

Four Essays in Nonlinear Economic Dynamics

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Preface

In the mid-20th century, N. Kaldor, R. M. Goodwin, J. R. Hicks, and others studied the autonomous and sustainable economic fluctuations caused by the nonlinearity inherent in an economy. The research field that emerged can be called either the “theory of nonlinear business cycles” or the “theory of endogenous business cycles.” A major reason for the popularity of these studies was the progress made in the theory of nonlinear oscillations, which was then being actively pursued in science and engineering.

However, the nonlinear oscillation theory—or at least the kind practiced by the economists of that era—was mainly concerned with relatively simple vibration patterns, such as limit cycles in a continuous time system. Thus, such studies proved inadequate in identifying those characteristics of the economy that are *recurrent but not periodic*. In addition, after the “microfoundations” began to be used widely for research in the latter half of the 20th century, studies of nonlinear dynamics in economics, which often lacked microfoundations, eventually got sidelined by more mainstream topics in economics.

In stark contrast to the “decline” in use of nonlinear dynamics in economics, research related to autonomous fluctuations of dynamical systems has been vigorously pursued since the 20th century in the mathematical sciences as “dynamical system theory.” The growing interest in a complex nonlinear phenomenon called *chaos* is a particularly significant development. The term chaos became popular after the publication of by an article titled “Period three implies chaos” by Li and Yorke (1975). However, research into the nature of the phenomenon is even older. The existence theorem of “chaotic invariant set” was presented in a general and comprehensive manner in the *horseshoe map* of Smale (1967) and a series of homoclinic theorems. However, these studies seem to have appeared too early for economists to take notice.

Against the backdrop of such achievements in mathematics, the importance of complex fluctuations caused by nonlinearity has again begun to be recognized in economics. At the same time, amid an emphasis on microfoundations, it was shown that the optimal path exhibits either cyclical fluctuations or chaotic fluctuations in an optimal growth model in the context of dynamic general equilibrium. One of the most prominent works in this field is an article written by Boldrin and Montrucchio (1986). They have proposed that for any given policy function, there exists an optimization problem such that the optimal path can become the given policy function. That is, any fluctuation that may produce chaos would not contradict the optimizing behavior. Meanwhile, there are also studies using an overlapping generations model. For example, Benhabib and Day (1982) and Grandmont (1985) proposed the existence of a chaotic competitive equilibrium path. When the economy is seen as a dynamical system on the interval, it is relatively easy to elucidate its complexity and identify its characteristics. For this reason, the possibility of chaotic fluctuations was proposed through various models—including the optimal growth model or the overlapping generations model—until the 2000s.

The author’s research is part of the theoretical studies of nonlinear dynamics discussed above. In other words, the author focuses on identifying the characteristics of complex

fluctuation patterns caused by the nonlinearity inherent in an economic model.

With respect to research methods, the author uses *piecewise linearity* and *piecewise smoothness* in the model for analysis. In general, the dynamic characteristics of a particular model with piecewise features can be understood in depth because analysis becomes easier. This method allows for analysis of “global dynamics,” which are generally difficult to examine. In particular, in the case where the model is represented by piecewise linear, it is not unusual that a “complete” understanding of the model’s dynamic characteristics can be attained. Another method uses the bifurcation theory, which has yet gain widespread acceptance in economics. In particular, the author studies a *codimension-two bifurcation* phenomenon. This refers to a bifurcation that requires two parameters, rather than one, for a complete understanding of its structure. In the theory of dynamics, it is known as a bifurcation that provide information on the global properties. That is, a dynamical system around the codimension-two point may create multiple attractors and complex behavior derived from global bifurcation such as homoclinic bifurcation and heteroclinic bifurcation.

The thesis consists of four essays; each of them is independent and self-contained.

In chapter one, we develop a simple, computable overlapping generations model that exhibits endogenous fluctuations. The key assumption is that a firm can choose from multiple technologies of production. Since the model reduces to a piecewise linear map on the unit interval, it allows us to conduct an in-depth analysis of its dynamic properties. Particularly, this piecewise linearization reveals the ability of the model to exhibit periodic attracting cycles of an arbitrarily large period as well as non-periodic attractors. Furthermore, it is demonstrated that the occurrence of periodic patterns is completely characterized by the rotation number or the “devil’s staircase.”

In chapter two, we investigate the dynamics of a two-dimensional, discrete-time inflation model with a piecewise linear Phillips curve and Okun’s law. Piecewise linearity of the model allows us to derive numerous analytical and geometrical results, which is difficult to obtained using a smooth model. Next, using a perturbation argument, we show that, for a large set of parameter values, our model exhibits multiple attractors that coexist with chaotic invariant sets. This implies that without any exogenous shocks, the rate of inflation and other relevant macroeconomic variables exhibit long-lasting, complicated fluctuations before they settle down into a periodic cycle or a steady state in the long run.

In chapter three, we extend the model in chapter two to the model which can virtually reproduce the chaotic fluctuations in the long run. As a result, the model reduces to a piecewise smooth map, which is tractable enough to analytically investigate the dynamics in depth. To study chaotic behaviors in the model, we adopt two approaches, border collision bifurcation and Markov property. The border collision bifurcation theory characterizes the routes from a globally attracting steady state to other non-stationary behaviors. On the other hand, Markov property reveals that the model has the capability to exhibit chaotic dynamics for much large set of parameter values.

In chapter four, we investigate a discrete-time version of logit dynamics, as applied

to the rock-paper-scissors (RPS) game. First, we show that around the Nash equilibrium point, an attracting closed invariant curve appears due to the Neimark-Sacker bifurcation. Next, near the resonance point, we find a period-three attracting cycle, which can be thought of as a counterpart to the cyclically stable set in the RPS game with best response dynamics. Moreover, we show that the cycle can coexist with an attracting closed invariant curve, a period-three saddle cycle, and the attracting or repelling Nash equilibrium point. Finally, we use the codimension-two bifurcation theory to specify the set of heteroclinic bifurcations that destroy coexistence of the attractors.

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1 A Simple Model of Growth Cycles with Technology Choice

1.1 Introduction

Apparently, the overlapping generations (OLG, hereafter) model is one of the most popular dynamic economic models in the literature. It has been widely used in many fields of economics. Especially, the OLG setting has been used as a building block for *endogenous growth (or business) cycle*¹ models, as is used in this study. The endogenous growth cycle theory argues that the internal economic excitement, rather than the exogenous shocks, is responsible for the perpetual fluctuations of the economy. A few prominent examples of chaotic (i.e., random-looking but deterministic) economic dynamics in an OLG model, in the early literature, can be found in the studies by Benhabib and Day (1982) and Grandmont (1985).

An OLG model used for demonstrating endogenous fluctuations often contains additional assumptions that make its functional form less tractable. For example, the existence of indeterminacy was evident in the aforementioned model studied by Grandmont (1985). Michel and de la Croix (2000) and Chen et al. (2008) demonstrated that an OLG model with myopic expectations can exhibit chaotic behavior when the utility function and/or production function has a constant elasticity of substitution (CES) form.

The study aims to develop an explicit OLG model that is tractable and has rich dynamics that would allow the endogenous cycles to emerge in a deterministic way. In fact, the model provided in this study comprises only of the Cobb-Douglas utility and Cobb-Douglas production technologies. Furthermore, unlike Michel and de la Croix (2000) and Chen et al. (2008), we do not assume imperfect foresight, such as set myopic expectations, or employ complicated learning mechanisms to achieve rich dynamic results for the model.

Instead, we assume that, unlike in ordinary textbook-type OLG models with production, a representative firm (or its owner) can choose from multiple production technologies.² In other words, the firm faces a discrete choice problem³ before it commences the production of a good. Through this simple case, the study reveals that the firm faces a binary choice problem that results in the emergence of endogenous cycles induced by strong nonlinearity. Our model reduces into a simple, first-order piecewise linear difference equation with one endogenous discontinuity, which allows us to provide an elaborate characterization of the periodic and non-periodic dynamics of the model.

¹Some authors argue that the two-period-lived OLG model is not appropriate for studying business cycles because the time span of even one period in the model is very long for a business cycle. Although we do not necessarily agree with this criticism, we have considered this factor and preferred using the term *growth cycle* instead of *business cycle*.

²Since we are considering a single or aggregate good market, an individual technology, (in a literal sense), can be referred to as an *industry*. For instance, we can consider the choice between technology 1 and technology 2 to be the same as that between “agriculture” and “manufacture.” Thus, it is also reasonable to refer to our model as a *growth model with industrial structural change*.

³For discrete choice theory in economics, see Anderson et al. (1992).

However, it must be noted that the relationship between discrete choice and complex dynamics has been discussed in different contexts in economics. Ishida and Yokoo (2004) developed a macroeconomic model wherein newly entered firms choose whether to invest in a time-consuming project, giving rise to a piecewise linear dynamic model that can generate periodic cycles of any arbitrarily large period. Matsuyama (2007) presented an OLG-type growth model with an endogenous technology switch that is induced by financial imperfections; he showed that the model has the ability to exhibit several dynamic growth patterns including perpetual fluctuations. Asano et al. (2012) focused on a special case of the growth model of Matsuyama (2007) and showed that the piecewise-linearized model can exhibit periodic and non-periodic fluctuations. The model presented in this study is constructed along the lines of the study by Matsuyama (2007) in terms of the discrete choice. However, our model is simpler than Matsuyama's model in terms of both the story behind the modeling and the functional form of the model. Prior to this study by Matsuyama (2007), Iwaisako's (2002) study presented an OLG model with a technology choice; by using a graphical argument, Iwaisako suggested the possible occurrence of several growth patterns including cycles. Iwaisako (2002) investigated a situation in which investors choose one of the two technologies: constant returns to scale and increasing returns to scale. On the other hand, the present study assumes that all the available technologies are in the form of the Cobb-Douglas production function. In this sense, the model presented in this study is simpler than Iwaisako's model.

Despite the simplicity of our model, results derived from the analysis of the model seem to be sharper than the earlier related works in terms of the characterization of dynamic patterns. A piecewise-linearization, similar to Asano et al. (2012), helps us to directly apply some useful results borrowed from the studies on the mathematical neuron models to our model; ⁴ the employment of this method helps us to conduct a detailed characterization of the cyclical regime-switching patterns arising in our model.

Particularly, we show that the occurrence of a periodic attractor can be completely characterized by the *rotation number* or its visualization, referred to as the devil's staircase, which is the graph of a function that is monotonic but flat almost everywhere. Furthermore, we also show that the set of parameter values for which the non-periodic attractors appear is very small. In other words, the non-periodic motions in our model cannot be virtually observed.

The organization of the paper is as follows. Section 1.2 derives the piecewise linear model by incorporating the technology choice into the double Cobb-Douglas OLG model. Section 1.3 shows that the model can exhibit periodic as well as non-periodic fluctuations. Section 1.4 investigates the relationship between the parameters and the periodic patterns by introducing the rotation number. Section 1.5 concludes the study. Some mathematical proofs are delegated to appendices.

⁴Refer to Nagumo and Sato (1972) for their seminal work on the mathematical neuron model. Refer to the studies by Hata (1982, 2014) for learning about the recent developments in the field.

1.2 The discrete choice model

We consider a Diamond-type one-sector OLG model that is modified further. Time is discrete, that is, $t = 0, 1, 2, \dots$, and the agents live for two periods. A young household supplies one unit of labor inelastically. We keep the utility function, u , of the household as simple as possible; this allows us to assume that it is in the form of a log-linearized Cobb-Douglas production function, that is,

$$u(c_t^y, c_{t+1}^o) = (1-s) \log c_t^y + s \log c_{t+1}^o, \quad s \in (0, 1), \quad (1)$$

where c_t^y denotes the amount of consumption of the young generation born at the time t and c_{t+1}^o denotes the amount of consumption of the old generation living at the time $t+1$. The utility given by (1) is maximized under the following constraints:

$$c_t^y + s_t = w_t \quad \text{and} \quad c_{t+1}^o = r_{t+1} s_t, \quad (2)$$

where s_t , w_t , and r_{t+1} are the amount of saving, real wage rate, and real gross rate of return, respectively. The maximization yields

$$s_t = s w_t. \quad (3)$$

The final good Y_t , which is perishable, is produced by the firm. Unlike the common OLG models, we assume that there are M types of production technologies with $M \geq 2$; we also assume that, at the beginning of every period, the firm faces a discrete choice problem related to the choice of technology. For the sake of simplicity, all the technologies are specified as Cobb-Douglas of constant returns to scale:

$$Y_t = F_i(K_t, L_t) = A_i K_t^{\alpha_i} L_t^{1-\alpha_i}, \quad i = 1, 2, \dots, M \quad (4)$$

where $A_i > 0$ is the total factor productivity and $\alpha_i \in (0, 1)$ is the capital share of the production of technology i . In the per-capita form, we can write:

$$y_t = f_i(k_t) = F_i(k_t, 1), \quad (5)$$

where $y_t = Y_t/L_t$ and $k_t = K_t/L_t$. We assume that the owner of the firm, who belongs to the old generation, chooses technology that earns the highest return. For simplicity's sake, we assume that when the highest rates of return are tied among multiple technologies, then the technology with the smallest index is chosen. Subsequently, the usual first order conditions with the technology choice are represented by

$$r_t = f'_{J_t}(k_t), \quad (6)$$

$$w_t = f_{J_t}(k_t) - k_t f'_{J_t}(k_t), \quad \text{and} \quad (7)$$

$$J_t = \arg \max_{1 \leq j \leq M} f'_j(k_t). \quad (8)$$

The market clearing condition

$$k_{t+1} = s_t, \quad (9)$$

with the Eq.(3), (7) and (8) generate the dynamic model in the following form:

$$\begin{cases} k_{t+1} = s A_{J_t} (1 - \alpha_{J_t}) k_t^{\alpha_{J_t}} \equiv h_{J_t}(k_t), \\ J_t = \arg \max_{1 \leq j \leq M} \{A_j \alpha_j k_t^{\alpha_j - 1}\}, \quad \text{and} \quad k_0 > 0. \end{cases} \quad (10)$$

1.2.1 Further specifications in the binary choice setting

We focus on the cyclical behavior of the model in a simple setting. We first assume that there exist only the following two production technologies: technology 1, which is represented by (A_1, α_1) , and technology 2, (A_2, α_2) . To avoid unnecessary subscriptions, we rewrite the technology parameters as $(A_1, \alpha_1) = (A, a)$, for technology 1, and $(A_2, \alpha_2) = (B, b)$, for technology 2.

We impose some assumptions on the parameters of technologies for making the model capable of generating cyclical dynamics. To this end, we first derive the threshold value of per-capita capital stock at which the choice of technology switches. By solving $f'_1(\hat{k}) = f'_2(\hat{k})$, we obtain this unique threshold

$$\hat{k} = \left[\frac{aA}{bB} \right]^{1/(b-a)} \quad (11)$$

as long as $a \neq b$. Without loss of generality, we assume that

$$a < b. \quad (12)$$

It is evident that (12) implies

$$\frac{|f''_1(\hat{k})|}{|f''_2(\hat{k})|} = \frac{1-a}{1-b} > 1. \quad (13)$$

The situation described by (13) is depicted in Fig.1. Since the graphs of $f'_1(k)$ and $f'_2(k)$, both of which are negatively sloping, have only one intersection at $k = \hat{k}$, the (13) implies that technology 1 is chosen for $k \leq \hat{k}$ and technology 2 is chosen for $k > \hat{k}$.

Thus, Eqs.(10) turn into the following equation:

$$k_{t+1} = h(k_t) = \begin{cases} h_1(k_t) = s(1-a)Ak_t^a, & \text{if } 0 \leq k_t \leq \hat{k}, \\ h_2(k_t) = s(1-b)Bk_t^b, & \text{if } \hat{k} < k_t. \end{cases} \quad (14)$$

We also find a point $k^* > 0$ such that

$$h_1(k^*) = h_2(k^*), \quad (15)$$

which is uniquely given by

$$k^* = \left[\frac{A(1-a)}{B(1-b)} \right]^{1/(b-a)}. \quad (16)$$

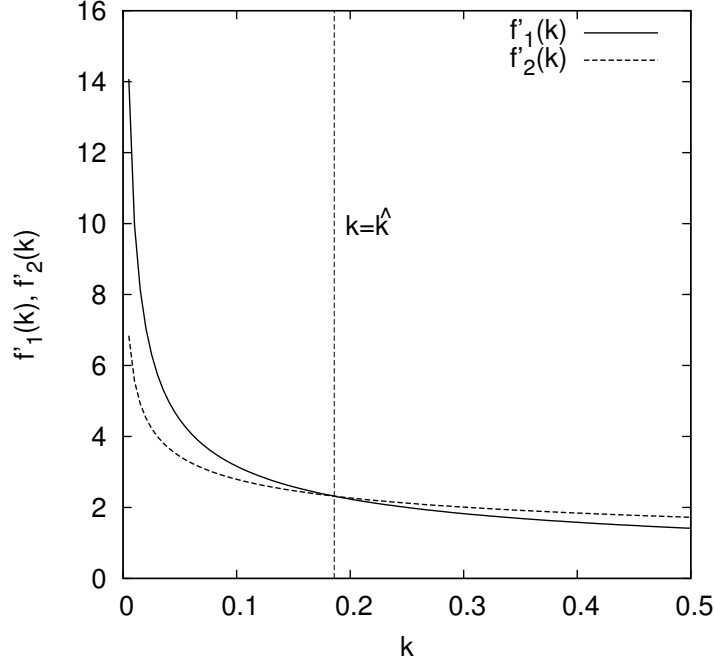


Figure 1: The graphs of $f'_1(k)$ and $f'_2(k)$. They have only one intersection at $k = \hat{k}$. $A = B = 2$, $a = 0.5$, and $b = 0.7$.

As

$$\frac{\hat{k}}{k^*} = \left[\frac{a(1-b)}{b(1-a)} \right]^{1/(b-a)} < 1 \quad (17)$$

by (12), we have $\hat{k} < k^*$. It must be noted that $h'_i(k) > 0$ for $k > 0$, we have

$$\frac{h'_1(k^*)}{h'_2(k^*)} = \frac{a}{b} < 1. \quad (18)$$

It implies that the graph of $h_2(k)$ cuts that of $h_1(k)$ from below at $k = k^*$. This situation is depicted in Fig.2.

The graphical argument shows that if Eq.(14) has a positive steady state, then it would be globally attracting. As we are interested in non-stationary behaviors, we examine the case where Eq.(14) has no positive steady state. Let $e_i > 0$ be such that $h_i(e_i) = e_i$ for $i = 1, 2$. A simple computation shows that

$$e_1 = [s(1-a)A]^{1/(1-a)} \quad \text{and} \quad e_2 = [s(1-b)B]^{1/(1-b)}. \quad (19)$$

It would be sufficient for the following inequalities to hold to ensure that Eq.(14) does not have a positive steady state:

$$e_2 < \hat{k} < e_1. \quad (20)$$

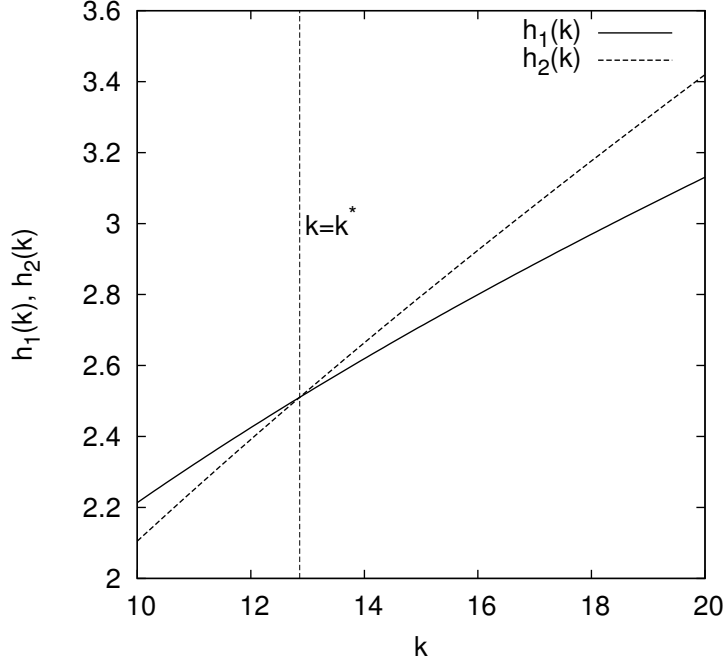


Figure 2: The graphs of $h_1(k)$ and $h_2(k)$. The graph of $h_2(k)$ cuts that of $h_1(k)$ from below at $k = k^*$. $A = B = 2$, $a = 0.5$, $b = 0.7$, and $s = 0.7$.

The following lemma shows that the set of parameter values that realize the inequalities given by (20) is not empty.

Lemma 1. *Let $a, b \in (0, 1)$ with $a < b$, $B > 0$ and $s \in (0, 1)$ be given. Subsequently, there exist $\underline{A} > 0$ and $\bar{A} > 0$ with $\underline{A} < \bar{A}$ such that the inequalities given by (20) hold for $A \in (\underline{A}, \bar{A})$.*

Proof. See Appendix. □

For $A \in (\underline{A}, \bar{A})$, it is evident that every trajectory generated by Eq.(14) eventually enters the trapping interval $T = [h_2(\hat{k}), h_1(\hat{k})]$ and never leaves it. Thus, in order to study the long-run behavior of the model of this situation, it is sufficient to focus on the dynamics of the trapping interval T .

In what follows, we assume that the parameters are taken as in Proposition 1. It must be noted that $h_2(\hat{k}) < \hat{k} < h_1(\hat{k})$. Let $h|_T : T \rightarrow T$ be the restriction of the mapping h to T . As $h(T) \subset T$, $h|_T$ is well-defined. Therefore, the goal of this study is to provide a detailed characterization of the dynamics of the mapping $h|_T : T \rightarrow T$.

In order to simplify the analysis of the dynamics of $h|_T$, we use a variable change via a homeomorphism (i.e., a continuous, one-to-one, and onto mapping) $\varphi : T \rightarrow I = [0, 1]$

such that

$$x_t = \varphi(k_t) = \frac{\log [k_t/s(1-b)\hat{B}\hat{k}^b]}{\log b(1-a)/a(1-b)} \quad (21)$$

to obtain a mapping $\tau : I \rightarrow I$, which is topologically equivalent to $h|_T : T \rightarrow T$, defined by

$$x_{t+1} = \tau(x_t) = \begin{cases} 1 + a(x_t - c) & \text{if } 0 \leq x_t \leq c, \\ b(x_t - c) & \text{if } c < x_t \leq 1, \end{cases} \quad (22)$$

where

$$c = c(s, A) = \varphi(\hat{k}) = \frac{\frac{1-b}{b-a} \log [aA/bB] - \log s(1-b)B}{\log b(1-a)/a(1-b)}. \quad (23)$$

Therefore, we focus on the analysis of Eq.(22) in the sequel. It must be noted that the aforementioned model turns *piecewise linear* through the variable change, which makes our model significantly tractable.

It is also worth noting that the graph of Eq.(22) has two branches with different slopes, a and b , while Ishida and Yokoo (2004) and Asano et al. (2012) deal with a case of $a = b$.

