps (x, b) to (c, z). Therefore, Lemma 3.1 gives

$$\begin{aligned} t_y t_a^{-1} t_{t_x(y)} t_a^{-1} &= [t_y t_a^{-1}, \phi'], \\ t_{t_x^{2m+1}(y)} t_a^{-1} &= [t_{t_x^{2m+1}(y)}, \psi'], \\ t_x^k t_b^{-k} t_z^k t_c^{-k} &= [t_x^k t_b^{-k}, \tau'], \end{aligned}$$

particularly for $k = 2m$ or $2m+1$. Since the conjugation of a commutator is a commutator, this finishes the proof. □

REMARK 8.3. *The above proof was given in the first draft of [**5**]. Using Proposition 8.2 it was shown in [**5**] that for a boundary curve ∂ of Σ_g^r, $\mathrm{cl}_{\mathcal{M}_g^r}(t_\partial^n) = [(n+3)/2]$ if $g \geq 2$ and $r \geq 1$, and therefore, $\mathrm{scl}_{\mathcal{M}_g^r}(t_\partial) \leq 1/2$, in fact it is known that $\mathrm{scl}_{\mathcal{M}_g^r}(t_\partial) = 1/2$.*

Proof of Theorem D (3). Suppose that $g \geq 3$ and $h \geq 2$. Let S_h^1 be the genus-h subsurface of Σ_g with one boundary component s_h. When we write $g = hk + g'$ where $0 \leq g' \leq h-1$, that is $k = [\frac{g}{h}]$, there is a word ρ_k in $\mathcal{M}_g$ such that the subsurfaces $S_h^1, \rho_k(S_h^1), \ldots, \rho_k^{k-1}(S_h^1)$ are disjoint from each other. In the notation of Proposition 8.2, we write

$$\begin{aligned} s_{h,j} &:= \rho_k^{j-1}(s_h), \\ X_{i,j} &:= {}_{\rho_k^{j-1}}(X_i), \qquad\qquad Y_{i,j} := {}_{\rho_k^{j-1}}(Y_i) \end{aligned}$$

for $j = 1, 2, \ldots, k$. Then, Proposition 8.2 and the primitive braid relations give

$$t_{s_{h,j}}^n = [X_{1,j}, Y_{1,j}][X_{2,j}, Y_{2,j}] \cdots [X_{[\frac{|n|+3}{2}],j}, Y_{[\frac{|n|+3}{2}],j}].$$

for $j = 1, 2, \ldots, k$. Since $\rho_k^{j-1}(S_h^1)$ is disjoint from $\rho_k^{j'-1}(S_h^1)$ if $j \neq j'$, any words e_j in $\mathcal{M}(\rho_k^{j-1}(S_h^1))$ and any words $f_{j'}$ in $\mathcal{M}(\rho_k^{j'-1}(S_h^1))$ satisfy $e_j f_{j'} = f_{j'} e_j$ from the commutative relations. Therefore, from Lemma 3.6, we have

$$t_{s_{h,1}}^n t_{s_{h,2}}^n \cdots t_{s_{h,k}}^n = [\mathcal{X}_1', \mathcal{Y}_1'][\mathcal{X}_2', \mathcal{Y}_2'] \cdots [\mathcal{X}_{[\frac{|n|+3}{2}]}', \mathcal{Y}_{[\frac{|n|+3}{2}]}'],$$

where $\mathcal{X}_i' = X_{i,1} X_{i,2} \cdots X_{i,k}$ and $\mathcal{Y}_1' = Y_{i,1} Y_{i,2} \cdots Y_{i,k}$. Moreover, Lemma 3.5 gives

$$[t_{s_{h,1}}^n t_{s_{h,2}}^{2n} \cdots t_{s_{h,k-1}}^{(k-1)n}, \rho_k] t_{s_{h,k}}^{kn} = [\mathcal{X}_1', \mathcal{Y}_1'][\mathcal{X}_2', \mathcal{Y}_2'] \cdots [\mathcal{X}_{[\frac{|n|+3}{2}]}', \mathcal{Y}_{[\frac{|n|+3}{2}]}'].$$

In particular,

$$t_{s_{h,k}}^{kn} = [t_{s_{h,1}}^n t_{s_{h,2}}^{2n} \cdots t_{s_{h,k-1}}^{(k-1)n}, \rho_k]^{-1} [\mathcal{X}_1', \mathcal{Y}_1'][\mathcal{X}_2', \mathcal{Y}_2'] \cdots [\mathcal{X}_{[\frac{|n|+3}{2}]}', \mathcal{Y}_{[\frac{|n|+3}{2}]}'].$$

Since $X_{i,j}, \mathcal{X}_i'$ (resp. $Y_{i,j}, \mathcal{Y}_i'$) and ρ_k satisfy the assumption of Lemma 3.9 from their definitions and the primitive braid relations, we obtain $\mathcal{X}_1' \rho_k = \rho_k \mathcal{X}_1'$ (resp. $\mathcal{Y}_1' \rho_k = \rho_k \mathcal{Y}_1'$). Note that $s_{h,j}$

is a boundary curve of $\rho_k^{j-1}(S_h^1)$ and that $s_{h,1}, s_{h,2}, \ldots, s_{h,k}$ are disjoint curves. Therefore, $t_{s_{h,1}}^n t_{s_{h,2}}^{2n} \cdots t_{s_{h,k-1}}^{(k-1)n}$ commutes with $\mathcal{X}_1'$ and $\mathcal{Y}_1'$ by the property of boundary curves and the commutative relations. By Lemma 3.6, we have

$$\begin{aligned}[t_{s_{h,1}}^n t_{s_{h,2}}^{2n} \cdots t_{s_{h,k-1}}^{(k-1)n}, \rho_k]^{-1}[\mathcal{X}_1', \mathcal{Y}_1'] &= [\rho_k, t_{s_{h,1}}^n t_{s_{h,2}}^{2n} \cdots t_{s_{h,k-1}}^{(k-1)n}][\mathcal{X}_1', \mathcal{Y}_1'] \\ &= [\rho_k \mathcal{X}_1', t_{s_{h,1}}^n t_{s_{h,2}}^{2n} \cdots t_{s_{h,k-1}}^{(k-1)n} \mathcal{Y}_1'],\end{aligned}$$

and the proof is complete. □

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