1.3 Two types of dynamics

It is interesting to note that Eq.(22) can be identified with a simplified version of Caianiello's equation in the neural networks, whose original model is proposed by Caianiello to describe the behavior of a "model of brain" or "thinking machine"; this model is studied by Nagumo and Sato (1972) in detail. Additionally, Hata (2014) comprehensively analyzes this sort of equation, and we utilize results of Hata's (2014) analysis for the mathematical neuron model to study economic growth.

When we investigate Eq.(22), the position of the threshold $c(s, A)$ on the unit interval plays an important role. The position of the threshold characterizes the dynamical property of the model. Generally, for a given value of c , Eq.(22) has either a globally attracting periodic cycle or a non-periodic attractor. We examine the periodic case in section 1.3.1 and the non-periodic case in section 1.3.3.

First, we show that $c(s, A)$ can take any value within the range $(0, 1)$, independently of the parameters a and b .

Lemma 2. *For any $a, b \in (0, 1)$ with $a < b$ and for any $c^* \in (0, 1)$, there exist $s^* \in (0, 1)$ and $A^* \in (\underline{A}, \bar{A})$ such that $c(s^*, A^*) = c^*$.*

Proof. See Appendix. □

1.3.1 Periodic dynamics

All propositions in section 1.3.1 and section 1.3.3 can be almost directly⁵ derived by the results of Hata (2014).

⁵In what follows, we provide mathematical formulae for $a < b$, while results of Hata's (2014) analysis work for $b < a$. Thus, we need slight modifications.

The following proposition ensures the existence of a globally attracting periodic cycle in Eq.(22).

Proposition 1. *For each irreducible fraction $p/q \in (0, 1)$, there exists a closed interval $\Delta(p/q) \subset (0, 1)$ such that if $c(s, A) \in \Delta(p/q)$, for any $x \in (0, 1)$, then the orbit of x converges to some periodic orbit of period q .*

Proof. See Hata (2014), Theorem 4.2 in p.36 and Theorem 10.1 in pp.117–118. $\square$

To be more precise, if $c(s, A) \in \Delta(p/q)$, then the Eq.(22) has a globally attracting cycle of period q , which visits the interval $(0, c)$ p times and the interval $(c, 1)$ $q - p$ times. Later, in the study, we refer to the closed interval $\Delta(p/q)$ as the *periodic interval* for p/q . Subsequently, we can almost completely grasp the dynamical features of the model by verifying to which periodic interval the threshold c belongs.

Due to the piecewise linearity of the model, we can exactly calculate the left and right endpoints of the periodic interval $\Delta(p/q)$ as follows. Let $\Delta(p/q) = [L_{p,q}, R_{p,q}]$ and

$$P_{p,q}(w, z) = \sum_{k=1}^q w^{k-1} z^{\lfloor pk/q \rfloor}, \quad (24)$$

$$P_{p,q}^+(w, z) = \sum_{k=1}^q w^{k-1} z^{\lceil pk/q \rceil - 1}, \quad (25)$$

where $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ represent the floor and ceiling function, respectively. In other words, $\lfloor x \rfloor$ is the largest integer that is not greater than x . On the other hand, $\lceil x \rceil$ is the smallest integer that is not less than x . Subsequently, we have

$$L_{p,q} = \frac{1}{b} \frac{P_{q,p}(a/b, b)}{P_{p,q}^+(b, a/b)}, \quad (26)$$

$$R_{p,q} = \frac{1}{b} \frac{P_{q,p}^+(a/b, b)}{P_{p,q}(b, a/b)}. \quad (27)$$

See Hata (2014) for more details.

Using Eq.(26) and (27), we can calculate the periodic interval for any fraction $q/p \in (0, 1)$. Given q , we can use lemma 2 to construct the model that has an attracting cycle of period q .

Finally, we provide a few important properties of the periodic interval $\Delta(p/q)$.

Proposition 2. *The interval $\Delta(p/q)$ has the following properties.*

1. *Let p/q and r/s be irreducible fraction in $(0, 1)$ with $p/q < r/s$. Subsequently, $\Delta(p/q) \cap \Delta(r/s) = \emptyset$ and $\sup \Delta(p/q) \leq \inf \Delta(r/s)$.*
2. $\left| \bigcup_{p/q \in [0,1] \cap \mathbb{Q}} \Delta(p/q) \right| = 1$.

Proof. See Hata (2014), Theorem 10.1 in pp.117–118 and Theorem 10.7 in p.132. $\square$

Proposition 2 states that the union of all the intervals is disjoint and its Lebesgue measure equals to 1. However, as shown later, the family of closed intervals is *not* a cover of the unit interval. Therefore, one might want to examine the case where the value $c(s, A)$ does not lie in any periodic interval $\Delta(p/q)$. We investigate such a case in section 1.3.3.

1.3.2 Examples

In this section, we provide a few examples for the periodic intervals. Given $p = 1$, we consider the cases $\Delta(1/q)$ and $\Delta((q-1)/q)$. Subsequently, we have

$$\begin{aligned} P_{q,1}(a/b, b) &= (a/b)^0 b^{\lfloor q \rfloor} \\ &= b^q, \end{aligned}$$

and

$$\begin{aligned} P_{1,q}^+(b, a/b) &= \sum_{k=1}^q b^{k-1} (a/b)^{\lceil k/q \rceil - 1} \\ &= \sum_{k=1}^q b^{k-1}. \end{aligned}$$

Moreover, we have

$$\begin{aligned} P_{q,1}^+(a/b, b) &= (a/b)^0 b^{\lceil q \rceil - 1} \\ &= b^{q-1}, \end{aligned}$$

and

$$\begin{aligned} P_{1,q}(b, a/b) &= \sum_{k=1}^q b^{k-1} (a/b)^{\lfloor k/q \rfloor} \\ &= (a/b)^0 \sum_{k=1}^{q-1} b^{k-1} + (a/b) b^{q-1} \\ &= \sum_{k=1}^{q-2} b^{k-1} + (1+a) b^{q-2}. \end{aligned}$$

Using Eq.(26) and (27), we can derive the left and right endpoints of the interval $\Delta(1/q)$ as

$$L_{1,q} = \frac{b^{q-1}}{\sum_{k=1}^q b^{k-1}}, \quad (28)$$

$$R_{1,q} = \frac{b^{q-2}}{\sum_{k=1}^{q-2} b^{k-1} + (1+a) b^{q-2}}. \quad (29)$$

Similarly, by taking $p = q - 1$, we can calculate the left and right endpoints of the interval $\Delta((q - 1)/q)$ in the following explicit forms:

$$L_{q-1,q} = 1 - \frac{a^{q-2}}{\sum_{k=1}^{q-2} a^{k-1} + (1+b)a^{q-2}}, \quad (30)$$

$$R_{q-1,q} = 1 - \frac{a^{q-1}}{\sum_{k=1}^q a^{k-1}}. \quad (31)$$

If we further set $q = 3$, we have

$$L_{1,3} = \frac{b^2}{1+b+b^2}, \quad (32)$$

$$R_{1,3} = \frac{b}{1+(1+a)b}. \quad (33)$$

Additionally, we have

$$L_{2,3} = 1 - \frac{a}{1+(1+b)a} = \frac{1+ab}{1+(1+b)a}, \quad (34)$$

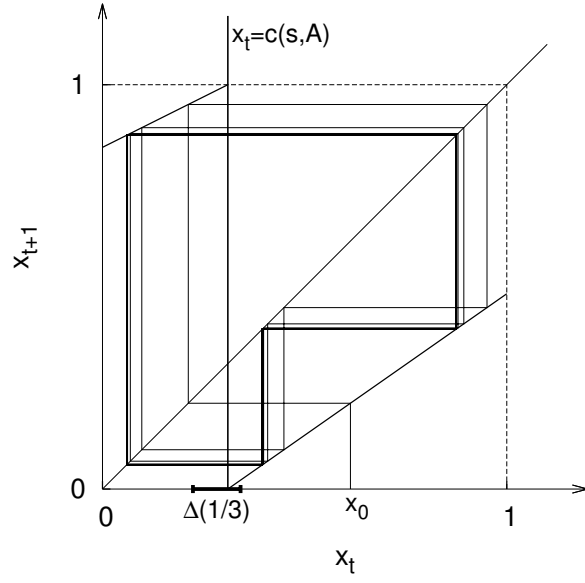
$$R_{2,3} = 1 - \frac{a^2}{1+a+a^2} = \frac{1+a}{1+a+a^2}. \quad (35)$$

By letting $a = 0.5$ and $b = 0.7$ and using Eq.(32) through (35), we have

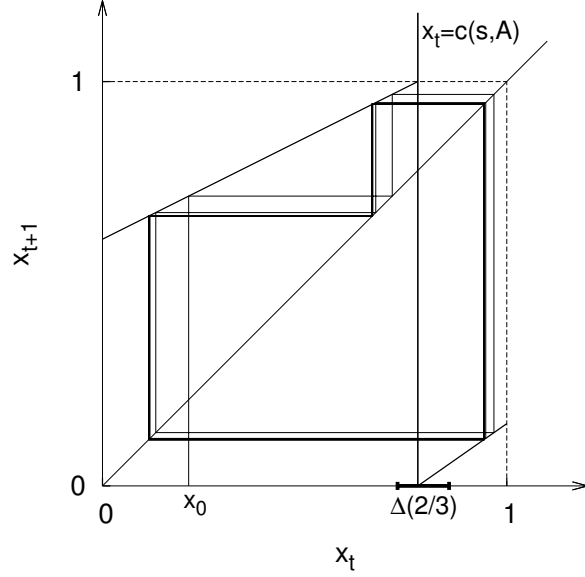
$$\Delta\left(\frac{1}{3}\right) = [L_{1,3}, R_{1,3}] \approx [0.22374, 0.34146],$$

$$\Delta\left(\frac{2}{3}\right) = [L_{2,3}, R_{2,3}] \approx [0.72973, 0.85714].$$

Fig.3 (a) depicts the trajectory generated by Eq.(22) with $c \in \Delta(1/3)$ for an initial condition $x_0 \in (0, 1)$. On the other hand, Fig.3 (b) depicts the case where $c \in \Delta(2/3)$. Indeed, the trajectory of each case converges to a period-3 cycle. We can also observe that the cycle visits the interval $(0, c)$ once within one cycle in case of (a) and twice in case of (b), depending on the numerator of $1/3$ and $2/3$, respectively. It must be noted that the dynamic features described above are independent of initial conditions x_0 .



(a) The trajectory from initial value x_0 for $c(s, A) \in \Delta(1/3)$.



(b) The trajectory from initial value x_0 for $c(s, A) \in \Delta(2/3)$.

Figure 3: Trajectories under Eq.(22) for $a = 0.5$ and $b = 0.7$. In (a), the trajectory converges to the period-3 cycle that visits $(0, c)$ once within one cycle. In (b), the trajectory converges to the period-3 cycle that visits $(0, c)$ twice within one cycle.

1.3.3 Non-periodic dynamics

In this section, we discuss the non-periodic dynamics of our model.

Let

$$\Gamma = [0, 1] \setminus \bigcup_{p/q \in [0, 1] \cap \mathbb{Q}} \Delta(p/q).$$

The set Γ is the remainder obtained from the unit interval $[0, 1]$ by deleting infinite disjoint closed intervals considered in section 1.3.1.

We can show the following.

Proposition 3. *For any a and b with $0 < a < b < 1$, the set Γ is non-empty and uncountable.*

Proof. See Appendix. □

If $c(s, A) \in \Gamma$, Eq.(22) does not have any more periodic cycle. Let

$$\Omega_\tau = \bigcap_{n=0}^{\infty} \text{cl} \tau^n([0, 1]),$$

where $\text{cl}X$ represents the closure of X . Subsequently, as shown in Hata (2014, Theorem 7.1 in p.76 and Theorem 8.5 in p.90), Ω_τ is a compact and totally disconnected uncountable set, such as a Cantor set. Subsequently, we have the following fact:

Proposition 4. *If $c(s, A) \in \Gamma$, for any $x \in (0, 1)$, then the ω -limit set of x , i.e., $\bigcap_{n=0}^{\infty} \bigcup_{k=n}^{\infty} \text{cl} \tau^k(x)$, equals to Ω_τ .*

Proof. See Hata (2014), Theorem 7.4 in p.80 and Theorem 8.5 in p.90. □

We can conclude that, for $c(s, A) \in \Gamma$, Eq.(22) has a *global* non-periodic attractor; it implies that once such a threshold value is chosen, then the economy would fluctuate in a non-periodic manner in the long run for *any* initial condition.

It should be noted, however, that the set Γ is extremely “thin.” In fact, we can assert the following:

Proposition 5. *The Hausdorff dimension of the set Γ is zero.*

Proof. See Hata (2014), Theorem 10.10 in p.139. □

See, for example, Falconer (2003) for the definition of the Hausdorff dimension. Considering the fact that the Cantor Middle Third set (with Lebesgue measure zero) has a positive Hausdorff dimension, while that of Γ is zero, we may say that the set Γ is much sparser than the Cantor set mentioned above. It implies that our model cannot virtually reproduce the non-periodic fluctuations in the long run.

1.4 Rotation number

We introduce a useful tool for describing the dynamical behavior of our model and investigate how the parameter values affect the dynamics of the model. In what follows, we represent Eq.(22) as $\tau_{a,b}$ to emphasize the parameter dependence. For any $x \in (0, 1)$, we define

$$\epsilon(x) = \begin{cases} 0 & (x \in [0, c)), \\ 1 & (x \in [c, 1]). \end{cases}$$

Moreover, we also define

$$\epsilon_k(x) = \epsilon \circ \tau_{a,b}^k(x).$$

It is known that, for any $\tau_{a,b}$, the limit

$$\rho(\tau_{a,b}) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \epsilon_k(x)$$

exists and is independent of the choice of $x \in (0, 1)$. See, for example, Hata (2014), Theorem 3.2 in pp.23-24. The value $\rho(\tau_{a,b})$ is called the *rotation number*⁶ of $\tau_{a,b}$. Clearly, if $c(s, A) \in \Delta(p/q)$, then the rotation number equals to p/q . On the other hand, if $c(s, A) \in \Gamma$, then the rotation number is irrational.

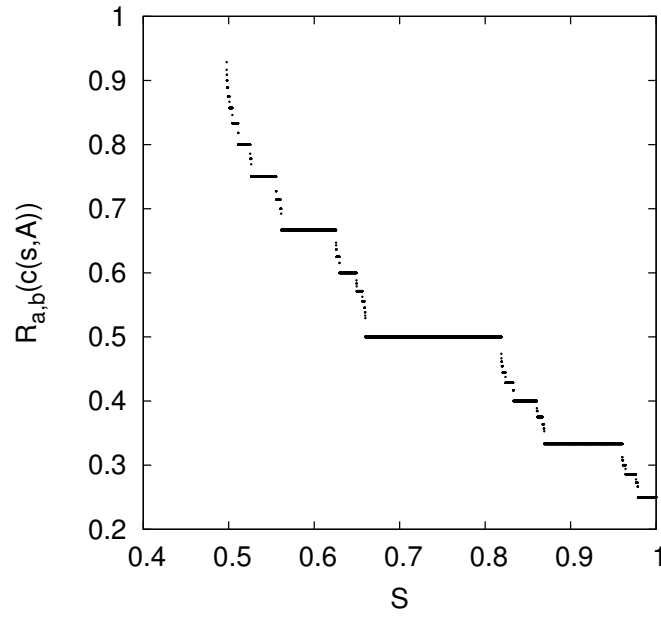
As expected, for the given a and b , we examine the relationship between the $\rho(\tau_{a,b})$ and the value c , and eventually examine the parameter values s and A . In what follows, we regard the rotation number as a function of $c(s, A)$. In this sense, we define

$$R_{a,b}(c(s, A)) = \rho(\tau_{a,b}).$$

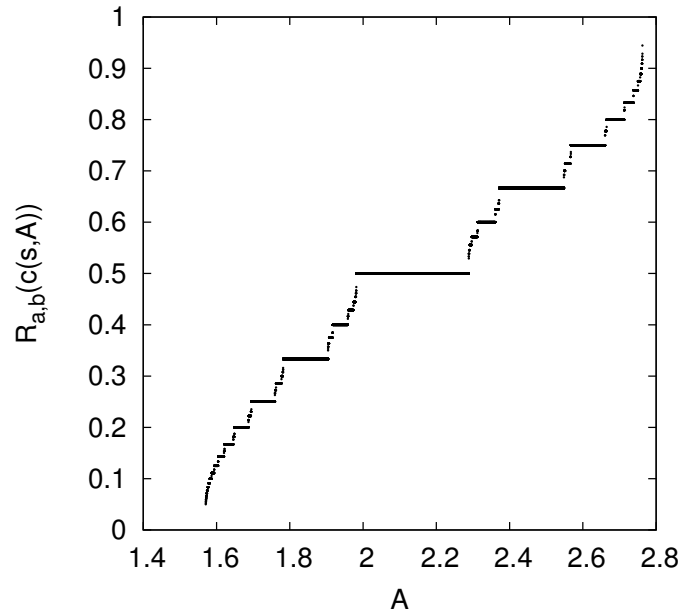
It is known that $R_{a,b}(c(s, A))$ is continuous with respect to c , and its derivative exists and vanishes almost everywhere. See Hata (2014), Theorem 10.11 in p.140. Taking the continuity and the result of Proposition 2 into account, we can see that $R_{a,b}(c)$ is monotonically increasing with respect to the threshold c . Now, we examine the relationship of the threshold c and the parameters s and A . Since $c(s, A)$ is continuous with respect to s and A , $R_{a,b}(c(s, A))$ is also continuous with respect to s and A . Moreover, since $c(s, A)$ is monotonically decreasing with respect to s and increasing with respect to A , we have the following proposition.

Proposition 6. *The rotation number function $R_{a,b}(c(s, A))$ is continuous and monotonically decreasing with respect to s and monotonically increasing with respect to A . Moreover, the partial derivatives of $R_{a,b}(c(s, A))$ exist and vanish almost everywhere.*

⁶The term *rotation number* seems odd in this context. This is a terminology originally used in the dynamical systems theory to characterize the rotating dynamics of a homeomorphism on the circle. The same concept is referred to in different ways, depending on the contexts, such as the *expansion rate* in Ishida and Yokoo (2004) and the *average firing rate* in the neuron context (Nagumo and Sato, 1972).



(a) The graph of rotation number function $R_{a,b}(c(s,A))$ with respect to s .



(b) The graph of rotation number function $R_{a,b}(c(s,A))$ with respect to A .

Figure 4: The devil's staircases: the graphs of the rotation number function $R_{a,b}(c(s,A))$ for $a = 0.5, b = 0.7$, and $B = 2$. (a) $A = 2.2$. (b) $s = 0.7$.

Fig.4 (a) and (b) depict the graph of $R_{a,b}(c(s, A))$ with respect to s and A , respectively. Fig.4 may invoke the well-known Cantor's function. However, contrary to the Cantor's function, the rotation number function is not Hölder continuous for any order. In other words, for any $C, \alpha \in \mathbb{R}_+$, there exist $x, y \in (0, 1)$ such that $|R_{a,b}(x) - R_{a,b}(y)| > C|x - y|^\alpha$.⁷ In this sense, the rotation number function discussed here should be distinguished from the Cantor's function.

Finally, we present our remarks on the rotation number. In Fig.4, we can see the rotation number $2/5$ between the rotation numbers $1/2$ and $1/3$. It is clear that the mediant $(1+1)/(2+3)$ implies the number $2/5$. This is reminiscent of the construction of the *Farey sequence* in the number theory.⁸ The following proposition immediately follows from the continuity and monotonicity of the rotation number function.

Proposition 7. *For any two rotation numbers, $p/q < r/s$, there is a rotation number of the form $(p + r)/(q + s)$ between p/q and r/s .*

Precisely, if $R_{a,b}(c_1) = p/q$ and $R_{a,b}(c_2) = r/s$, then there would exist $c^* \in (c_1, c_2)$ such that $R_{a,b}(c^*) = (p + r)/(q + s)$. Let u/v be the irreducible form of $(p + r)/(q + s)$. Subsequently, proposition 7 says that if an attracting period- q cycle and an attracting period- s cycle are detected for some thresholds c_1 and c_2 , respectively, then one will find another attracting period- v cycle between them that would visit u times the interval $(0, c^*)$, corresponding to the regime of technology 1, and $v - u$ times the interval $(c^*, 1)$, corresponding to the regime of technology 2.

Remark 1. Here, we briefly discuss the mechanism that generates periodic cycles and attempt to provide insight into the economic intuition behind the model.

Fig.5 shows the time series obtained from the model with rotation number $6/7$. At a low level of capital stock, technology 1, which is characterized by (A, a) , yields a higher rate of return to capital than that obtained from technology 2, characterized by (B, b) . Therefore, for an initially low capital stock, the owner of the firm chooses technology 1 and the capital stock increases, as in the case of the traditional OLG model with no technology choice. On the other hand, if the capital stock exceeds some level, technology 2 provides a higher rate of return to capital than that of technology 1. As a result, the owner switches technology 1 to technology 2.

However, it should be noted that a high rate of return to capital does not necessarily imply a high level of wage, nor eventually the amount of saving. Indeed, when the owner changes technology (see box A in Fig.5), the wage rate drops down, which induces a decrease in capital stock in the next period. Subsequently, if the decrease in capital stock is sufficient, the owner will switch back to technology 1. Hence, the story repeats, and the capital stock in the economy exhibits periodic cycles.

For rotation numbers that are not of $\frac{1}{n}$ or $\frac{n-1}{n}$, persistent fluctuations also occur in the same way. However, the switches of technology will occur more frequently in such a case.

⁷See, for example, Lawrence (1998).

⁸See, for example, Hardy and Wright (1979) for the Farey sequence.

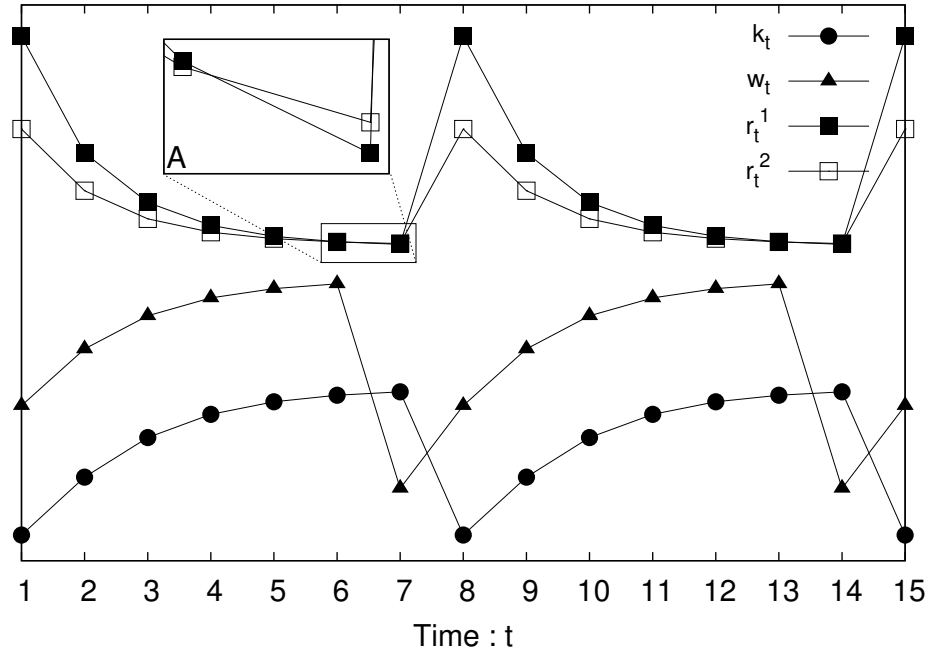


Figure 5: The time series of capital stock k_t , wage rate w_t , and two rates of return to capital, denoted by r_t^1 and r_t^2 for technology 1 and 2, respectively, for rotation number $6/7$.

1.5 Concluding remarks

We have developed a tractable endogenous growth cycle model that is described by a piecewise linear difference equation. Nonlinearity generating perpetual fluctuations are attributed to the choice of technology. Despite the simplicity of the model, its dynamics are rich enough to generate a cycle of any arbitrarily large period. Moreover, owing to the results from the study of mathematical neuron models, we can completely specify the set of parameters for which a specific type of periodic cycle appears. It is also interesting that our purely economic model can be linked with the mathematical “brain,” or neuron, model as well as to the number theory.

We have also shown that the model can exhibit non-periodic behavior for a non-empty and uncountable set of parameters. Nonetheless, such a set of parameters is extremely thin, and hence the non-periodic perpetual fluctuations in the long run are relatively pathological in our model. In other words, our model is virtually incapable of reproducing economic fluctuations that are considered recurrent but not periodic in nature.

However, there might be several ways of extending the model in this study to reproduce non-periodic perpetual fluctuations, such as the *chaotic* behavior. For instance, it is

possible to smooth the discontinuous map in our model by introducing the “misperception” idea of Yokoo and Ishida (2008) and Asano and Yokoo (2019) and thus to determine if the modified model can exhibit chaos as a typical phenomenon. Alternatively, it would also be interesting to introduce a continuum of technologies through which the model might be smoothed without losing strong nonlinearity. These areas could serve as the focus of future research. We hope that the simple model presented here may be used as a building block for other potential endogenous growth cycle or business cycle models in more complex settings.

1.6 Appendix

1.6.1 Proof of Lemma 1

Proof. By arranging $e_2 < \hat{k}$ and $\hat{k} < e_1$ for A , we have $\underline{A} < A$ and $A < \bar{A}$, respectively, where

$$\underline{A} = m_1^{b-a} B^{(1-a)/(1-b)} \quad \text{and} \quad \bar{A} = m_2^{(b-a)(1-a)/(1-b)} B^{(1-a)/(1-b)} \quad (\text{A.1})$$

with

$$m_1 = [s(1-b)]^{1/(1-b)} (b/a)^{1/(b-a)} \quad \text{and} \quad m_2 = [s(1-a)]^{1/(1-a)} (b/a)^{1/(b-a)}. \quad (\text{A.2})$$

It remains to show that $m_1 < m_2^{(1-a)/(1-b)}$. Suppose that this is not the case. Subsequently, we have $m_1 \geq m_2^{(1-a)/(1-b)}$, or

$$[s(1-b)]^{1/(1-b)} (b/a)^{1/(b-a)} \geq [s(1-a)]^{1/(1-a)} (b/a)^{(1-a)/[(b-a)(1-b)]}, \quad (\text{A.3})$$

implying that

$$\frac{1-b}{1-a} \geq \frac{b}{a}, \quad (\text{A.4})$$

or $a \geq b$, which is a contradiction. $\square$

1.6.2 Proof of Lemma 2

Proof. Taking (A.1) and (A.2) of Appendix A into account, a simple calculation shows that, for any $s \in (0, 1)$, $c(s, \underline{A}) = 0$ and $c(s, \bar{A}) = 1$. By the continuity of $c(s, A)$ with respect to A , for any $s^* \in (0, 1)$ and $c^* \in (0, 1)$, there exist $A^* \in (\underline{A}, \bar{A})$ such that $c(s^*, A^*) = c^*$. $\square$

1.6.3 Proof of Proposition 3

Proof. The proposition follows from the fact that *the closed interval* $[0, 1]$ *cannot be partitioned (in the sense of set theory) into a countably infinite number of non-empty closed intervals*. See Sierpiński (1918) or other mathematical textbooks of topology for more details.

The above fact implies that the union of periodic intervals, $\bigcup_{p/q \in [0,1] \cap \mathbb{Q}} \Delta(p/q)$, is not a cover for the closed interval $[0, 1]$. Therefore, the set $\Gamma = [0, 1] \setminus \bigcup_{p/q \in [0,1] \cap \mathbb{Q}} \Delta(p/q)$ is non-empty. Now, suppose that the set Γ is countable. For each $x \in \Gamma$, let $\{x\}$ be a singleton set. Subsequently, the union of the set of periodic intervals and the set of $\{x\}$ becomes countable and gives a partition of the closed interval $[0, 1]$, which contradicts the above fact. Thus, the set Γ is uncountable. $\square$

2 Complex Dynamics of an Inflation Model with a Piecewise Linear Phillips Curve

2.1 Introduction

In this study, we construct a simple inflation-unemployment model exhibiting complex behaviors of macroeconomic variables. The model is composed by simple macroeconomic relations such as the expectations-augmented Phillips curve, static expectations, and a dynamic version of Okun's law, all of which can be found in undergraduate macroeconomic textbooks such as Blanchard (1997).

In addition to these traditional relations, we assume that the incorporated Phillips curve is piecewise linear for the purpose of introducing slight nonlinearity into the model with all other relations being linear. The nonlinearity of the Phillips curve in a macroeconomic model can affect the qualitative features of its dynamics. The topic of nonlinearity has a long history in the literature; however, little attention has been paid to nonlinear dynamics from the viewpoint of the Phillips curve, leaving a dearth of analytical research to this effect. Few exceptions include Soliman (1996a, 1996b) and Chiarella, Flaschel, Gong, and Semmler (2003). Chiarella et al. (2003) introduce a kinked Phillips curve into a Keynesian monetary macroeconomic model to demonstrate the complex dynamics in the model using numerical analysis. Soliman (1996a, 1996b) develop similar inflation model to ours. He introduces a smooth nonlinear Phillips curve, and suggests that the model generates chaotic dynamics using computer simulations.

Adopting piecewise linearity to the Phillips curve has a distinct advantage over other types of nonlinearities. In fact, we show in the sequel that the piecewise linearity of the Phillips curve allows us to derive certain exact analytical results using a direct calculation, which otherwise could not be obtained when the Phillips curve is assumed to have smooth nonlinearity. For instance, our two-dimensional model is rigorously shown to have a transverse homoclinic point (explained in Section 2.3), which implies the existence of a horseshoe, that is, a chaotic invariant set. However, the same could hardly be shown for the Soliman's model, which is equipped with a specific smooth Phillips curve, without relying on numerical methods.

It is worth emphasizing here that establishing the existence of chaos in a higher-dimensional model is not impossible but difficult without computer assistance, even when the model is completely specified. In fact, the economic literature contains few higher-dimensional economic models that are analytically established to be chaotic. See Yokoo (2000) and Chen and Ming (2008) for OLG models; Onozaki, Sieg, and Yokoo (2003) for a cobweb model; Brock and Hommes (1997) for the heterogeneous belief model; and Droste, Hommes, and Tuinstra (2002) for a Cournot-type duopoly with heterogeneous belief. These economic models are established, in a certain sense, as chaotic when the systems are two dimensional but close to a one dimensional. However, owing to piecewise linearity, we can show that, without relying on numerical methods, our model exhibits chaos as well as other nonlinear phenomena, including coexisting periodic attrac-

tors when the system is far from one dimensional. In this sense, the model we present in this study is a tractable exception to the double meaning.

The remainder of this paper is organized as follows. Section 2.2 constructs our model. In section 2.3, we show that our model can exhibit infinitely many periodic orbits coexisting with chaotic invariant sets for a certain set of parameter values. In section 2.4, we show that our model is capable of exhibiting periodic attractors coexisting with chaotic invariant sets. Section 2.5 provides concluding remarks. Some mathematical proofs are delegated to the appendices.

2.2 Model

First, we formalize three macroeconomic relations as in Blanchard (1997).

$$\pi_t = f(u_t) + \alpha\pi_t^e, \quad 0 \leq \alpha \leq 1; \quad (1)$$

$$u_{t+1} = -\beta(g_t - g_n) + u_t, \quad b > 0; \quad (2)$$

$$g_t = m - \pi_t. \quad (3)$$

Eq.(1) describes the expectations-augmented Phillips curve. The gap between the actual inflation rate π_t and the expected inflation rate π_t^e is related with the unemployment rate u_t through the function f . The subscript t denotes time that discretely extends from 0 to infinity. Eq.(2) represents the dynamic version of Okun's law. According to this formula, if the output growth rate is above the natural rate of growth g_n , then the time variance of unemployment decreases. Eq.(3) presents the equilibrium condition in the money market. We assume here that the growth rate of nominal money, m , is constant over time.

We will call the graph of the function f the Basic Phillips curve, because if, instead of Eq.(1), we would relate π_t to u_t directly as $\pi_t = f(u_t)$, it would represent the Phillips curve prior to augmentation with expectations. We assume that the Basic Phillips curve has a piecewise linear form represented by¹

$$f(u_t) = \begin{cases} -a_1 u_1 + b_1 & \text{if } u_t \leq T \\ -a_2 u_1 + b_2 & \text{if } u_t > T, \end{cases} \quad (4)$$

where $a_1 > a_2 > 0, b_1 > b_2 > 0$, and

$$T = \frac{b_1 - b_2}{a_1 - a_2}.$$

In addition to these formulations, we adopt a framework of backward-looking expectations for the inflation rate. To focus on the role of nonlinearity in the Phillips curve in

¹There have been debates about the shape of the Phillips curve of the United States whether it is convex or concave with respect to the unemployment level. See, for example, Laxton et al. (1999) about such debates. Since we merely focus on a set of specific parameter values for the Phillips curve, we assume that our Phillips curve has convex shape in terms of unemployment for the time being.

terms of unemployment, we keep the formulation of expectations for the inflation rate as simple as possible. Thus, we choose the static expectations to describe the dynamics of the inflation rate, that is, we assume

$$\pi_{t+1}^e = \pi_t. \quad (5)$$

Eqs.(1) through (5) can be reduced to the following second-order difference equation:

$$u_{t+2} = (1 + \alpha - \beta a_i)u_{t+1} - \alpha u_t + [(b_i + m(\alpha - 1))], \quad (6)$$

where $i = 1$ if $u_{t+1} \leq T$ and $i = 2$ if $u_{t+1} \geq T$.

Now, we introduce new variables as follows:

$$u_t = Cy_t + T \quad \text{and} \quad y_{t+1} = x_t$$

with

$$C = \frac{\beta[(a_1 b_2 - a_2 b_1) - m(a_1 - a_2)(1 - \alpha)]}{a_1 - a_2}.$$

The change of variables transforms equation (6) into the map:

$$F : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} (1 + \alpha - \beta a_i)x - \alpha y + 1 \\ x \end{pmatrix}, \quad (7)$$

where $i = 1$ if $Cx \leq 0$ and $i = 2$ if $Cx > 0$. We can easily see that the map F is invertible. In fact, the inverse map F^{-1} is represented by

$$F^{-1} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ -\frac{1}{\alpha}(x - (1 + \alpha - \beta a_i)y - 1) \end{pmatrix},$$

where $i = 1$ if $Cx \leq 0$ and $i = 2$ if $Cx > 0$.

In addition, note that the dynamics of (7) directly depends on the parameters a_1, a_2, β , and α . Other parameters such as b_1, b_2 , and m influence the dynamics through C . In this study, we focus on the case where $C < 0$, which occurs, for example, when the growth rate of nominal money m is relatively high.

2.3 Area preserving case: $\alpha = 1$

We begin our analysis drawing on special case where $\alpha = 1$. In this case, the expectation-augmented Phillips curve (1) with Eq.(5) is represented by $\pi_t - \pi_{t-1} = f(u_t)$. Since it relates the change in inflation to the level of unemployment, it is often called the *Accelerationist Phillips curve*.

In the rest of this paper, we present the map F as F_α to emphasize parameter dependence. For $\alpha = 1$, the map F_α becomes

$$F_1 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} (2 - \beta a_i)x - y + 1 \\ x \end{pmatrix}, \quad (8)$$

where $i = 1$ if $x > 0$ and $i = 2$ if $x \leq 0$. For each point in the plane, the Jacobian determinant of the map (8) is unity. In this case, the map is referred to as an area preserving map.

Now we deal with a more specific case. We assume $a_1 = 3$, $a_2 = 1$, and $\beta = 1$. Then, the map (8) is given by

$$\bar{F}_1 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -|x| - y + 1 \\ x \end{pmatrix}. \quad (9)$$

The overline of F represents the restrictions of the parameter values. This specific map (9) is well studied by Aharonov et al. (1997).

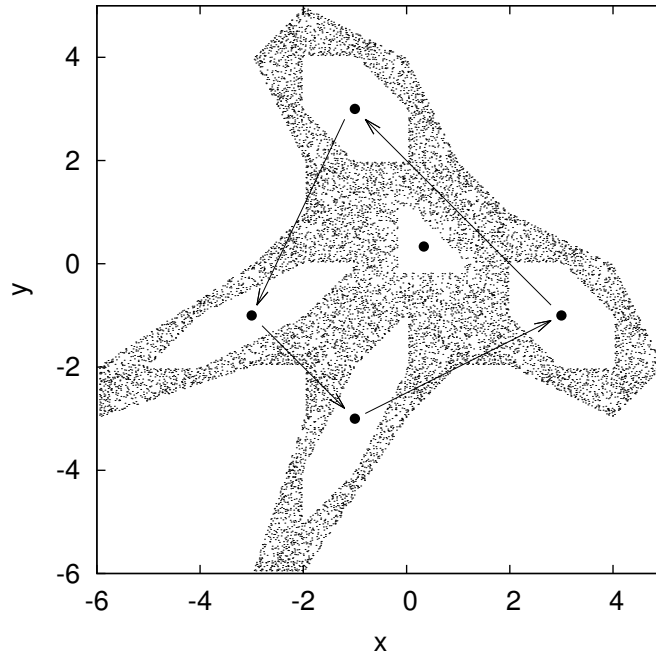


Figure 1: Fixed point, periodic points of period four, and an orbit for some initial condition of the map (9).

Fig.1 depicts the fixed point, periodic points of period four, and an orbit for some initial condition of map (9).¹ We briefly discuss the features of these orbits.

Fig.1 illustrates the polygonal regions and that an orbit wandering on the plane in a complex manner avoids the polygonal regions. The dynamic behaviors on these polygonal regions, however, turn out quite simple. The map (9) has a fixed point at $(\frac{1}{3}, \frac{1}{3})$. Let J be a Jacobian matrix of map (9). In the neighborhood of the fixed point, J is given by

$$J = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \quad (10)$$

¹The fixed point and periodic orbit depicted in the figure are neither an attractor nor a repeller.

and thus, $J^3 = \text{Id}$. It is easy to see that the maximal neighborhood of $(\frac{1}{3}, \frac{1}{3})$, on which $J^3 = \text{Id}$, is a triangle with vertices at $(1, 0)$, $(0, 1)$, and $(0, 0)$. Therefore, all points within the triangular regions, except the fixed point, have period 3. The map (9) has the periodic points of period four at $(3, -1)$, $(-1, 3)$, $(-3, -1)$, and $(-1, -3)$. Let J_4 be a Jacobian matrix of the fourth iterate of the map (9). In the neighborhood of the point $(3, -1)$, for instance, J_4 is given by

$$J_4 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \quad (11)$$

and thus, $J_4^6 = \text{Id}$. Therefore, all points in the neighborhood of each periodic point have period 24.

On the other hand, it can be slightly difficult to understand the dynamics outside of the polygons. Here, we provide an elementary and geometrically intuitive proof of the occurrence of chaotic behavior. More specifically, we attempt to find a transverse homoclinic point, which is a point on which the stable and unstable manifolds of a fixed point intersect transversely. If such a point exists, then according to the Birkhoff-Smale homoclinic theorem, there exists a “horseshoe,” that is, a chaotic invariant set. See, for example, Guckenheimer and Holmes (1997) for more details.

Proposition 1. *The map F given by (9) is chaotic. In fact, F^4 has a transverse homoclinic point and therefore a horseshoe.*

Proof. First, we can easily check that the map F has a period four saddle cycle that consists of periodic points $p_1 = (1, 1)$, $p_2 = (-1, 1)$, $p_3 = (-1, -1)$ and $p_4 = (1, -1)$ such that $p_{i+1} = F(p_i)$ for $i = 1, 2$, and 3 and $p_1 = F(p_4)$. Let A be the set bounded by the threshold lines $y = -2x + 2$, $y = -x + 1$, and $x = 2$ with $x > 0$. And let B be the set bounded by the threshold lines $y = \frac{1}{2}x + 1$ and $y = 2$ with $x < 0$. See Fig.2. Minor consideration shows that the fourth iterate of F restricted to A , $F_{|A}^4$, is given by

$$F_{|A}^4 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 3x + y - 3 \\ -x + 2 \end{pmatrix}. \quad (12)$$

Similarly, the fourth iterate of F restricted to B , $F_{|B}^4$, is given by

$$F_{|B}^4 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x - y + 1 \\ -x + 2y - 2 \end{pmatrix}. \quad (13)$$

Now, we calculate the stable manifolds W^s and the unstable manifolds W^u of the fixed points p_i ($i = 1, 2, 3, 4$) for F^4 . A simple calculation shows that $W^u(p_1)$ crosses the line segment $y = -2x + 2$ at $c = (\frac{3-\sqrt{5}}{2}, \sqrt{5}-2)$. However, (12) maps the point c to $F(c) = (\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}) \in B$, which implies that $W^u(p_1)$ does not kink up to $F(c)$ from p_1 . Since the slope of $W^s(p_2)$ is $\frac{\sqrt{5}-1}{2}$, $W^u(p_1)$ transversely intersects with $W^s(p_2)$ at $p_h = (2 - \sqrt{5}, -3 + 2\sqrt{5})$. That is, $p_h \in W^u(p_1) \cap W^s(p_2)$ is a transverse heteroclinic point. See Fig.2.

By the same argument, together with the symmetric nature of the map, we can see that $W^u(p_2) \cap W^s(p_3)$, $W^u(p_3) \cap W^s(p_4)$, and $W^u(p_4) \cap W^s(p_1)$ each has a transverse heteroclinic point. Take a disc $D^{HU} \subset W^u(p_1)$ such that $p_h \in D^{HU}$. Take another disc $D^u \subset W^u(p_1)$ such that $p_1 \in D^u$. Then, by successive application of Palis's λ -lemma (see e.g. Palis and de Melo (1982), especially, Corollary 1, p.85.), we can see that for sufficiently large n , $F^{4n}(D^{HU})$ contains a disc D^{UU} which is sufficiently C^1 -close to D^u . This says that there is a transverse intersection of $W^u(p_1) \cap W^s(p_1)$. $\square$

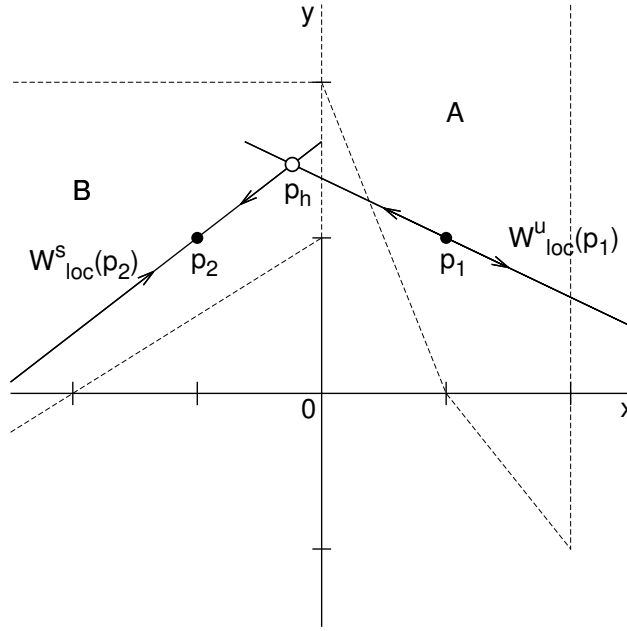


Figure 2: Formation of a transverse heteroclinic point $p_h \in W^u_{loc}(p_1) \cap W^s_{loc}(p_2)$. Dotted lines represent the thresholds of the fourth iterate of map (9).

Fig.2 presents the local unstable manifold $W^u_{loc}(p_1)$ and local stable manifolds $W^s_{loc}(p_2)$.

Here, with the help of computer simulation, we are able to observe the structure that generates complex dynamics. Fig.3 depicts $W^s(p_i)$ and $W^u(p_i)$ ($i = 1, 2, 3, 4$). The invariant manifolds form a complicated structure called homoclinic tangles, which are accompanied by the Smale's horseshoe.

Finally, it should be noted that the long-run behaviors of macroeconomic variable are *qualitatively different*, depending on initial conditions. Since our dynamical system is area-preserving, there are no attractors. Therefore, depending on the initial conditions, one trajectory can be periodic while the other is chaotic even in the long run.

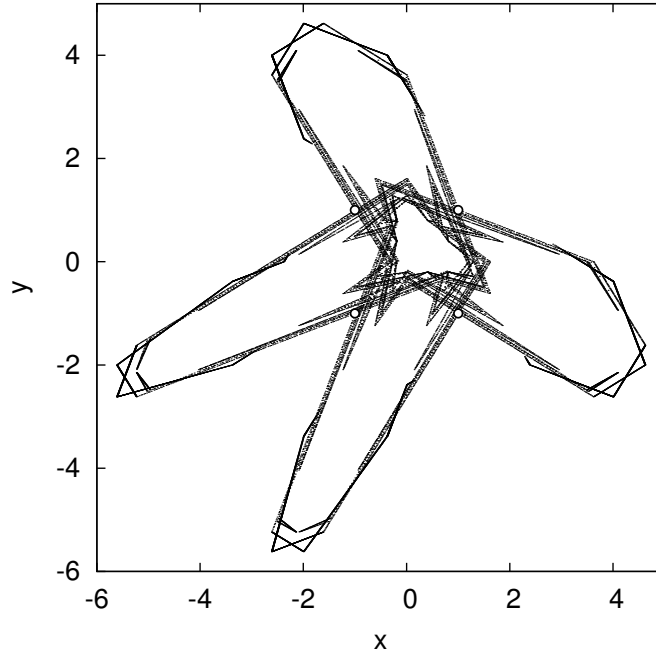


Figure 3: Stable and unstable manifolds of periodic points of map (9) (The invariant manifolds transversely intersect infinitely many times and form homoclinic/heteroclinic structures that induce chaotic behaviors.)

2.4 Dissipative case : $\alpha \neq 1$

In the previous section, we analyzed the area preserving case. Although there are some interesting dynamical features, such a case may be regarded as somewhat pathological in economics. Indeed, some properties in the area preserving case disappear with slight perturbation of degree α . In this section, we analyze the case where $\alpha < 1$. In particular, we focus on the case where degree α is slightly perturbed from unity. As a result, we demonstrate the coexistence of periodic attractors and a chaotic invariant set.

As in the previous section, we let $a_1 = 3$, $a_2 = 1$, and $\beta = 1$. In this setting, the map $\bar{F}_\alpha$ is represented in the following two parts:

$$\bar{F}_\alpha^- : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \alpha x - \alpha y + 1 \\ x \end{pmatrix} \quad (14)$$

if $x \leq 0$ and

$$\bar{F}_\alpha^+ : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} (\alpha - 2)x - \alpha y + 1 \\ x \end{pmatrix} \quad (15)$$

if $x > 0$.

The following propositions demonstrate the coexistence of an attracting fixed point and periodic cycle whose period is larger than one.

Proposition 2. *For $\alpha < 1$, but sufficiently close to 1, the map $\bar{F}_\alpha$ given by (14) and (15) has an attracting fixed point.*

Proof. See Appendix. □

Proposition 3. *For $\alpha < 1$, but sufficiently close to 1, the map $\bar{F}_\alpha$ given by (14) and (15) has an attracting period-four cycle.*

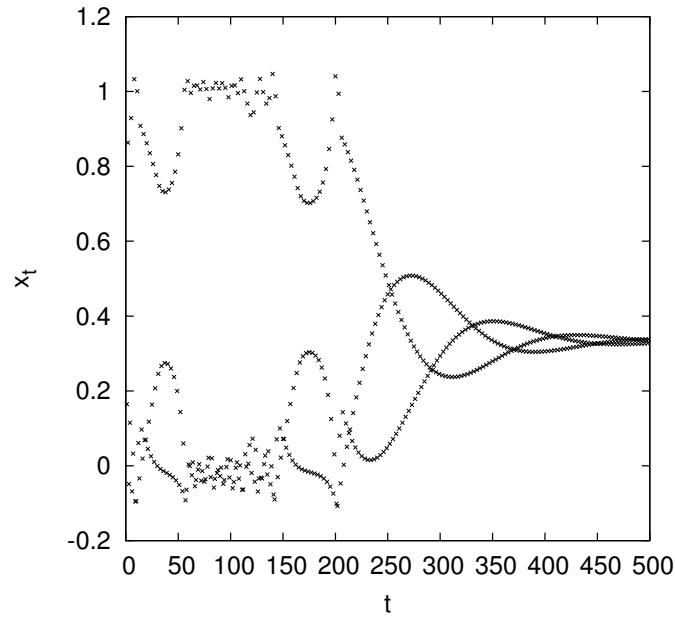
Proof. See Appendix. □

Note that since the periodic points in the propositions are hyperbolic, the propositions are true for (a_1, a_2, β) sufficiently near $(3, 1, 1)$. More importantly, the horseshoes shown in the previous section persist for any (a_1, a_2, β) sufficiently near $(3, 1, 1)$ owing to hyperbolicity. Accounting for this fact, we summarize propositions 1 through 3 as the following theorem.

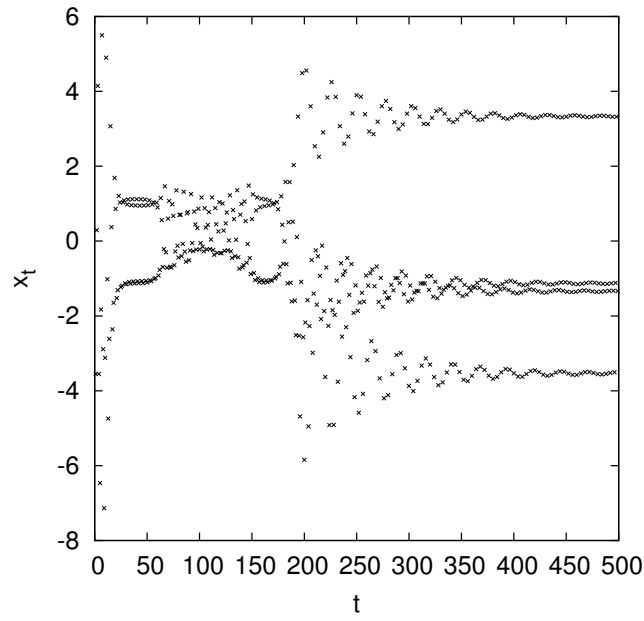
Proposition 4. *Let $C < 0$. If $(\alpha, a_1, a_2, \beta) \approx (1, 3, 1, 1)$, and $\alpha < 1$, then map F given by (7) has multiple attracting cycles and chaotic invariant sets.*

The proposition implies *transient chaos*. That is, even if a trajectory ends up converging to an attracting cycle, it may continue to fluctuate in a complicated manner for a long time. Fig.4 (a) and (b) depict a time series that converges to an attracting fixed point and an attracting period-four cycle, respectively. In both cases, we observe complex transient motions in the early stage.

Remark 1. Even if $C > 0$, we show the same result as the theorem above using proofs similar to those for propositions 1 through 3. In such a case, we find that an attracting fixed point, attracting period-five cycle, and chaotic invariant sets coexist. However, we omit such a case because it significantly increases the length of this paper.



(a) Unemployment levels that converge to an attracting fixed point.



(b) Unemployment levels that converge to an attracting period-four cycle

Figure 4: Time series of unemployment levels for fixed parameters $(a_1, a_2, \beta) = (3, 1, 1)$ and $\alpha = 0.95$.

2.5 Concluding remarks

In this study, we introduced a piecewise linear Phillips curve in a simple inflation–unemployment model to show that it exhibits rich dynamics. Piecewise linearity allowed us to visually capture the geometric structure that generates complex behaviors.

More specifically, we first found that our model is an area preserving map for some set of parameter values. In this case, the model exhibits infinitely many periodic and chaotic orbits.

Next, we showed that the model close to the area preserving case has coexisting periodic attractors with transient chaos because of the Smale’s horseshoes. This implies that a trajectory with its initial point near a horseshoe may converge to a periodic (including period one) attractor but exhibits complicated transient fluctuations. If exogenous shocks are added to the model, the complicated behaviors inherently induced by the chaotic invariant sets might be more persistent. However, since we cannot ignore such exogenous effects in the real economy, our findings can, to a certain extent, explain the mechanism of non-stationary and complex behaviors of macroeconomic variables.

Finally, we point out that our model is expected to generate another type of chaotic behavior. In fact, computer simulations by the author suggest that the model exhibits strange attractors for other sets of parameter values. We, however, leave for future research.

2.6 Appendix

2.6.1 Proof of Proposition 2

Proof. Clearly, the map F_α has a fixed point $(\frac{1}{3}, \frac{1}{3})$. Now, we can easily check that the Jacobian matrix of $\bar{F}_1^+$ has conjugate complex eigenvalues

$$\lambda_{1,2} = \frac{-1 \pm i\sqrt{3}}{2}.$$

For a smooth α dependence of the eigenvalues, the Jacobian matrix of $\bar{F}_\alpha^+$ also has conjugate complex eigenvalues for α sufficiently near 1. In addition, the Jacobian determinant of $\bar{F}_\alpha^+$ is $\alpha < 1$. Therefore, the absolute value of the eigenvalues is less than 1. $\square$

2.6.2 Proof of Proposition 3

Proof. Recall that the map $\bar{F}_1$ has periodic points in period four at

$$(3, -1), (-1, 3), (-3, -1), \text{ and } (-1, -3).$$

We expect that the periodic points corresponding to those of the map $\bar{F}_1$ exist for α sufficiently near 1. In fact, we directly calculate such a point because of linearity.

Recall matrix (11). The Jacobian matrix of $\bar{F}_1^4$ restricted to the neighborhood of the point $(3, -1)$ has conjugate complex eigenvalues

$$\lambda_{1,2} = \frac{1 \pm i\sqrt{3}}{2},$$

where F^n represents the n -th iterate of the map F . For a smooth α dependence of the eigenvalues, the Jacobian matrix of $\bar{F}_\alpha^4$ restricted to the neighborhood of the point $(3, -1)$ has conjugate complex eigenvalues for α sufficiently near 1. The straightforward calculation shows that the Jacobian determinant of $\bar{F}_\alpha^4$ restricted to the neighborhood of the point $(3, -1)$ is $\alpha^4 < 1$. Therefore, the absolute value of the eigenvalues is less than 1. $\square$

3 Chaotic Dynamics of a Piecewise Smooth Overlapping Generations Model with a Multitude of Technologies

3.1 Introduction

There is no doubt that the overlapping generations (OLG, hereafter) model is one of the most popular dynamic economic models in the literature. Especially, the OLG setting plays an important role in constructing an endogenous growth (or business) cycle model, which exhibits, without external shocks, a perpetual periodic or chaotic (i.e., random-looking but deterministic) fluctuation. In early literature, a few prominent examples of such perpetual fluctuation in an OLG model can be found in the studies by Benhabib and Day (1982) and Grandmont (1985).

Matsuyama (2007) presents an OLG-type model of credit cycles induced by financial imperfections; he shows that the model has the capability to exhibit several dynamic growth patterns, including perpetual fluctuations. Asano et al. (2012) focus on a special case of the model by Matsuyama (2007) and show that, in such a case, the model can exhibit periodic and non-periodic fluctuations. Iwaisako (2002) investigates a situation in which investors can choose from among two production technologies—constant returns to scale and increasing returns to scale—and shows several dynamic patterns such as perpetual fluctuation and permanent economic growth. It should be noted that, in the above studies, each model can eventually reduce to a one-dimensional map with discontinuity. In these models, the set of parameter values for which a complex fluctuation can appear is extremely “thin” even if there exists a complex fluctuation theoretically. In other words, the chaotic motions cannot be observed virtually.

Along this line of discrete choice in an OLG model, Umezaki and Yokoo (2018) construct an explicit OLG model in which the firm faces a binary choice problem related to the choice of technology (represented by the Cobb-Douglas production function of constant returns to scale). Similar to the above studies, our model can eventually reduce to a piecewise linear map with discontinuity. Such piecewise-linearization, similar to Asano et al. (2012), helps us to directly apply to our model some useful results borrowed from prior studies on mathematical neuron models. The employment of this method completely characterizes the existence and occurrence of periodic fluctuations in the model. However, the model also cannot virtually reproduce complex fluctuations, because it has a small set of parameter values for which there exists non-periodic fluctuation in the model.

Of course, we can frequently observe *non*-periodic, complex fluctuation of economic variables in the real world. Therefore, it is natural that, along this line, we develop a model that can virtually reproduce the chaotic fluctuations in the long run. Especially, this study aims to extend the model in Umezaki and Yokoo (2018). To do this, our idea is to smooth the map with discontinuity through some modifications. If this new piecewise smooth map has sufficient nonlinearity, we can easily conjecture that the model has the capability to generate observable complex fluctuations.

It must be noted that the approach to smooth the discontinuous model has been dis-

cussed in context. Yokoo and Ishida (2008) introduce imperfect observability, which they call misperception. It is represented by a parameterized random variable in the state variable to smooth Ishida and Yokoo's (2004) piecewise linear model with discontinuity, which is a macroeconomic, but not an OLG, model with discrete choice. Asano and Yokoo (2019) also adopt this idea into the model in Asano et al. (2012) and investigate the chaotic dynamics. However, in general, the abovementioned smooth model with imperfect observability is too general to investigate its dynamics in detail. Indeed, these studies make a somewhat strict assumption about misperception, in which the model can reduce to a piecewise *linear* map. Although such linearization gives us depth and clear results about the complex dynamics of the model, there is a tradeoff between tractability and generality in this method.

On the other hand, in this study, we use a more primitive and intuitive way to smooth Umezaki and Yokoo's (2018) piecewise linear map with discontinuity. There is only one modification: we assume that the firm faces a *continuous* choice problem of technology rather than a binary choice problem. As a result, we can transform, without additional assumption, a piecewise linear model with discontinuity into a continuous piecewise smooth model that can virtually reproduce the chaotic motion and is tractable enough to investigate such a fluctuation. In this sense, our model can cope with both generality and tractability. Matsuyama (2013) and Matsuyama et al. (2016) adopt the same method, which assumes a continuous choice instead of discrete choice, to smooth the discontinuous model in Matsuyama (2007). However, our model is simpler than Matsuyama's in terms of the story and motivation behind the modeling.

To study the complex dynamics in continuous piecewise smooth maps, we adopt two approaches: *border collision bifurcation* and *Markov property*. That is, we obtain two different characterizations of chaotic dynamics in the model. The first approach includes a bifurcation that occurs in piecewise smooth, piecewise linear, and piecewise non-linear maps. This theory reveals the bifurcation from a globally attracting steady state to other non-stationary behaviors using only local information in the system. Especially, we can completely characterize the bifurcation from an attracting steady state to another non-stationary attractor, such as periodic cycles or *chaotic bundles*. Gardini et al. (2008) and Matsuyama et al. (2016) use the border collision bifurcation theory to investigate the chaotic dynamics in their models. Second, we find that our model can have a special property—a Markov property for some sets of parameter values. If the model has such a property, we can easily establish and characterize its complex dynamics. Yokoo and Ishida (2008) and Asano and Yokoo (2019) find the Markov property in their piecewise linear models to analyze the chaotic dynamics. Therefore, from a technical viewpoint, our model has an advantage against the preceding models because we can characterize the chaotic fluctuations in the model using two different approaches.

The remainder of this paper is organized as follows. Section 3.2 re-examines the OLG model with a binary choice of technologies. Section 3.3 derives the piecewise smooth model by introducing a new assumption, a multitude of choices in the technologies, into the OLG model with a binary choice setting. Section 3.4 describes the border collision bi-

furcation theory characterized by routes from the globally attracting steady state to other non-stationary cycles, including chaotic fluctuation. Section 3.5 reveals that the capability of the model has a Markov property for some set of parameter values. Section 3.6 concludes the study. Some mathematical proofs are delegated to the appendices.

3.2 The model under binary technology choice

In this section, for the reader's convenience, we re-examine the base model of this study. (See Umezaki and Yokoo (2018) for more details.) We consider a Diamond-type one-sector OLG model that is modified further. Time is discrete, that is, $t = 0, 1, 2, \dots$, and the agents live for two periods. A young household supplies one unit of labor inelastically. We keep the utility function, u , of the household as simple as possible; this allows us to assume that it is in the form of a log-linearized Cobb-Douglas production function, that is,

$$u(c_t^y, c_{t+1}^o) = (1 - s) \log c_t^y + s \log c_{t+1}^o, \quad s \in (0, 1), \quad (1)$$

where c_t^y denotes the amount of consumption of the young generation born at time t , and c_{t+1}^o denotes the amount of consumption of the old generation living at time $t + 1$. The utility given by Eq.(1) is maximized under the following constraints:

$$c_t^y + s_t = w_t \quad \text{and} \quad c_{t+1}^o = r_{t+1} s_t, \quad (2)$$

where s_t , w_t , and r_{t+1} are the amounts of saving, real wage rate, and real gross rate of return, respectively. The maximization yields

$$s_t = s w_t. \quad (3)$$

The final good, Y_t , which is perishable, is produced by the firm. Unlike the common OLG models, we assume that there are two types of production technologies. We also assume that, at the beginning of every period, the firm faces a discrete choice problem related to the choice of technology. For simplicity, all the technologies are specified as the Cobb-Douglas of constant returns to scale:

$$\begin{aligned} Y_t &= F_a(K_t, L_t) = A K_t^a L_t^{1-a} \quad \text{and} \\ Y_t &= F_b(K_t, L_t) = B K_t^b L_t^{1-b}, \end{aligned}$$

where $A, B > 0$ is the total factor productivity, and $0 < a < b < 1$ is the capital share of the production of technology. In the per-capita form, we can write

$$y_t = f_i(k_t) = F_i(k_t, 1), \quad (4)$$

where $y_t = Y_t/L_t$, $k_t = K_t/L_t$, and $i \in \{a, b\}$. We assume that the firm's owner, who belongs to the old generation, chooses technology that earns the highest return. For simplicity, we assume that when the highest rates of return are tied among two technologies, technology

1 characterized by (a, A) is chosen. Subsequently, the usual first-order conditions with the technology choice are represented by

$$r_t = f'_{J_t}(k_t), \quad (5)$$

$$w_t = f_{J_t}(k_t) - k_t f'_{J_t}(k_t), \text{ and} \quad (6)$$

$$J_t = \arg \max_{j \in \{a, b\}} f'_j(k_t). \quad (7)$$

The market clearing condition,

$$k_{t+1} = s_t, \quad (8)$$

with Eqs.(3), (6) and (7) generate the dynamic model in the following form:

$$k_{t+1} = \begin{cases} sA(1-a)k_t^a & 0 \leq k_t \leq \hat{k}, \\ sB(1-b)k_t^b & \hat{k} < k_t, \end{cases} \quad (9)$$

where

$$\hat{k} = \left[\frac{aA}{bB} \right]^{1/(b-a)}. \quad (10)$$

It is clear that if Eq.(9) has a positive fixed point, it would be globally attracting. As we are interested in non-stationary behaviors, we examine the case where Eq.(9) has no positive fixed point. The following lemma shows that there is a parameter set for which Eq.(9) has no positive fixed point.

Lemma 1. *Let $a, b \in (0, 1)$ with $a < b$, $B > 0$, and $s \in (0, 1)$ be given. Subsequently, there exist $\underline{A} > 0$ and $\bar{A} > 0$ with $\underline{A} < \bar{A}$ such that map (9) has no positive fixed point for $A \in (\underline{A}, \bar{A})$.*

Proof. See Umezaki and Yokoo (2018). □

Finally, by defining a new variable x_t as

$$x_t = \frac{\log [k_t/s(1-b)B\hat{k}^b]}{\log b(1-a)/a(1-b)}, \quad (11)$$

Eq.(9) can reduce to the following piecewise linear map on the unit interval:

$$x_{t+1} = \tau(x_t) = \begin{cases} 1 + a(x_t - c) & \text{if } 0 \leq x_t \leq c, \\ b(x_t - c) & \text{if } c < x_t \leq 1, \end{cases} \quad (12)$$

where

$$c = c(s, A) = \frac{\frac{1-b}{b-a} \log [aA/bB] - \log s(1-b)B}{\log b(1-a)/a(1-b)}. \quad (13)$$

It is interesting to note that Eq.(12) can be identified with a simplified version of Caianiello's equation in the neural networks, whose original model is proposed by Caianiello to describe the behavior of a “model of brain” or “thinking machine”; this model

is studied by Nagumo and Sato (1972) in detail. Additionally, Hata (2014) comprehensively analyzes this type of equation, and we utilize this author's analysis results for the mathematical neuron model to study economic growth.

When investigating Eq.(12), the position of threshold $c(s, A)$ on the unit interval plays an important role. The position of the threshold characterizes the dynamical property of the model. Generally, for a given value of c , Eq.(12) has either a globally attracting periodic cycle or a non-periodic attractor.

The following lemma shows that threshold $c(s, A)$ can take any value within the range of $(0, 1)$, independently of parameters a and b .

Lemma 2. *For any $a, b \in (0, 1)$ with $a < b$ and any $c^* \in (0, 1)$, there exist $s^* \in (0, 1)$ and $A^* \in (\underline{A}, \bar{A})$ such that $c(s^*, A^*) = c^*$.*

Proof. See Umezuki and Yokoo (2018). □

Now, we restate the existence of a periodic orbit in the model.

Proposition 1. *For each irreducible fraction $p/q \in (0, 1)$, there exists a closed interval $\Delta(p/q) \subset (0, 1)$ such that if $c(s, A) \in \Delta(p/q)$, for any $x \in (0, 1)$, the orbit of x converges to some periodic orbit of period q .*

Proof. See Hata (2014), Theorem 4.2 on p.36 and Theorem 10.1 on pp.117–118. □

Thus, we can almost completely grasp the periodic dynamical features of the model by verifying to which closed interval threshold c belongs. Moreover, owing to the piecewise linearity of the model, we can exactly calculate the left and right endpoints of the closed interval $\Delta(p/q)$ for any p/q . See Umezuki and Yokoo (2018) for more details on this point.

Next, we can consider the following set:

$$\Gamma = [0, 1] \setminus \bigcup_{p/q \in [0, 1] \cap \mathbb{Q}} \Delta(p/q).$$

Set Γ is the remainder obtained from the unit interval $[0, 1]$ by Deleting the infinite closed intervals. If $c \in \Gamma$, Eq.(12) has a non-periodic attractor.

Proposition 2. *If $c(s, A) \in \Gamma$, for any $x \in (0, 1)$, the ω -limit set of x ; that is, $\bigcap_{n=0}^{\infty} \bigcup_{k=n}^{\infty} \text{cl}\tau^k(x)$, is some compact and totally disconnected uncountable set on the unit interval.*

Proof. See Hata (2014), Theorem 7.4 on p.80 and Theorem 8.5 on p.90. □

In the proposition, $\text{cl}A$ represents closure of set A . We can conclude that, for $c(s, A) \in \Gamma$, Eq.(12) has a *global* non-periodic attractor; it implies that once such a threshold value is chosen, the economy would fluctuate in a non-periodic manner in the long run for *any* initial condition,. However, the fact is that set Γ is extremely “thin.” The Hausdorff dimension of set Γ is zero. This implies that this base model cannot virtually reproduce non-periodic fluctuations in the long run.

3.3 The model under continuum technology choice

In what follows, we modify the base model examined in the previous section to construct a simple, tractable, OLG model with technology choice to virtually reproduce non-periodic fluctuations in the long run.

To do this, we make only one modification. We assume that the firm faces a continuum of technologies. That is, there is a continuum of production functions,

$$Y_t = F_\alpha(K_t, L_t) = AK_t^\alpha L_t^{1-\alpha}, \quad \alpha \in [a, b],$$

and the firm can choose this before the production of a good. Here, we also assume that the parameter of total factor productivity, given as A and B in the binary choice setting, is constant through all technologies.

Therefore, we face the dynamic model in the following form:

$$\begin{cases} k_{t+1} = sA(1-\alpha)k_t^\alpha, \\ \alpha = \arg \max_{\alpha^* \in [a, b]} \{A\alpha k_t^{\alpha^*-1}\}, \end{cases} \quad \text{and } k_0 > 0. \quad (14)$$

It is easily shown that α is given by

$$\alpha = \begin{cases} a, & k_t \leq e^{-\frac{1}{a}}, \\ -\frac{1}{\log k_t}, & \text{if } e^{-\frac{1}{a}} < k_t \leq e^{-\frac{1}{b}}, \\ b, & e^{-\frac{1}{b}} < k_t. \end{cases} \quad (15)$$

Subsequently, we have the following piecewise-smooth dynamic model:

$$k_{t+1} = \begin{cases} \beta(1-a)k_t^a, & k_t \leq e^{-\frac{1}{a}}, \\ \beta\left(1 + \frac{1}{\log k_t}\right)k_t^{-\frac{1}{\log k_t}}, & \text{if } e^{-\frac{1}{a}} < k_t \leq e^{-\frac{1}{b}}, \\ \beta(1-b)k_t^b, & e^{-\frac{1}{b}} < k_t, \end{cases} \quad (16)$$

where $\beta = sA$.

By defining a new variable $y_t = \log k_t$, Eq.(16) turns out to be

$$y_{t+1} = h(y_t) = \begin{cases} h_1(y_t) = ay_t + \log \beta(1-a), & y_t \leq -\frac{1}{a}, \\ h_2(y_t) = \log\left(1 + \frac{1}{y_t}\right) + \log \beta - 1, & \text{if } -\frac{1}{a} < y_t \leq -\frac{1}{b}, \\ h_3(y_t) = by_t + \log \beta(1-b), & -\frac{1}{b} < y_t. \end{cases} \quad (17)$$

Fig.1 depicts the graph of map (17), which has a fixed point in $\left(-\frac{1}{a}, -\frac{1}{b}\right)$.

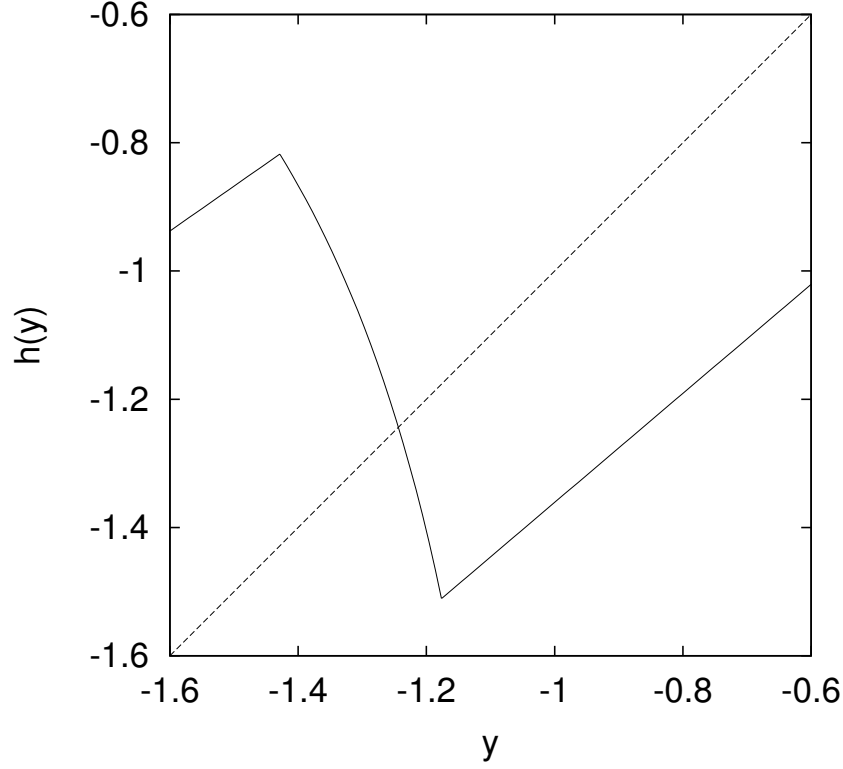


Figure 1: The graph of Piecewise smooth map (17), which has a fixed point in $(-\frac{1}{a}, -\frac{1}{b})$. $a = 0.7$, $b = 0.85$, and $\beta = 4$.

The graphical argument shows that if Eq.(17) has a fixed point on $(-\infty, -\frac{1}{a}]$ or $[-\frac{1}{b}, \infty)$, it would be globally attracting. Elementary algebra shows the following proposition that detects the set of parameter $\beta = sA$, for which the model exhibits the globally attracting steady state.

Proposition 3. *Let*

$$\beta^*(u, v) = \frac{1}{1-v} \exp\left(-\frac{1-u}{u}\right), \quad 0 < u, v < 1.$$

For any $a < b$, if $\beta < \beta^(a, a)$ or $\beta^*(b, b) < \beta$, map (17) has a fixed point on $(-\infty, -\frac{1}{a}]$ or $[-\frac{1}{b}, \infty)$, respectively.*

Here, it should be noted that, for any $a < b$, we have $\beta^*(a, a) < \beta^*(b, b)$.

As we are interested in non-stationary behaviors, we examine the case wherein Eq.(17) has a positive steady state on the interval $(-\frac{1}{a}, -\frac{1}{b})$. That is, we mainly focus on the following case:

$$\beta^*(a, a) < \beta < \beta^*(b, b).$$

Let us further classify the model. It is evident that every trajectory generated by (17) eventually enters the trapping interval $T = \left[h_2(-\frac{1}{b}), h_1(-\frac{1}{a}) \right]$ and never leaves it. Thus, to study the long-run behavior of the model in this situation, it is sufficient to focus on the dynamics of the trapping interval, I . It must be noted that, depending on parameter values a and b , we face the following two scenarios.

1. If $\frac{a-b}{ab} > \log \frac{(1-b)}{(1-a)}$, then $\beta^*(b, a) < \beta^*(a, b)$. We have the following cases:
 - If $\beta^*(a, a) < \beta < \beta^*(b, a)$, then T contains the right threshold only.
 - If $\beta^*(b, a) < \beta < \beta^*(a, b)$, then T contains all the thresholds.
 - If $\beta^*(a, b) < \beta < \beta^*(b, b)$, then T contains the left threshold only.
2. If $\frac{a-b}{ab} < \log \frac{(1-b)}{(1-a)}$, then $\beta^*(a, b) < \beta^*(b, a)$. We have the following cases:
 - If $\beta^*(a, a) < \beta < \beta^*(a, b)$, then T contains the right threshold only.
 - If $\beta^*(a, b) < \beta < \beta^*(b, a)$, then T contains no thresholds.
 - If $\beta^*(b, a) < \beta < \beta^*(b, b)$, then T contains the left threshold only.

If T contains only the right (left) threshold, the model can be regarded as a unimodal map. On the other hand, if T contains both thresholds, the model reduces to a bimodal map on the unit interval. Finally, if T contains no thresholds, the model can be regarded as a monotonically decreasing map, which can have a periodic cycle at most period two. It is reasonable that we omit such an insignificant case.

As we will see later, both unimodal and bimodal maps in our model are capable of generating chaotic behaviors. We investigate the chaotic dynamics of the unimodal and bimodal cases in sections 3.4 and section 3.5, respectively.

3.4 Chaotic dynamics : Border collision bifurcation

In this study, we adopt two approaches to investigate the chaotic behaviors in the model. In this section, we deal with *border collision bifurcation* to investigate the dynamical property of the unimodal case. In this case, we can locally, but almost completely, characterize the bifurcation from a globally attracting steady state to non-stationary behaviors; that is, bifurcation from a globally attracting steady state to a periodic cycle or a *chaotic bundle*. Roughly speaking, a one-dimensional piecewise map may undergo border collision bifurcation when the 45 degree line touches the graph of the map at its non-smooth boundaries. Therefore, in our model, border collision bifurcation occurs when β exceeds $\beta^*(a, a)$ as well as β approaches $\beta^*(b, b)$. To avoid increasing the length of the paper, we focus on the former case, in which the model can be represented by a single peaked unimodal map. Of course, the same analytical method is adopted for the latter case, in which the model can be represented by a V-shaped unimodal map. In section 3.4.1, we briefly discuss some mathematical results for border collision bifurcation. In section 3.4.2, we adopt the results to our OLG model with technology choice.

3.4.1 Some mathematical results from border collision bifurcation

All results in this section follow from Nusse and Yorke (1995).

Before we examine the piecewise smooth map (17), we consider the following piecewise linear map:

$$F_{v_1, v_2} = \begin{cases} v_1 x + 1 & x \leq 0, \\ v_2 x + 1 & 0 < x, \end{cases} \quad (18)$$

with $0 < v_1 < 1$ and $v_2 < -1$. For convenience, let

$$D = \{(v_1, v_2) : 0 < v_1 < 1 \text{ and } v_2 < -1\}.$$

Note that the above map is topologically conjugated with the following map:

$$F_\mu = \begin{cases} v_1 x + \mu & x \leq 0 \\ v_2 x + \mu & 0 < x, \end{cases} \quad (19)$$

for all $\mu > 0$, which undergoes border collision bifurcation when $\mu = 0$.

First, we state the existence of the periodic orbit in map (18).

Theorem 1. (Nusse and Yorke, 1995)

For integer $m \geq 2$, let

$$P_m = \left\{ (v_1, v_2) \in D : -v_1^{1-m} < v_2 < \frac{1 - v_1^{m-1}}{v_1^{m-1} - v_1^{m-2}} \right\}.$$

If $(v_1, v_2) \in P_m$, F_{v_1, v_2} has a period- m attractor. Moreover, the set

$$\{x : \omega\text{-limit set of } x \text{ is not the period-}m \text{ attractor}\}$$

Has Lebesgue measure zero.

Proof. See Nusse and Yorke (1995). □

Let γ_m be the intersection of $v_2 = -v_1^{1-m}$ and $v_2 = \frac{1 - v_1^{m-1}}{v_1^{m-1} - v_1^{m-2}}$. Note that, for any integer $m \geq 2$, $\gamma_{m+1} < \gamma_m$. Moreover, if $v_1 \leq \gamma_N$ for some N ,

$$-v_1^{1-(m+1)} < \frac{1 - v_1^{(m+1)-1}}{v_1^{(m+1)-1} - v_1^{(m+1)-2}} < -v_1^{1-m} < \frac{1 - v_1^{m-1}}{v_1^{m-1} - v_1^{m-2}}$$

for any $m < N$. Therefore, the family of sets P_m is pairwise disjoint. Fig.2 depicts the P_2, P_3, P_4 , and P_5 on the (v_1, v_2) -plane.

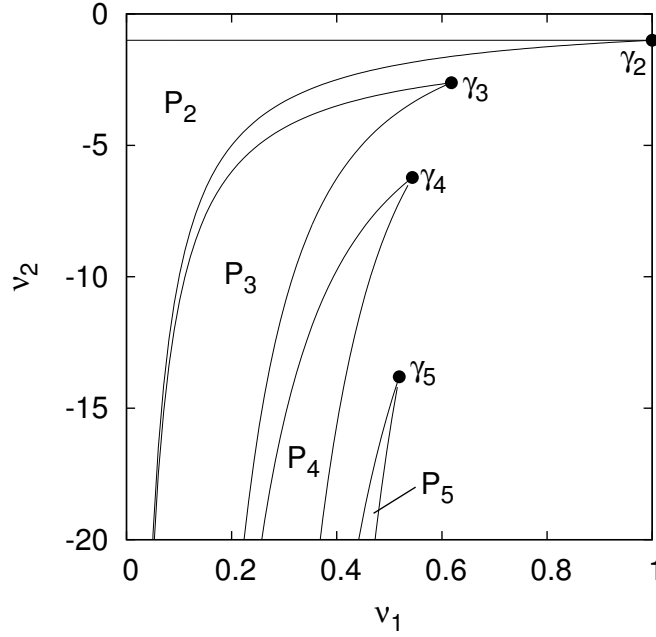


Figure 2: Sets P_2, P_3, P_4 , and P_5 on the (v_1, v_2) -plane. If $(v_1, v_2) \in P_k$, map (18) has an attracting period- k cycle.

Next, it is reasonable that we examine the case wherein $(v_1, v_2) \in D \setminus \bigcup_{m \geq 2} P_m := \Sigma$. That is, if $(v_1, v_2) \in \Sigma$, map F_{v_1, v_2} has a chaotic attractor in the sense that there exist no periodic attractors, and, for any x , the trajectory of x has a positive Lyapunov exponent. Hereafter, we call set Σ the *chaotic region*. Moreover, if $(v_1^0, v_2^0) \in \Sigma$, we call such a pair the *chaotic pair*.

We discuss the chaotic region for more details. Let (v_1^0, v_2^0) be any given chaotic pair. Define $L(v_1^0)$ for the intersection of the vertical line and chaotic region, that is,

$$L(v_1^0) = \{(v_1, v_2) \in \Sigma : v_1 = v_1^0\}.$$

Let m be the integer for which the “top” of the component of $L(v_1^0)$ that contains (v_1^0, v_2^0) is the boundary of set P_m . Moreover, let $C_m(v_1^0)$ be this component of $L(v_1^0)$. It is clear that there are two cases wherein $C_m(v_1^0)$ is bounded or unbounded. Fig.3 depicts some examples of set C_m on the (v_1, v_2) -plane.

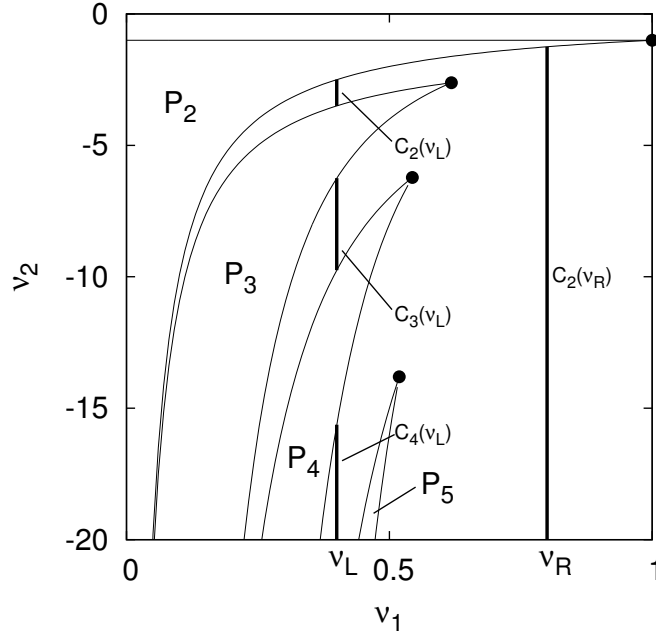


Figure 3: Some examples of set C_m on the (v_1, v_2) -plane. Sets $C_2(v_L)$, $C_3(v_L)$ and $C_4(v_L)$ are bounded, while set $C_2(v_R)$ is unbounded. Note that $L(v_R) = C_2(v_R)$ and $L(v_L) = C_2(v_L) \cup C_3(v_L) \cup C_4(v_L) \cup \dots$.

As we will see later, in our model, we only have to consider the case wherein $m = 2$ and $C_m(v_1^0)$ is unbound. Therefore, we simply define $C_m(v_1^0)$ as $C(v_1^0)$. Considering this, we state the existence of the chaotic orbit in map (18) as follows.

Theorem 2. (Nusse and Yorke, 1995, with some modifications¹)

Let $(v_1, v_2) \in C$ be a chaotic pair and assume that $(v_1, v_2) \in C(v_1)$. There exist $(v_1, \bar{v}_2) \in C(v_1)$ and $(v_1, \hat{v}_2) \in C(v_1)$, for which $\bar{v} < \hat{v} < -v_1^{-1}$ such that the following hold.

1. If $\hat{v} < v_2 < -v_1^{-1}$, map F_{v_1, v_2} has a 2^{k+1} -piece chaotic attractor for some $k \geq 1$ (depending on v_1 and v_2). In addition, if $v_1^2 < \frac{1}{2}(\sqrt{5} - 1)$, then $k = 1$.
2. If $\bar{v} < v_2 < \hat{v}$, then F_{v_1, v_2} has a two-piece chaotic attractor.
3. If $v_2 < \bar{v}$, then F_{v_1, v_2} has an one-piece chaotic attractor.

Proof. See Nusse and Yorke (1995). □

We have seen the periodic and chaotic dynamics generated from border collision bifurcation on the piecewise linear map (18). The following theorem translates the above results to general border collision bifurcation results for piecewise, smooth, one-dimensional nonlinear maps.

¹Here, we only consider the case wherein $m = 2$ and $C_m(v_1^0)$ is unbounded.

Theorem 3. (Nusse and Yorke, 1995)

Let $G : \mathbb{R} \times J \rightarrow \mathbb{R}$ be a piecewise smooth map. Assume that G has the isolated crossing fixed-point property at (x_0, μ_0) . Let $m \geq 2$ and $k \geq 2$ be any integers. Define $\nu_1 = \lim_{x \uparrow x_0} \frac{\partial G}{\partial x}(x, \mu_0)$ and $\nu_2 = \lim_{x \downarrow x_0} \frac{\partial G}{\partial x}(x, \mu_0)$. Then,

- (i) if $(\nu_1, \nu_2) \in P_m$, G exhibits border-collision bifurcation from a fixed-point attractor to a period- m attractor at $(x, \mu) = (x_0, \mu_0)$.
- (ii) if $(\nu_1, \nu_2) \in \Sigma_k$, where $\Sigma_k = \text{Int} \{ (\nu_1, \nu_2) \in \Sigma : F_{\nu_1, \nu_2} \text{ has a } k\text{-piece chaotic attractor} \}$, G exhibits a border-collision bifurcation from a fixed point attractor to a k -piece chaotic attractor at $(x, \mu) = (x_0, \mu_0)$.

Proof. See Nusse and Yorke (1995). □

The property called the *isolated crossing fixed-point property* may be a somewhat unfamiliar notion; however, it is not an unusual property. Indeed, our model has this property and can be adopted to Theorem 3.

Proposition 4. Map (17) has the isolated crossing fixed-point property at $(y, \beta) = (-\frac{1}{a}, \beta^*(a, a))$.

Proof. See Appendix. □

3.4.2 Border collision bifurcation on the OLG model.

In this section, we detect the periodic and chaotic dynamics generated from the border collision bifurcation in our model.

We now consider the following unimodal map:

$$y_{t+1} = h(y_t) = \begin{cases} h_1(y_t) = ay_t + \log \beta(1 - a) & y_t \leq -\frac{1}{a}, \\ h_2(y_t) = \log(1 + \frac{1}{y_t}) + \log \beta - 1 & -\frac{1}{a} < y_t \leq -\frac{1}{b}. \end{cases} \quad (20)$$

As we have seen in Proposition 4, map (20) has the isolated crossing fixed-point property at $(y, \beta) = (-\frac{1}{a}, \beta^*(a, a))$, and therefore we can adopt Theorem 3 into the model. Now, we have the following:

$$\nu_1 = \lim_{y \uparrow -\frac{1}{a}} \frac{\partial h}{\partial y}(x, \beta^*(a, a)) = a \quad \text{and} \quad (21)$$

$$\nu_2 = \lim_{y \downarrow -\frac{1}{a}} \frac{\partial h}{\partial y}(x, \beta^*(a, a)) = -\frac{a^2}{1 - a}. \quad (22)$$

It should be noted that, in our model, we cannot take ν_1 and ν_2 independently. Fig.4 depicts line

$$\nu_2 = -\frac{\nu_1^2}{1 - \nu_1}, \quad (23)$$

on the (ν_1, ν_2) -plane. We have only to examine the parameter set of (ν_1, ν_2) on this line. In the figure, the left white circle represents intersections of two lines, $\nu_2 = -1$ and (23).

On the other hand, the right white circle represents intersections of two lines, $v_2 = -\frac{1}{v_1}$ and (23). The v_1 coordinate of the left and the right white circles are given by v^* and v^{**} , respectively. Subsequently, we easily see that $v^* = \frac{-1+\sqrt{5}}{2} \approx 0.618$, and $v^{**} \approx 0.682$ which is a root of the equation $v_1^3 + v_1 - 1 = 0$ in $[0, 1]$.

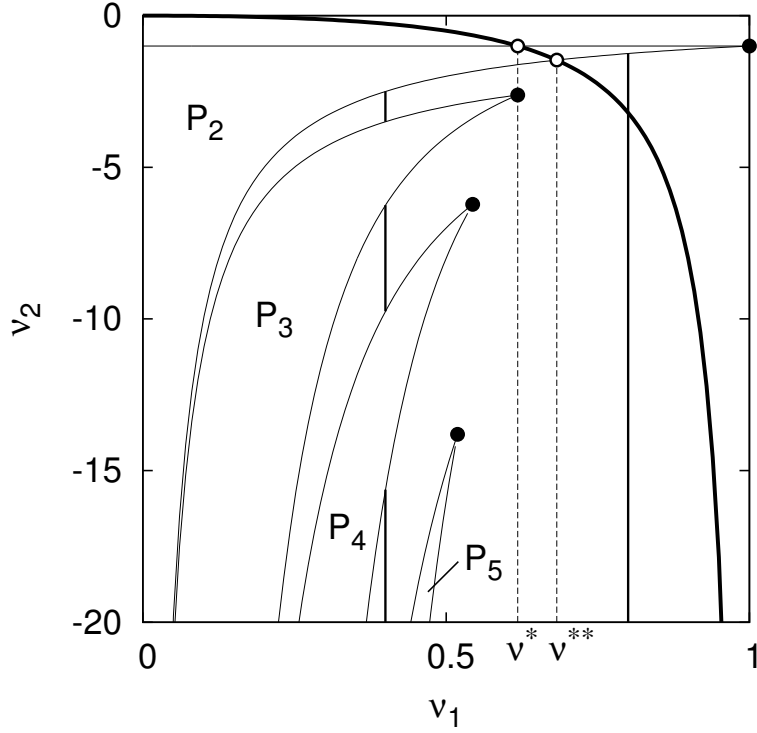


Figure 4: Line $v_2 = -\frac{v_1^2}{1-v_1}$ on the (v_1, v_2) -plane. For $a > v^{**}$, our model has the capability to generate chaotic dynamics induced by border collision bifurcation.

First, we can immediately state the periodic cycle in our model.

Proposition 5. *If $a \in (v^*, v^{**})$, our model exhibits border collision bifurcation from an attracting fixed point to a period-two attractor at $(y, \beta) = (-1/a, \beta^*(a, a))$.*

We now examine the chaotic dynamics in the model, which is the main focus of this section. It is easy to see that that our model exhibits the chaotic attractor for $a > v^{**}$. Moreover, we see that we have to only consider sets $C_2(a)$ for $a > v^{**}$, which are unbounded. According to Theorems 2 and 3, we expect that the model exhibits a 2^{k+1} -piece chaotic attractor for some $k \geq 1$ (depending on v_1 and v_2), a two-piece chaotic attractor, or a one-piece chaotic attractor.

Fortunately, Sushko et al. (2015) comprehensively analyze the border collision bifurcation in piecewise linear map (18). Their results are too technical to refer to all of them

here; however, we need to refer to a few results to investigate the chaotic dynamics in our model. Now, we define the following sets:

$$\begin{aligned} Q_1 &= \{(v_1, v_2) \in \Sigma : v_1 v_2^2 - v_1 + v_2 < 0\}, \\ Q_2 &= \{(v_1, v_2) \in \Sigma : v_1 v_2^2 - v_1 + v_2 > 0, v_1^2 v_2^3 + v_1 - v_2 < 0\}, \quad \text{and} \\ Q_4 &= \{(v_1, v_2) \in \Sigma : v_1^2 v_2^3 + v_1 - v_2 > 0, v_1^6 v_2^6 - v_1 + v_2 < 0\}. \end{aligned}$$

Subsequently, we have the following.

Theorem 4. *For $k = 1, 2$, and 4 , if $(v_1, v_2) \in Q_k$, map (18) has the k -piece chaotic attractor.*

Proof. See Sushko et al. (2015). □

Fig.5 depicts the sets Q_1 , Q_2 , and Q_4 on the (v_1, v_2) -plane. Black circles represent the intersections of line $v_2 = -\frac{v_1^2}{1-v_1}$ and the boundaries of Q_k ($k = 1, 2, 4$). The v_1 coordinate of these intersections, represented by σ^* and σ^{**} , are as follows:

- $\sigma^* \approx 0.705$ is a root of the equation $v_1^8 - v_1^3 + 2v_1^2 - v_1 = 0$ in $[0, 1]$.
- $\sigma^{**} \approx 0.724$ is a root of the equation $v_1^5 + v_1^2 - v_1 = 0$ in $[0, 1]$.

Now, we can completely characterize the chaotic dynamics generated from border collision bifurcation in our OLG model.

Proposition 6. *For map (20), we have the following three scenarios of bifurcation from the attracting fixed point to a chaotic attractor.*

1. *For $a \in (v^{**}, \sigma^*)$, our model exhibits border collision bifurcation from the attracting fixed point to a four-piece chaotic attractor at $(y, \beta) = (-1/a, \beta^*(a, a))$.*
2. *For $a \in (\sigma^*, \sigma^{**})$, our model exhibits border collision bifurcation from the attracting fixed point to a two-piece chaotic attractor at $(y, \beta) = (-1/a, \beta^*(a, a))$.*
3. *For $a \in (\sigma^{**}, 1)$, our model exhibits border collision bifurcation from the attracting fixed point to an one-piece chaotic attractor at $(y, \beta) = (-1/a, \beta^*(a, a))$.*

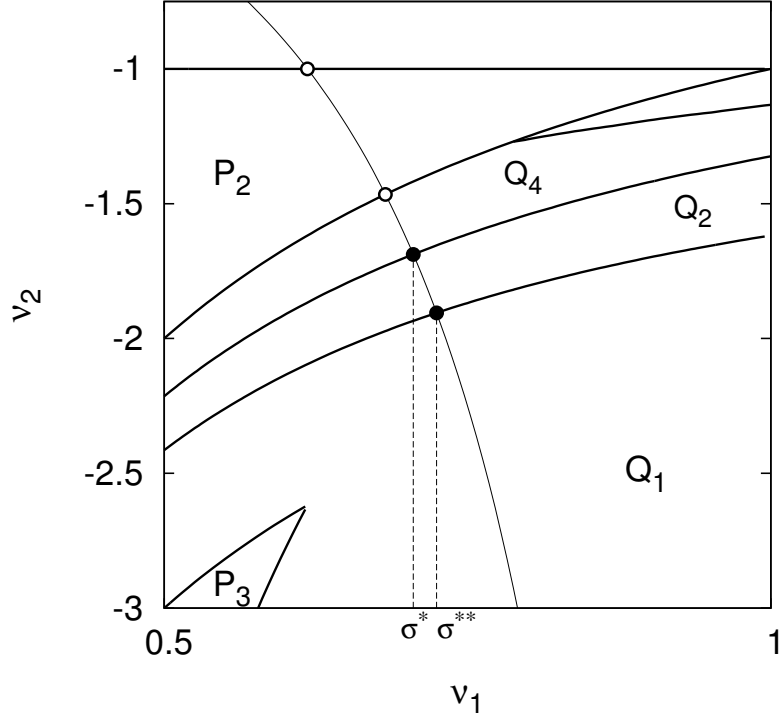
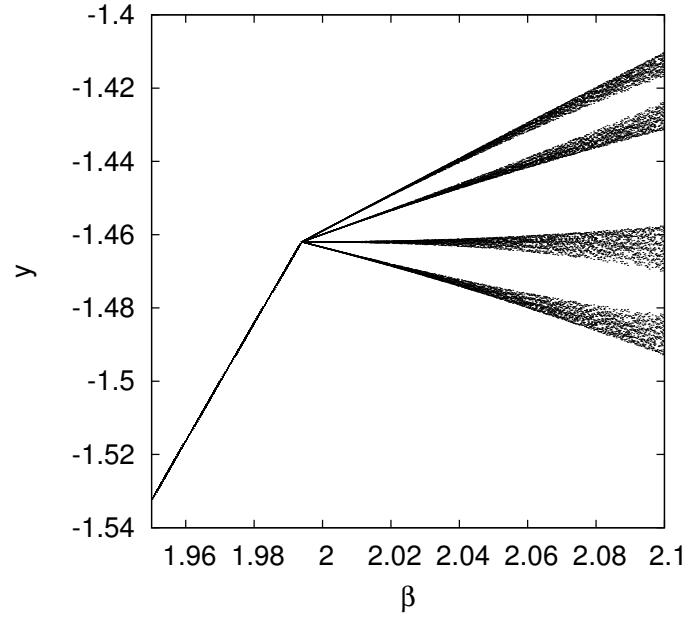


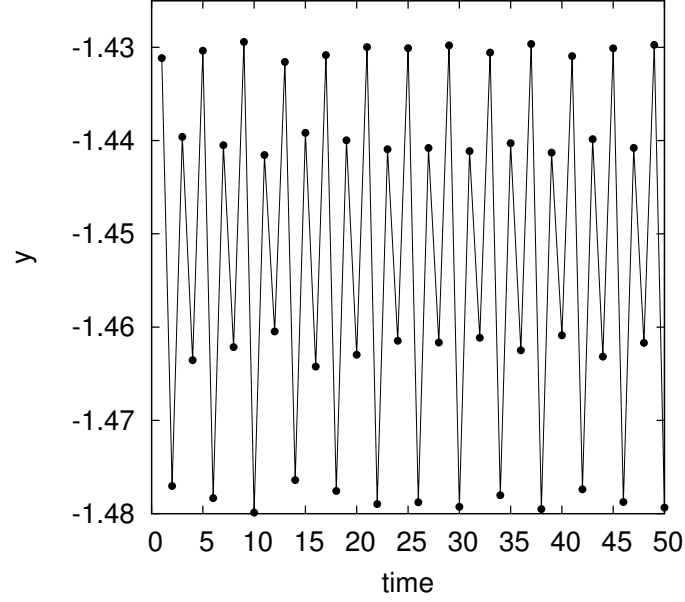
Figure 5: Classification of the chaotic region. Depending on the value of a , our model exhibits four-piece, two-piece, and one-piece chaotic attractors that relate to regions Q_4 , Q_2 , and Q_1 , respectively.

On a k -piece chaotic attractor, the trajectory goes through each interval every k -th period. Of course, when a trajectory returns to the same interval, it never takes the same value. Interestingly, when observing such a trajectory, one may regard the trajectory as a k -cycle with random noises. Fig.6 (a) depicts the bifurcation diagram of map (20) respect to β , where the model has the capability to exhibit four-piece chaotic attractor. Fig.6 (b) depicts time series obtained from map (20) with four-piece chaotic attractor.

In this section, we observed the chaotic dynamics where $\beta \in (\beta^*(a, a), \beta^*(a, a) + \epsilon)$ and, eventually, $\beta \in (\beta^*(b, b) - \epsilon, \beta^*(b, b))$. In other words, we have investigated the *beginning* and *end* of the chaotic dynamics in the OLG model with technology choice. In the next section, we examine the case wherein the β is not sufficiently near $\beta^*(a, a)$ or $\beta^*(b, b)$.



(a) Bifurcation diagram of map (20) with respect to β .



(b) Time series obtained from map (20) with four-piece chaotic attractor.

Figure 6: Panel (a) depicts the bifurcation diagram of map (20) respect to β , where $a = 0.684 \in (\nu^{**}, \sigma^*)$. Panel (b) depicts time series obtained from map (20) with $a = 0.684$ and $\beta = 2.06$.

3.5 Chaotic dynamics : Markov property

In this section, we show that our model has the *Markov property* for some sets of parameters, wherein the model can reduce to a bimodal map on the unit interval. If the map has such a property, we can easily reproduce chaotic dynamics. In what follows, we assume

$$\frac{a-b}{ab} > \log \frac{1-b}{1-a} \quad \text{and} \quad \beta^*(b, a) < \beta < \beta^*(a, b),$$

where the model can be reduce to a bimodal map on the unit interval.

To simplify the analysis of the dynamics, we first use a variable change to reduce our model to the bimodal map in the unit interval.

By defining a new variable x_t as

$$x_t = \frac{y_t + 1 - \log \beta(1-b)}{\log(1-a) - \log(1-b)},$$

Eq.(16) can be transformed into the following piecewise smooth map:

$$x_{t+1} = f(x_t) = \begin{cases} f_L(x_t) = 1 + a(x_t - \theta_L), & x_t \leq \theta_L, \\ \phi(x_t), & \theta_L < x_t \leq \theta_R, \\ f_R(x_t) = b(x_t - \theta_R), & \theta_R < x_t, \end{cases} \quad (24)$$

where

$$\begin{aligned} \phi(x_t) &= \log \left[\frac{g(x_t)}{(1-b)(g(x_t) - 1)} \right]^{\frac{1}{\lambda}}, \quad \text{with} \\ \lambda &= \log \frac{(1-a)}{(1-b)}, \\ g(x_t) &= \lambda x_t + \log \beta(1-b), \end{aligned}$$

and

$$\begin{aligned} \theta_L &= -\frac{(1-a) + a \log \beta(1-b)}{\lambda a}, \\ \theta_R &= -\frac{(1-b) + b \log \beta(1-b)}{\lambda b}. \end{aligned}$$

Now, we briefly discuss some property of map (24). It is easily shown that

$$\begin{aligned} \frac{d\phi(x)}{dx}(\theta_L) &= -\frac{a^2}{1-a} \quad \text{and} \\ \frac{d\phi(x)}{dx}(\theta_R) &= -\frac{b^2}{1-b}. \end{aligned}$$

Moreover, we have the following.

Proposition 7. *Map $\phi(x)$ is a monotonically decreasing and concave function on (θ_L, θ_R) .*

Proof. See Appendix. $\square$

Therefore, the derivatives of ϕ have lower bound $-\frac{b^2}{1-b}$ and upper bound $-\frac{a^2}{1-a}$ for $x \in [\theta_L, \theta_R]$.

3.5.1 Mathematical definition of the Markov property

First, we give some mathematical definitions of the Markov property. See, for example, Boyarsky and Góra (1997) for more details. Let $I = [0, 1]$ and $\psi : I \rightarrow I$ be a map on I onto itself. Let $\mathcal{P}$ be a finite partition of I and I_k ($k = 1, 2, \dots, n$) be subintervals of the partition. Moreover, let ψ_k be the restriction of ψ to I_k . Now, we can define the following.

Definition 1. *If each ψ_k is a homeomorphism from I_k onto some connected union of subintervals I_m , then ψ is said to be Markov. Partition $\mathcal{P}$ is said to be a Markov partition with respect to ψ .*

To examine complex dynamics in the piecewise smooth map, we also need to discuss the following property. Here, let ψ be a piecewise smooth map, which is not necessarily Markov, and let ψ^n be an n -th iterate of map ψ . Then, we define the following.

Definition 2. *If there is an integer $n \geq 1$ such that $\inf \left| \frac{d\psi^n(x)}{dx} \right| > 1$ on each I_k whenever the derivative exists, then ψ is said to be eventually expanding.*

It is readily seen that the eventual expansion implies that the map has no attracting periodic cycles and exhibits chaotic dynamics with long-run observability.

3.5.2 Markov partition on the OLG model

We first show that our model (24) can exhibit the following periodic cycle for some set of parameter values:

$$0 = f^5(0) < f^2(0) = \theta_L < f^4(0) = \theta_R < f(0) = c < f^3(0) = 1. \quad (25)$$

Fig.7 depicts the graph of map (24) with the periodic cycle (25). We can show that such a specific cycle implies chaotic dynamics.

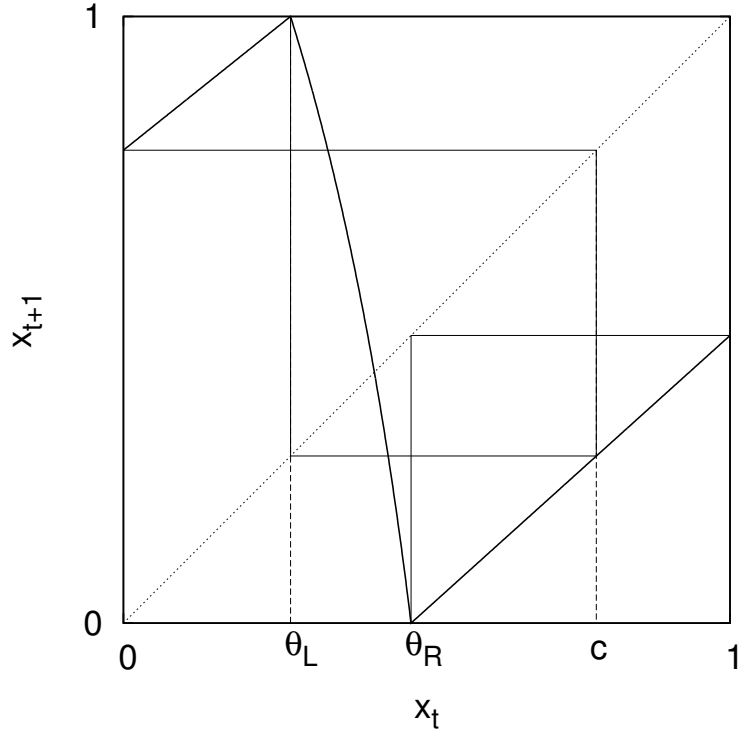


Figure 7: The graph of map (24) with a period-5 Markov partition. $a = 0.8, b \approx 0.902$ and $\beta \approx 6.507$.

Proposition 8. (*Observable chaos on a period-5 Markov partition*)

There exist a_L and a_R with $0 < a_L < a_R < 1$. If $a \in (a_L, a_R)$, there exists $b \in (a, 1)$ and $\beta > 0$ such that for (a, b, β) , we have a period-5 Markov partition of the unit interval with respect to f . Moreover, f is eventually expanding and thus exhibits observable chaos.

Proof. To calculate the period-5 cycle given by (25), θ_L and θ_R must solve the following equations,

$$f_R(1) = \theta_R \quad \text{and} \quad f_R(f_L(0)) = \theta_L.$$

A straightforward calculation shows that

$$\beta = \frac{1}{1-b} \exp - \left[\frac{b\lambda}{1+b} + \frac{1-b}{b} \right] \quad \text{and} \quad (26)$$

$$\lambda = \frac{(b-a)(ab+1)(1+b)}{a^2b^3}. \quad (27)$$

First, we check that

$$\beta^*(b, a) < \frac{1}{1-b} \exp - \left[\frac{b\lambda}{1+b} + \frac{1-b}{b} \right] < \beta^*(a, b). \quad (28)$$

We see that the inequality

$$\beta^*(b, a) < \frac{1}{1-b} \exp - \left[\frac{b\lambda}{1+b} + \frac{1-b}{b} \right]$$

holds if and only if $0 < \frac{\lambda}{1+b}$. Thus, clearly, this inequality holds. Moreover, we see that the inequality

$$\frac{1}{1-b} \exp - \left[\frac{b\lambda}{1+b} + \frac{1-b}{b} \right] < \beta^*(a, b)$$

holds if and only if $\log \frac{1-b}{1-a} < \frac{a-b}{ab} \frac{1+b}{b}$. Let

$$\eta_1(b) = \log \frac{1-b}{1-a} \quad \text{and} \quad \eta_2(b) = \frac{a-b}{ab} \frac{1+b}{b}.$$

It is easily seen that $\eta_1(b)$ is a monotonically decreasing and concave function in $[0, 1]$. Moreover, we have

$$\frac{d\eta_2(a)}{db} = -\frac{1+a}{a^3} < 0 \quad \text{and} \quad (29)$$

$$\frac{d^2\eta_2(a)}{db^2} = \frac{2(a+2)}{a^4} > 0. \quad (30)$$

Therefore, there exists $\epsilon > 0$ such that for any $a \in (0, 1)$, $\eta_2(b)$ is a monotonically decreasing and convex function for $b \in (a - \epsilon, a + \epsilon)$. Note that $\eta_1(a) = \eta_2(a)$ and

$$\frac{d\eta_1(a)}{db} = -\frac{1}{1-a} \quad \text{and} \quad \frac{d\eta_2(a)}{db} = -\frac{1+a}{a^3}.$$

Considering the concavity of η_1 and convexity of η_2 for neighborhood a , if

$$-\frac{1}{1-a} < -\frac{1+a}{a^3} \Rightarrow a^3 + a^2 - 1 > 0,$$

there exists $a < \bar{a}$ such that $\log \frac{1-b}{1-a} < \frac{a-b}{ab} \frac{1+b}{b}$ for $b \in (a, \bar{a})$. Let $a_L \approx 0.755$ be a root of the equation $a^3 + a^2 - 1 = 0$ in $(0, 1)$. We conclude that if $a \in (a_L, \bar{a})$, then inequality (28) holds.

Next, we examine Eq.(27). Let

$$\varphi_1(b) = \log \frac{1-a}{1-b} \quad \text{and} \quad \varphi_2(b) = \frac{(b-a)(ab+1)(1+b)}{a^2b^3}.$$

Clearly, $\varphi_1(a) = \varphi_2(a)$, and we can easily see that $\varphi_1(b)$ is a monotonically increasing convex function and $\lim_{b \rightarrow 1} \varphi_1(b) = \infty$. Moreover, we have

$$\begin{aligned} \varphi_2(1) &= \frac{2(1-a^2)}{a^2} \quad \text{textbounded for } a \in (0, 1) \quad \text{and} \\ \frac{d\varphi_2(a)}{db} &= \frac{a^3 + a^2 + a + 1}{a^5} > 0. \end{aligned}$$

Therefore, if

$$\frac{d\varphi_1(a)}{db} = \frac{1}{1-a} < \frac{a^3 + a^2 + a + 1}{a^5} \Rightarrow a^5 + a^4 - 1 < 0,$$

$\varphi_1(b)$ intersects with $\varphi_2(b)$ at least one point $b \in (a, 1)$. Let $\hat{a} \approx 0.857$ be a root of the equation $a^5 + a^4 - 1 = 0$ in $(0, 1)$. We conclude that if $a < \hat{a}$, there exists $b \in (a, 1)$ such that, for a and b , Eq.(27) holds.

Let $a_R = \min\{\bar{a}, \hat{a}\}$. Now, we can state that if $a \in (a_L, a_R)$, there exists $b \in (a, 1)$ and $\beta > 0$ such that for (a, b, β) , we have a period-5 Markov partition of the unit interval with respect to f .

Finally, we show that f is eventually expanding. Let $I_1 = [0, \theta_L]$, $I_2 = [\theta_L, \theta_R]$, $I_3 = [\theta_R, c]$, $I_4 = [c, 0]$, and $\bar{I}_k$ be an interior of I_k . Note that every point $x \in \bigcup_{k=1}^k \bar{I}_k$ will visit I_2 at least once every fourth iteration. Therefore, for $x \in \bigcup_{k=1}^k \bar{I}_k$,

$$|(f^4)'(x)| \geq a^3 |\phi'| \geq \left| -\frac{a^5}{1-a} \right|,$$

and if

$$\left| -\frac{a^5}{1-a} \right| > 1 \Rightarrow a^5 + a - 1 > 0, \quad (31)$$

map f is eventually expanding. Since we can easily check whether $a_L < a$, inequality (31) holds, which completes the proof. $\square$

Proposition 8 can be immediately extended to a more general case of a period- $(2n+3)$ Markov partition for $n \geq 1$.

Proposition 9. (*Observable chaos on a period- $(2n+3)$ Markov partition*)

For any $n \in \mathbb{N}$, there exist $a_{2n+3,L}$ and $a_{2n+3,R}$ with $0 < a_{2n+3,L} < a_{2n+3,R} < 1$. If $a \in (a_{2n+3,L}, a_{2n+3,R})$, there exists $b \in (a, 1)$ and $\beta > 0$ such that for (a, b, β) , we have a period- $(2n+3)$ Markov partition of the unit interval with respect to f . Moreover, f is eventually expanding and, hence, chaotic.

Proof. See Appendix. $\square$

3.6 Concluding remarks

Based on the model studied by Umezaki and Yokoo (2019), we develop a piecewise smooth OLG model. By assuming a continuous choice instead of discrete choice of the technologies, the model has the capability to exhibit several dynamic growth patterns, including chaotic dynamics, which cannot be observed virtually for the model in Umezaki and Yokoo (2019). Moreover, in spite of its generality, the model is tractable enough to analytically investigate such complex dynamics in depth.

First, by using the border collision bifurcation theory, we investigated the routes from a globally attracting steady state to other non-stationary behaviors. Then, we completely

characterized the chaotic dynamics around the bifurcation point; that is, we detected four-piece, two-piece, and one-piece chaotic attractors. Second, we calculated the parameter values for which the model has the Markov property. Then, we characterized the chaotic dynamics, which are different from those in the case of border collision bifurcation. However, one might argue that sets of the parameter value, for which the map is Markov, are too restricted. For future research, it is a natural task to check the *robustness* of the chaotic dynamics by characterizing the dynamics near the parameter values for which the map is Markov.

Finally, it should be pointed out that, compared with the output elasticity for capital (or labor) estimated in the real world, parameter values a and b , for which the model can exhibit chaotic dynamics, are relatively high. Although the intuition behind this remains unclear, from the technical view point, it could be explained by considering an inverse problem as follows. In this study, we assume that the total factor productivity of the technology, represented by A , is constant for all technologies. However, in general, we can consider different total factor productivity for each technology. In that case, the model has more flexibility, and various new problems might appear, including the following. “A set of which type of technologies is needed to reduce the model to a continuous piecewise linear map?” “For a given piecewise smooth map, is there a set of technologies such that the model can be reduced to the given piecewise smooth map?” “If so, what features does the set of technologies have?” In the future, we hope that our model is investigated more deeply with additional settings without losing its tractability and generality.

3.7 Appendix

3.7.1 Proof of Proposition 4

Proof. For a general statement of the isolated crossing fixed-point property, see Nusse and Yorke (1995). Map (20) has the isolated crossing fixed-point property at $(-\frac{1}{a}, \beta^*(a, a))$ if the following conditions hold.

1. There exists $\{(-\frac{1}{a}, \beta) : \beta \in J_0, J_0 \text{ is an open interval containing } \beta^*(a, a)\}$, and for any $\beta \in J_0$, map (20) is not differentiable at the point $(-\frac{1}{a}, \beta)$.
2. The set $\{(x(\beta), \beta) : h(x(\beta)) = x(\beta), \beta \in J_0\}$ is a continuous curve and intersects curve $\{(-\frac{1}{a}, \beta) : \beta \in J_0\}$ transversally at $(-\frac{1}{a}, \beta)$.
3. For $\beta \in J_0$, $\frac{dh(y)}{dy} > 0$ for $y < -\frac{1}{a}$ and $\frac{dh(y)}{dy} < 0$ for $y > -\frac{1}{a}$. Moreover, the left limit $\lim_{y \uparrow -\frac{1}{a}} \frac{dh(y)}{dy}$ and right limit $\lim_{y \downarrow -\frac{1}{a}} \frac{dh(y)}{dy}$ exist.
4. $h(-\frac{1}{a})$ is a smooth function of $\beta \in J_0$, and for $\beta = \beta^*(a, a)$, $\frac{dh(-\frac{1}{a})}{d\beta} > 0$.

It is obvious that map (20) satisfies these conditions. □

3.7.2 Proof of Proposition 7

Proof. We easily show that

$$\frac{d\phi(x)}{dx} = -\frac{1}{g(x)[g(x) - 1]}.$$

Since $\lambda > 0$ and $g(\theta_R) = -\frac{b}{1-b} < 0$, $g(x) < 0$ for $[\theta_L, \theta_R]$. Therefore, we have

$$\frac{d\phi(x)}{dx} < 0.$$

Moreover, we have

$$\frac{d^2\phi(x)}{dx^2} = \frac{\lambda[2g(x) - 1]}{g(x)^2[g(x) - 1]^2} < 0,$$

for $x \in [\theta_L, \theta_R]$. This completes the proof. $\square$

3.7.3 Proof of Proposition 8

Proof. First, we show that there exists the following $2n + 3$ cycle:

$$0 = f^{2n+3}(0) < f^{n+1}(0) = \theta_L < f^{2n+2}(0) = \theta_R < \dots < f^{n+3}(0) < f(0) < 1 = f^{n+2}(0). \quad (\text{A.1})$$

To calculate the cycle given by (A.1), θ_L and θ_R must solve the following equations:

$$f_R^n(1) = \theta_R \quad \text{and} \quad f_R^n(f_L(0)) = \theta_L.$$

A straightforward but tedious calculation shows that

$$\beta = \frac{1}{1-b} \exp - \left[\frac{b^n \lambda}{\sum_{k=0}^n b^k} + \frac{1-b}{b} \right] \quad \text{and} \quad (\text{A.2})$$

$$\lambda = \frac{(b-a)(1+ab^n)\sum_{k=0}^n b^k}{a^2 b^{2n+1}}. \quad (\text{A.3})$$

First, we check that

$$\beta^*(b, a) < \frac{1}{1-b} \exp - \left[\frac{b^n \lambda}{\sum_{k=0}^n b^k} + \frac{1-b}{b} \right] < \beta^*(a, b). \quad (\text{A.4})$$

We see that the inequality

$$\beta^*(b, a) < \frac{1}{1-b} \exp - \left[\frac{b^n \lambda}{\sum_{k=0}^n b^k} + \frac{1-b}{b} \right]$$

holds if and only if

$$0 < \lambda \left(\frac{b^n}{\sum_{k=0}^n b^k} \right).$$

Clearly, this inequality holds. Moreover, we also see that the inequality

$$\frac{1}{1-b} \exp - \left[\frac{b^n \lambda}{\sum_{k=0}^n b^k} + \frac{1-b}{b} \right] < \beta^*(a, b)$$

holds if and only if $\log \frac{1-b}{1-a} < \frac{a-b}{ab} \frac{\sum_{k=0}^n b^k}{b^n}$. Let

$$\eta_1(b) = \log \frac{1-b}{1-a} \quad \text{and} \quad \hat{\eta}_2(b) = \frac{a-b}{ab} \frac{\sum_{k=0}^n b^k}{b^n}.$$

It is easily seen that $\eta_1(b)$ is a monotonically decreasing and concave function in $[0, 1]$. Moreover, we have

$$\frac{d\hat{\eta}_2(a)}{db} = -\frac{\sum_{k=0}^n a^k}{a^{n+2}} < 0 \quad \text{and} \quad (A.5)$$

$$\frac{d^2\hat{\eta}_2(a)}{db^2} = \frac{(1+n)(1+a^{n+1}) + 2\sum_{k=1}^n (n-k+1)a^k}{a^{n+3}} > 0. \quad (A.6)$$

Therefore, there exists $\epsilon > 0$ such that for any $a \in (0, 1)$, $\eta_2(b)$ is a monotonically decreasing and convex function for $b \in (a - \epsilon, a + \epsilon)$. Note that $\eta_1(a) = \hat{\eta}_2(a)$, and

$$\frac{d\eta_1(a)}{db} = -\frac{1}{1-a} \quad \text{and} \quad \frac{d\hat{\eta}_2(a)}{db} = -\frac{\sum_{k=0}^n a^k}{a^{n+2}}.$$

Considering the concavity of η_1 and convexity of $\hat{\eta}_2$ for neighborhood a , if

$$-\frac{1}{1-a} < -\frac{\sum_{k=0}^n a^k}{a^{n+2}} \Rightarrow a^{n+2} + a^{n+1} - 1 > 0,$$

there exists $a < \bar{a}_{2n+3}$ such that $\log \frac{1-b}{1-a} < \frac{a-b}{ab} \frac{\sum_{k=0}^n b^k}{b^n}$ for $b \in (a, \bar{a}_{2n+3})$. Let $a_{2n+3,L}$ be a root of the equation $a^{n+2} + a^{n+1} - 1 = 0$ in $(0, 1)$. We conclude that if $a \in (a_{2n+3,L}, \bar{a}_{2n+3})$, inequality (A.4) holds.

Next, we examine Eq.(A.3). Let

$$\varphi_1(b) = \log \frac{1-a}{1-b} \quad \text{and} \quad \hat{\varphi}_2(b) = \frac{(b-a)(1+ab^n)\sum_{k=0}^n b^k}{a^2 b^{2n+1}}.$$

Clearly, $\varphi_1(a) = \varphi_2(a)$. We can easily see that $\varphi_1(b)$ is a monotonically increasing convex function and $\lim_{b \rightarrow 1} \varphi_1(b) = \infty$. Moreover, we have

$$\begin{aligned} \varphi_2(1) &= \frac{(n+1)(1-a^2)}{a^2} \quad \text{textbounded for } a \in (0, 1) \quad \text{and} \\ \frac{d\varphi_2(a)}{db} &= \frac{\sum_{k=0}^{2n+1} a^k}{a^{2n+3}} > 0. \end{aligned}$$

Therefore, if

$$\frac{d\varphi_1(a)}{db} = \frac{1}{1-a} < \frac{\sum_{k=0}^{2n+1} a^k}{a^{2n+3}} \Rightarrow a^{2n+3} + a^{2n+2} - 1 < 0,$$

$\varphi_1(b)$ intersects $\hat{\varphi}_2(b)$ at least one point $b \in (a, 1)$. Let $\hat{a}_{2n+3}$ be a root of the equation $a^{2n+3} + a^{2n+2} - 1 = 0$ in $(0, 1)$. We conclude that if $a < \hat{a}_{2n+3}$, there exists $b \in (a, 1)$ such that, for a and b , Eq.(27) holds. Let $a_{2n+3,R} = \min \{\bar{a}_{2n+3}, \hat{a}_{2n+3}\}$. Now, we can state that if $a \in (a_{2n+3,L}, a_{2n+3,R})$, there exists $b \in (a, 1)$ and $\beta > 0$ such that for (a, b, β) , we have a period- $2n + 3$ Markov partition of the unit interval with respect to f .

Finally, we show that f is eventually expanding. Let I_k ($k = 1, 2, \dots, n-1$) be subintervals of the period- $2n + 3$ Markov partition and $I^* = [\theta_L, \theta_R]$. Moreover, let $\bar{I}_k$ be an interior of I_k . A minor consideration shows that every point $x \in \bigcup_{k=1}^k \bar{I}_k$ will visit I^* at least once every $2n + 2$ th iteration. Therefore, for $x \in \bigcup_{k=1}^k \bar{I}_k$,

$$|(f^{2n+2})'(x)| \geq a^{2n+1} |\phi'| \geq \left| -\frac{a^{2n+3}}{1-a} \right|,$$

and if

$$\left| -\frac{a^{2n+3}}{1-a} \right| > 1 \Rightarrow a^{2n+3} + a - 1 > 0, \quad (\text{A.7})$$

map f is eventually expanding. We can easily see that if $a_{2n+3,L} < a$, inequality (A.7) holds. This completes the proof. $\square$

4 Bifurcation Analysis of the Rock-Paper-Scissors Game with Discrete-Time Logit Dynamics

4.1 Introduction

The rock-paper-scissors (RPS) game is a well-known three-strategy game. In a static setting, this game has only one Nash equilibrium in a mixed strategy. The paper investigates the RPS game from an evolutionary viewpoint, with particular emphasis on the dynamic behavior departing from the Nash equilibrium play.

One of the main tools in evolutionary game theory is replicator dynamics. See, for example, Weissing (1991), among others, for a comprehensive analysis of RPS replicator dynamics. It is known that depending on the payoff values, the (simplest possible) continuous-time version of the RPS replicator dynamics can be capable of convergence to the Nash equilibrium (when the game is “good”), infinitely many cyclical orbits, or convergence to the heteroclinic cycle (when the game is “bad”). It is also known that the discrete-time version of RPS replicator dynamics can exhibit more complicated dynamic behaviors such as an attracting closed invariant curve, due to the *Neimark-Sacker bifurcation*, on which typically higher-periodic or even quasi-periodic motions arise. In both cases, the replicator dynamics can fail to converge to the Nash equilibrium.

To capture the long-run evolutionary dynamics out of the Nash equilibrium play, Gilboa and Matsui (1991) examine the RPS game in the setting of *best response dynamics*. They provide a new solution concept, called a cyclically stable set (CSS), and show that such a solution always exists. Moreover, they claim that in a large population game, the CSS is a more appropriate solution concept to the RPS game than the Nash equilibrium.

In empirical research, RPS cycles are often observed in natural sciences. See, for instance, Sinervo and Lively (1996) and Kerr et al. (2002). In contrast, very few examples of such cycles are found in social sciences. However, Xu, Zhou, and Wang (2013) observe a mean cyclical frequency in laboratory experiments on the RPS game. They compare their frequency data with the theoretical results obtained from *discrete-time logit dynamics*, which is the focus in this study, and show that the two sets of data agree.

Here, the logit dynamics (also known as logit-response dynamics) introduced by Blume (1993) is a version of the perturbed best response dynamics. The perturbations represent some kind of “noise” such as payoff noise, errors in the players’ perception of the payoff or errors in the players’ performance of the best response. For foundations of deterministic or stochastic perturbations, see Benaïm and Weibull (2003) and Sandholm (2010). In the literature of evolution in games in various settings including logit dynamics, noise is often used to select the set of long-run outcomes. These long-run states, called *stochastically stable states* (see, among others, Kandori et al (1993)), are characterized by the set of states with positive probability in the limiting invariant distribution as noise vanishes. Alós-Ferrer and Netzer (2010) develop a characterization of stochas-

tically stable states for logit dynamics in games, with arbitrary specification of revision opportunities.

The logit model has several advantages over other evolutionary dynamics in games. First, the logit choice has a good microfoundation from the random utility model. Second, it is easy for the logit model to fully account for the magnitude of suboptimal payoffs. See Alós-Ferrer and Netzer (2010) and references therein for discussion of these points. Third, from a technical viewpoint, the introduction of perturbations to the pure best response dynamics can avoid intractabilities that may arise when we have to deal with pure best response dynamics in the form of a differential inclusion in continuous-time settings or a set-valued map in discrete-time settings.

Considering the empirical evidence and the aforementioned advantages, we find it worthwhile to comprehensively investigate the RPS game in the discrete-time logit dynamics. We, however, deviate from the most common approach of characterizing the stochastically stable states for vanishingly small noise. Instead, we take an approach complementary to the literature of stochastic evolution by characterizing the dynamic properties of evolution for a level of noise that is not negligible. If we think of noise or the limit of rationality of agents as inevitable in the real world, it can be useful to study the dynamics far from the limiting cases.

In a continuous-time setting, Hommes and Ochea (2012) use the *Hopf bifurcation* theorem to show that logit dynamics for the RPS game can generate a limit cycle. We will investigate the RPS game with discrete-time logit dynamics to show that the model can generate richer dynamics than in a continuous-time setting. First, we will show that, when the parameter value representing the players' rationality exceeds some critical value, our model can exhibit an attracting closed invariant curve around the Nash equilibrium, due to the Neimark-Sacker bifurcation. In addition, we will also examine another type of bifurcation, called *strong resonance*, which leads to more complicated dynamical patterns.

Persistent fluctuations due to Hopf or Neimark-Sacker type bifurcations are not unfamiliar to economists. However, strong resonance seems to have attracted little attention in economic literature, even though it usually accompanies a Neimark-Sacker bifurcation. The strong resonance we examine here is categorized as a *codimension-two* bifurcation, which is a bifurcation that needs two parameters to capture the overall bifurcation diagram. One motivation to studying codimension-two bifurcations is that they are local bifurcations, but provide information on the *global* properties of the dynamics. In other words, a discrete-time dynamical system around the codimension-two point may create multiple periodic cycles and complex behaviors derived from homoclinic bifurcations or heteroclinic bifurcations, which are usually categorized as global bifurcations.

Our model can generate strong resonance when the payoff value of losing the game is close to the payoff value of a tie. To study strong resonance in our model, we adopt two approaches, a direct analysis and a flow-approximation-based analysis. In the direct analysis, we give an elementary proof of the coexistence of the attracting or repelling fixed point corresponding to the Nash equilibrium, an attracting closed invariant curve, a

period-three attracting cycle, and a period-three saddle cycle near the resonance point. In the flow-approximation approach, we exploit a technique that approximates the original discrete-time system using ordinary differential equations. This approximation technique provides information on both local and global dynamics, most of which cannot be obtained from studying only the original discrete-time system. As a result, we specify that near the resonance, our dynamic RPS game exhibits a heteroclinic bifurcation, which causes the closed invariant curve to disappear.

The remainder of this paper is organized as follows. Section 4.2 introduces a discrete-time version of logit dynamics and applies it to the RPS game. Section 4.3 gives preliminary results, which are used in the sections that follow. Section 4.4 shows that the Nash equilibrium in our model undergoes a Neimark-Sacker bifurcation, giving rise to an invariant closed curve. Section 4.5 discusses periodic cycles near the resonance, and Section 4.6 characterizes the set of global bifurcations in our model. Finally, Section 4.7 concludes the paper. Some mathematical proofs are delegated to appendices.

4.2 Model

In Section 4.2.1, we will define a discrete-time version of logit dynamics, and in Section 4.2.2 we apply it to the RPS game.

4.2.1 Logit dynamics

Consider an n -strategy population. We assume that, given an opponent's mixed strategy in time t defined by $\sigma_t \in \mathbb{R}^n$, the probability σ_{t+1}^i of a player choosing strategy i in time $t + 1$ is defined by the following formula, called the *logit choice*:

$$\sigma_{t+1}^i = \frac{\exp(\beta(A\sigma_t)_i)}{\sum_{k=1}^n \exp(\beta(A\sigma_t)_k)}, \quad i = 1, \dots, n, \quad (1)$$

where A is a representation matrix of the game, $(A\sigma)_i$ represents the i -th component of the vector $A\sigma$, and β is the intensity of choice, which represents the rationality of players. Note that if $\beta = 0$, each strategy is chosen with a uniform probability of $1/n$. If $\beta \rightarrow \infty$ or, in other words, if agents become more rational, the logit choice is known to converge to the best response correspondence. Another interpretation of β , which is more common in the literature of stochastic evolution in games, is that it represents the inverse measure of the level of *noise*, by which players tremble the strategy. As β goes to infinity, the noise vanishes. See e.g. Sandholm (2010) and Benaïm and Weibull (2003) for derivation of this formula.

Suppose that a fraction of the players λ is picked at random and switches to the logit choice to their opponents' strategies, and that the remaining players continue their current strategies. Then, we can define the discrete-time logit dynamics, according to Fudenberg and Levine (1998), as:

Definition 3. (Discrete-time logit dynamics)

$$\sigma_{t+1}^i = \sigma_t^i + \lambda \left[\frac{\exp(\beta(A\sigma_t)_i)}{\sum_{k=1}^n \exp(\beta(A\sigma_t)_k)} - \sigma_t^i \right], \quad \lambda \in (0, 1]. \quad (2)$$

4.2.2 RPS game with logit dynamics

The RPS game is a game with three strategies: s_1 , s_2 , and s_3 , where s_i is a unique best response to s_{i+1} , for $i = 1, 2$, and s_3 is a unique best response to s_1 . In this study, we will use the following simple payoff matrix:

$$A = \begin{pmatrix} 0 & 1 & -\epsilon \\ -\epsilon & 0 & 1 \\ 1 & -\epsilon & 0 \end{pmatrix}, \quad \epsilon \in \mathbb{R}. \quad (3)$$

The winning player receives payoff unity, while the losing player receives payoff $-\epsilon$. If there is a tie, then each player receives a payoff of 0. The RPS game is said to be *good* if $\epsilon < 1$, and *bad* if $\epsilon > 1$. Now, the logit dynamics in (2), using (3), lead to

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x + \lambda \{ f_1(x, y, z) - x \} \\ y + \lambda \{ f_2(x, y, z) - y \} \\ z + \lambda \{ f_3(x, y, z) - z \} \end{pmatrix}, \quad (4)$$

where

$$f_i(x, y, z) = \frac{v_i(x, y, z)}{v_1(x, y, z) + v_2(x, y, z) + v_3(x, y, z)},$$

with

$$\begin{aligned} v_1(x, y, z) &= \exp(\beta(y - \epsilon z)), \\ v_2(x, y, z) &= \exp(\beta(-\epsilon x + z)), \\ v_3(x, y, z) &= \exp(\beta(x - \epsilon y)). \end{aligned}$$

By substituting $z = 1 - x - y$, we can reduce the map in (4) to the following two-dimensional map:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + \lambda \{ f_1(x, y, 1 - x - y) - x \} \\ y + \lambda \{ f_2(x, y, 1 - x - y) - y \} \end{pmatrix}. \quad (5)$$

A fixed point of the logit dynamics is called a *logit equilibrium*. In general, logit equilibria do not coincide with the Nash equilibrium. Owing to the symmetric property, however, the map (5) has the fixed point $(x^*, y^*) = (1/3, 1/3)$, independent of the parameter values, coinciding with the Nash equilibrium. Therefore, we hereinafter simply call the point $(x^*, y^*) = (1/3, 1/3)$ the Nash equilibrium.

4.3 Stability of the Nash equilibrium

Let $J(x, y)$ denote the Jacobian matrix of the map (5). Then, J , evaluated at the fixed point $(1/3, 1/3)$, is given by

$$J\left(\frac{1}{3}, \frac{1}{3}\right) = \begin{pmatrix} 1 + \lambda \left[\frac{1}{3}\beta\epsilon - 1\right] & \frac{1}{3}\lambda\beta(1 + \epsilon) \\ -\frac{1}{3}\lambda\beta(1 + \epsilon) & 1 - \lambda \left[\frac{1}{3}\beta + 1\right] \end{pmatrix}, \quad (6)$$

with non-real, conjugate complex eigenvalues

$$\mu_{1,2}(\beta) = (1 - \lambda) - \frac{1 - \epsilon}{6}\lambda\beta \pm i\frac{1 + \epsilon}{2\sqrt{3}}\lambda\beta,$$

where $i = \sqrt{-1}$. A simple calculation shows that if

$$\beta = \frac{3(1 - \lambda)(1 - \epsilon) + 3\sqrt{3\lambda(2 - \lambda)(1 + \epsilon)^2 + (1 - \epsilon)^2}}{2\lambda(1 + \epsilon + \epsilon^2)} \quad \beta_c, \quad (7)$$

then the Jacobian matrix has a pair of eigenvalues with modulus equal to unity. Moreover, we have

$$\frac{d}{d\beta}|\mu(\beta_c)| > 0.$$

Therefore, for $\beta \neq \beta_c$ ($\beta = \beta_c$), the fixed point $(1/3, 1/3)$ of the map (5) is hyperbolic (non-hyperbolic, respectively).

By applying the Hartman-Grobman theorem for maps (See e.g. Quandt (1986)) to the above expression, we immediately obtain the following proposition.

Proposition 1. (Stability of the Nash equilibrium)

The fixed point of the map given by (5) is an attractor if $\beta < \beta_c$ and a repeller if $\beta > \beta_c$.

4.4 Neimark-Sacker bifurcation

For all parameter values, J evaluated at the fixed point $(1/3, 1/3)$ has conjugate complex eigenvalues. Therefore, we can expect that our model undergoes Neimark-Sacker bifurcation, which creates a closed invariant curve around the Nash equilibrium on which periodic or quasi-periodic dynamics occur.

Before the main result in this section, we will briefly review the Neimark-Sacker bifurcation.

Theorem 1. (Neimark-Sacker bifurcation (e.g., Hale and Koçak (1991)))

Let

$$F : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}^2; \quad (\beta, \mathbf{x}) \mapsto F(\beta, \mathbf{x})$$

be a C^4 map depending on a real parameter β , and satisfying the following conditions:

1. $F(\beta, \mathbf{x}^*) = \mathbf{x}^*$ for β , near some fixed β_c ;
2. $DF(\beta, \mathbf{x}^*)$ has two non-real eigenvalues $\mu(\beta)$ and $\bar{\mu}(\beta)$, for β near β_c , with $|\mu(\beta_c)| = 1$;
3. $\frac{d}{d\beta}|\mu(\beta_c)| > 0$;
4. For $k = 1, 2, 3$, and 4 , $\mu(\beta_c)$ is not a primitive k -th root of unity (i.e., $\mu^k(\beta_c) \neq 1$)¹.

Then, there is a smooth β -dependent change of coordinates bringing F into the form

$$F(\beta, \mathbf{x}) = \mathcal{F}(\beta, \mathbf{x}) + O(\|\mathbf{x}\|^5),$$

and there are smooth functions $a(\beta)$, $b(\beta)$, and $\omega(\beta)$ so that in polar coordinates, the function $\mathcal{F}(\beta, \mathbf{x})$ is given by

$$\begin{pmatrix} r \\ \theta \end{pmatrix} \mapsto \begin{pmatrix} |\mu(\beta)|r - a(\beta)r^3 \\ \theta + \omega(\beta) + b(\beta)r^2 \end{pmatrix}.$$

Supercritical case: If $a(\beta_c) > 0$, then there is a neighborhood U of the fixed point $\mathbf{x}^*$ and a constant $\gamma > 0$ such that, for $|\beta - \beta_c| < \gamma$ and $\mathbf{x}^0 \in U$, the ω -limit set of $\mathbf{x}^0$ is $\mathbf{x}^*$ if $\beta < \beta_c$ and belongs to a closed invariant C^1 curve $\Gamma(\beta)$ encircling $\mathbf{x}^*$ if $\beta > \beta_c$. Furthermore, $\Gamma(\beta_c) = \mathbf{0}$.

Subcritical case: If $a(\beta_c) < 0$, then there is a neighborhood U of $\mathbf{x}^*$ and a constant $\gamma > 0$ such that, for $|\beta - \beta_c| < \gamma$ and $\mathbf{x}^0 \in U$, the α -limit set of $\mathbf{x}^0$ is $\mathbf{x}^*$ if $\beta > \beta_c$ and belongs to a closed invariant C^1 curve $\Gamma(\beta)$ encircling $\mathbf{x}^*$ if $\beta < \beta_c$. Furthermore, $\Gamma(\beta_c) = \mathbf{0}$.

The coefficient $a(\beta_c)$ is calculated as follows. See Hale and Koçak (1991). Now, we introduce new variables $\mathbf{y} = \mathbf{x} - \mathbf{x}^*$. We can consider a map $(\beta, \mathbf{y}) \mapsto \bar{F}(\beta, \mathbf{y})$ which has a fixed point at the origin. Clearly, $D\bar{F}(\beta, 0)$ has non-real, conjugate complex eigenvalues $\mu(\beta)$ and $\bar{\mu}(\beta)$ with $|\mu(\beta_c)| = 1$. Then, we can assume $\bar{F}$ to have the following form near the origin:

$$\mathbf{y} \mapsto \bar{F}(\mathbf{y}) = \begin{pmatrix} \text{Re}(\mu) & -\text{Im}(\mu) \\ \text{Im}(\mu) & \text{Re}(\mu) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} G_1(y_1, y_2) \\ G_2(y_1, y_2) \end{pmatrix}. \quad (8)$$

Then,

$$a = \text{Re} \left[\frac{(1 - 2\mu)\bar{\mu}^2}{1 - \mu} \xi_{11}\xi_{20} \right] + \frac{1}{2} |\xi_{11}|^2 + |\xi_{02}|^2 - \text{Re}(\bar{\mu}\xi_{21}), \quad (9)$$

where

$$\begin{aligned} \xi_{20} = & \frac{1}{8} \{ (G_1)_{x_1 x_1} - (G_1)_{x_2 x_2} + 2(G_2)_{x_1 x_2} \\ & + i[(G_2)_{x_1 x_1} - (G_2)_{x_2 x_2} - 2(G_1)_{x_1 x_2}] \}, \end{aligned}$$

¹This somewhat strange condition is related to another type of bifurcation, which we examine in the next section. We will briefly review this condition later.

$$\begin{aligned}\xi_{11} &= \frac{1}{4} \{ (G_1)_{x_1 x_1} + (G_1)_{x_2 x_2} + i [(G_2)_{x_1 x_1} + (G_2)_{x_2 x_2}] \}, \\ \xi_{02} &= \frac{1}{8} \{ (G_1)_{x_1 x_1} - (G_1)_{x_2 x_2} - 2(G_2)_{x_1 x_2} \\ &\quad + i [(G_2)_{x_1 x_1} - (G_2)_{x_2 x_2} + 2(G_1)_{x_1 x_2}] \},\end{aligned}$$

and

$$\begin{aligned}\xi_{21} &= \frac{1}{16} \{ (G_1)_{x_1 x_1 x_1} + (G_1)_{x_1 x_2 x_2} + (G_2)_{x_1 x_1 x_2} + (G_2)_{x_2 x_2 x_2} \\ &\quad + i [(G_2)_{x_1 x_1 x_1} + (G_2)_{x_1 x_2 x_2} - (G_1)_{x_1 x_1 x_2} - (G_1)_{x_2 x_2 x_2}] \}.\end{aligned}$$

Here the term $(G_i)_{x_j}$ denotes the partial derivative of G_i with respect to x_j , and so on. All the partial derivatives above are evaluated at the origin.

See Kuznetsov (1995) for a treatment of this bifurcation, especially the calculation of the coefficient $a(\beta_c)$ in a complex map setting and a complete proof of the theorem.

Thus, we obtain the following proposition.

Proposition 2. (The existence of a closed invariant curve)

Let

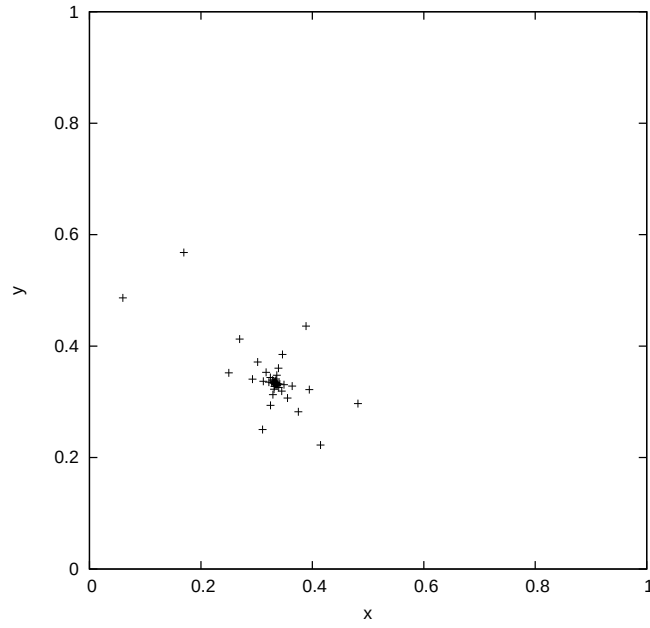
$$\begin{aligned}z &= (0, 1) \quad \text{and} \\ \Omega &= \{(\epsilon, \lambda) | 1 > \epsilon, \lambda = 1 - \sqrt{3}(1 - \epsilon)/3(1 + \epsilon)\}.\end{aligned}$$

If $(\epsilon, \lambda) \notin z \cup \Omega$, the map given by (5) exhibits a supercritical Neimark-Sacker bifurcation at $\beta = \beta_c$. That is, for $\beta > \beta_c$ and small $|\beta - \beta_c|$, the map given by (5) has an attracting closed invariant curve $\Gamma(\beta)$ encircling the Nash equilibrium.

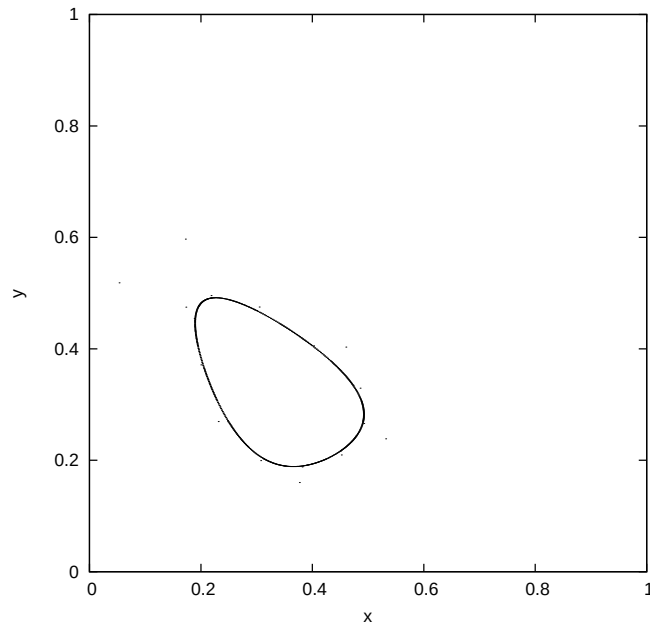
Proof. See Appendix. □

Fig.1 (a) illustrates the phase diagram on the (x, y) -plane for $\beta = 3 < \beta_c$, recalling that x denotes the share of , say, “rock” and y denotes that of “paper”. Fig.1 (b) illustrates the diagram for $\beta = 4 > \beta_c$. Of course, depending on the parameter values, the dynamic behavior on the closed invariant curve will change. It can be periodic or quasi-periodic in the topological equivalence to the rational or irrational rotation on the circle, respectively.

Contrary to the continuous-time logit dynamics studied by Hommes and Ochea (2012), who show that a limit cycle appears only if the game is bad, our discrete-time model can generate cyclical behavior around the Nash equilibrium not only in a bad game but also in a good game.



(a) Trajectory converging to the attracting fixed point corresponding to the Nash equilibrium. $\beta = 3 < \beta_c$.



(b) Trajectory converging to the attracting closed invariant curve generated by the Neimark-Sacker bifurcation. $\beta = 4 > \beta_c$.

Figure 1: Trajectories of the RPS game with logit dynamics for $\epsilon = 0.8$, and $\lambda = 0.5$.

4.5 Dynamics near the resonance: Multiple cycles

In this and the next section, we will analyze the case of strong resonance, which is a different type of bifurcation than in the previous section. Strong resonance occurs when condition (iv) of Theorem 1 is not satisfied. This condition² requires the eigenvalues, μ and $\bar{\mu}$, not to cross the unit circle on the complex plane at the points $(1, 0)$ for $k = 1$, $(-1, 0)$ for $k = 2$, $(-\frac{1}{2}, \pm \frac{\sqrt{3}}{2})$ for $k = 3$ and $(0, \pm 1)$ for $k = 4$.

Here, we will focus on the case where condition (iv) of Theorem 1 is not satisfied at $k = 3$ ³. That is, the eigenvalues cross a unit circle on the complex plane at the point $(-\frac{1}{2}, \pm \frac{\sqrt{3}}{2})$. Since we focus on the specific case of $k = 3$, we say that a dynamical system whose Jacobian matrix at the fixed point has eigenvalues $\mu_{1,2} = e^{\pm i \frac{2\pi}{3}}$ satisfies the resonance condition. See e.g. Arrowsmith and Place (1994) for more technical details related to the resonance in discrete-time dynamical systems.

In Sections 4.5.1 and 4.5.2, we will directly analyze the system in (5) as a *one*-parameter family of maps to show the existence of two types of period-three cycles. In Section 4.5.3, we will explain the appearance of the cycles. Finally, in Section 4.5.4, we will show the coexistence of cycles near the resonance.

4.5.1 Period-three attracting cycle

In what follows, we will assume $\lambda = 1$, for simplicity of calculation. In addition, we will first investigate the special case where $\epsilon = 0$, that is, the case where a losing player and a player in a tied game get the same payoff. These parametric restrictions will be relaxed somewhat using a perturbation argument. Note that for $\beta = \beta_c = 3$, the Jacobian matrix of the map (5) has eigenvalues $\mu_{1,2} = e^{\pm i \frac{2\pi}{3}}$.

In this subsection, we will detect an attracting periodic cycle far from the Nash equilibrium. First, we will show the existence of the cycle.

Lemma 1. *For $\beta = \beta_c$, the map given by (5) has a period-three cycle.*

Proof. Let $f_\beta : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ denote the map (5) and $F_\beta : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the third iteration of f_β (i.e., $F_\beta = f_\beta^3$). Moreover, let $V_i (i = 1, 2, 3)$ be

$$V_1 = \{(x, y) | y = x\}, \quad V_2 = \{(x, y) | y = -2x + 1\}, \quad \text{and} \quad V_3 = \{(x, y) | x = -2y + 1\}.$$

It is easy to show that f_β maps V_i to V_{i+1} for $i = 1, 2$ and V_3 to V_1 . Therefore, V_1 , V_2 , and V_3 are each invariant under F_β . Now, we can focus the analysis on the one-dimensional

²Suppose that the Jacobian matrix in Theorem 1 has eigenvalues of the form $\mu = \exp(\pm 2i\pi A)$ with $i = \sqrt{-1}$ and a rational number $A = p/q$ (irreducible). We say that weak resonance occurs if $q \geq 5$ and that strong resonance occurs if $q \leq 4$.

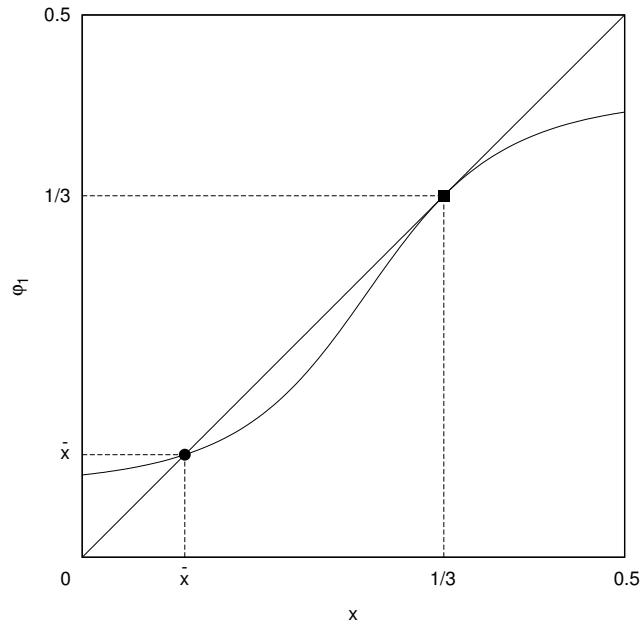
³Condition (iv) of Theorem 1 is satisfied at $k = 1, 2$ for any parameters in our model (see the proof of Proposition 2 in Appendix A). Moreover, we do not discuss the case where condition (iv) of Theorem 1 is not satisfied at $k = 4$, i.e., $(\epsilon, \lambda) \in \Omega$, because it seems to be less interesting. Indeed, in such a case, computer simulations show that our model displays quite similar dynamics to those of the standard Neimark-Sacker case.

dynamics. Let φ_i be the restriction of F_β to V_i . Then, we can show that $\varphi_i(x; \beta)$ has the following properties:

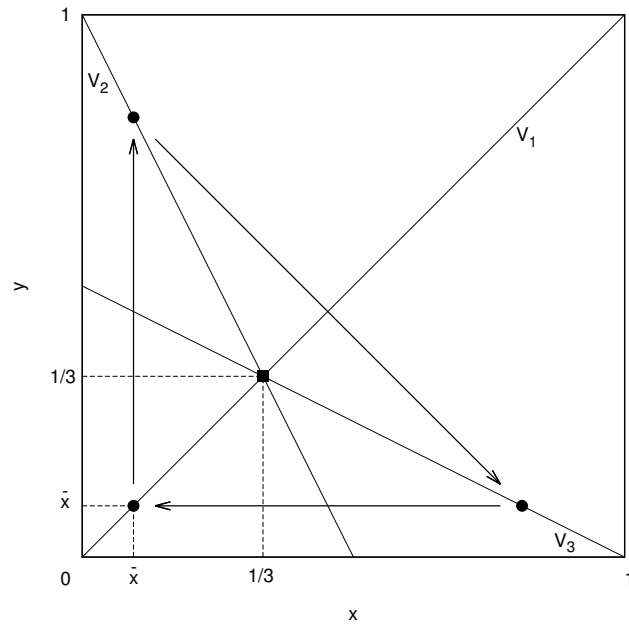
1. $\varphi_1(x; \beta) = \varphi_2(x; \beta) = \varphi_3(-2x + 1; \beta)$ and $\varphi_2(-2x + 1; \beta) = \varphi_3(x; \beta)$;
2. $\varphi_1(0; \beta) > 0$, $\varphi_1\left(\frac{1}{2}; \beta\right) < \frac{1}{2}$, and $\varphi_1\left(\frac{1}{3}; \beta\right) = \frac{1}{3}$;
3. $\varphi_1'(x; \beta) > 0$;
4. $\varphi_1'\left(\frac{1}{3}; \beta_c\right) = 1$ and $\varphi_1''\left(\frac{1}{3}; \beta_c\right) < 0$.

The second argument in the brackets indicates the dependence of parameter β . These properties, together with the continuity of $\varphi_1(x; \beta)$, imply that $\varphi_1(x; \beta_c)$, $\varphi_2(x; \beta_c)$, and $\varphi_3(-2x + 1; \beta_c)$ each cross the 45 degree line from above with a positive slope and at the same point. Therefore, F_{β_c} has three fixed points: $\mathbf{x}_1 = (\bar{x}, \bar{x})$, $\mathbf{x}_2 = (\bar{x}, -2\bar{x} + 1)$, and $\mathbf{x}_3 = (-2\bar{x} + 1, \bar{x})$, where $\bar{x} < \frac{1}{3}$.

Since f_β restricted to V_i has no other fixed point than the Nash equilibrium for $\beta = \beta_c$, the fixed point $\mathbf{x}_i$ corresponds to a period-three cycle of f_β . $\square$



(a) The graph of the map $\varphi_i(x; \beta_c)$ for $\beta = \beta_c$.



(b) Period-three cycle on the (x, y) -plane for $\beta = \beta_c$.

Figure 2: One-dimensional map $\varphi_i(x; \beta_c)$ has the fixed point $\bar{x}$ and, therefore, the original map has a period-three cycle corresponding to the fixed point $\bar{x}$: $\epsilon = 0, \lambda = 1$.

Fig.2 (a) shows the graph of $\varphi_1(x; \beta_c)$, and Fig.2 (b) depicts the periodic points on the (x, y) -plane. Note that the property (i) and the monotonicity of φ_i imply that $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ belong to the same period-three cycle.

Next, the following lemma gives the stability result of the fixed point $\mathbf{x}_i$.

Lemma 2. *For $i = 1, 2$ and 3 , the fixed point $\mathbf{x}_i$ in Lemma 1 is a hyperbolic attractor.*

Proof. See Appendix. □

Taking a smooth (ϵ, λ) -dependence of the map (5) into account, we can combine Lemma 1 with Lemma 2 to obtain the following proposition.

Proposition 3. (The existence of a period-three attracting cycle)

For sufficiently small $|\beta - \beta_c|$ and (ϵ, λ) near $(0, 1)$, the map given by (5) has a period-three attracting cycle.

4.5.2 Period-three saddle cycle

As we will show in Section 4.6, it is generally accepted that if a discrete-time dynamical system satisfies the resonance condition, then all nearby dynamical systems in the two-parameter space (excluding the codimension-two point) have a period-three saddle cycle.

Here, we will give an elementary proof of the existence and stability of the periodic cycle. This enables an intuitive understanding of the mechanism of the appearance of a period-three saddle cycle around the codimension-two point.

First, we will show the existence of the cycle.

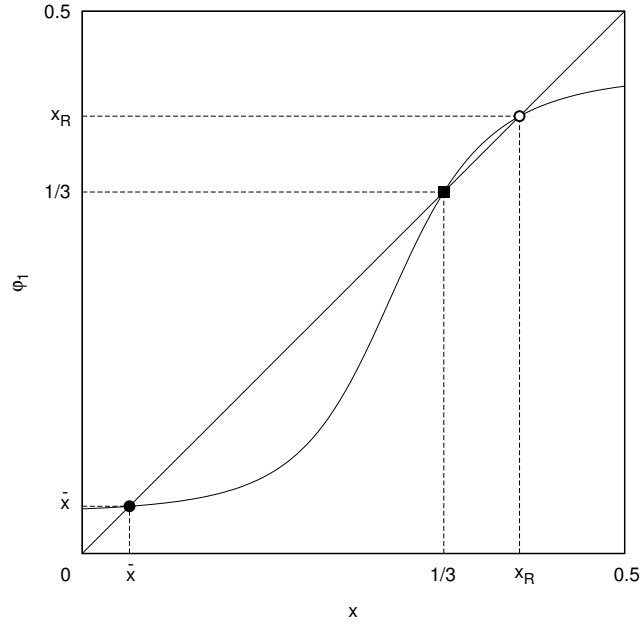
Lemma 3. *For sufficiently small $|\beta - \beta_c| \neq 0$, the map given by (5) has a period-three cycle near the Nash equilibrium.*

Proof. Differentiating $\varphi'_1(1/3; \beta)$ with respect to β , we obtain

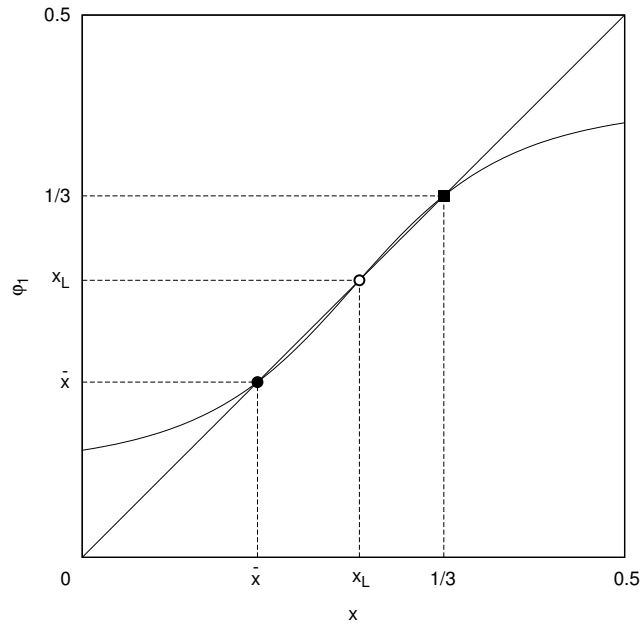
$$\left. \frac{\partial \varphi'_1(1/3; \beta)}{\partial \beta} \right|_{\beta=\beta_c} > 0.$$

Conditions (ii) and (iv) of Lemma 1 and the intermediate value theorem imply that the map $\varphi_1(x; \beta)$ has a fixed point x_R , for $\beta > \beta_c$, and x_L , for $\beta < \beta_c$, in the neighborhood of $x = 1/3$. Therefore, F_β has a fixed point $\mathbf{x}_R = (x_R, x_R)$, for $\beta > \beta_c$, and $\mathbf{x}_L = (x_L, x_L)$, for $\beta < \beta_c$, near the Nash equilibrium. By the same argument as Lemma 1, the fixed point corresponds to a period-three cycle of the map (5). □

Fig.3 displays the graph of $\varphi_1(x; \beta)$. Fig.4 depicts the periodic points on the (x, y) -plane.

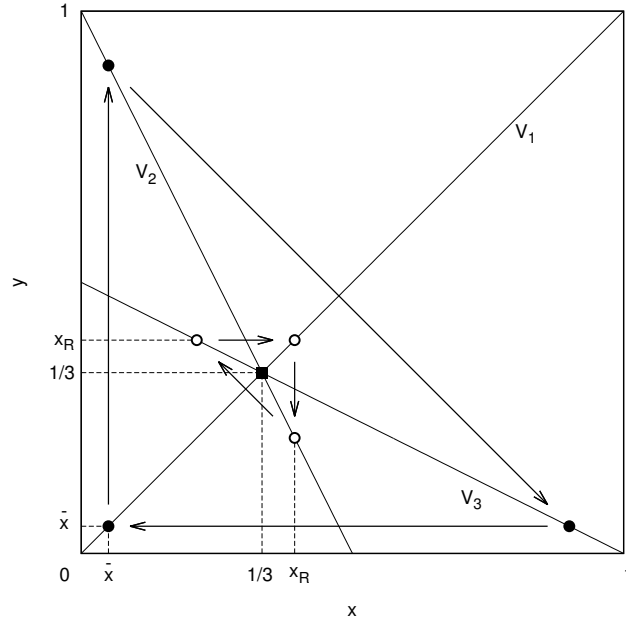


(a) The graph of the map $\varphi_i(x; \beta_c)$ for $\beta > \beta_c$.

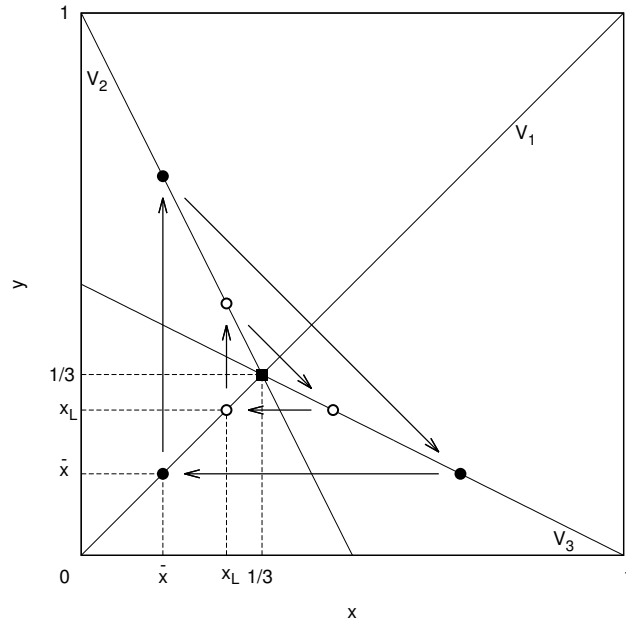


(b) Period-three cycle on the (x, y) -plane for $\beta < \beta_c$.

Figure 3: The map $\varphi_i(x; \beta_c)$ has a fixed point x_R for $\beta > \beta_c$ and x_L for $\beta < \beta_c$: $\epsilon = 0, \lambda = 1$.



(a) Multiple periodic cycles for $\beta > \beta_c$.



(b) Multiple periodic cycles for $\beta < \beta_c$.

Figure 4: For sufficiently small $|\beta - \beta_c| \neq 0$, the map (5) has two period-three cycles.

The following lemma gives the stability result of the fixed points $\mathbf{x}_R$ and $\mathbf{x}_L$.

Lemma 4. *The fixed points $\mathbf{x}_R$ and $\mathbf{x}_L$ in lemma 3 are hyperbolic saddle points.*

Proof. See Appendix. □

Using a similar perturbation argument to that in the previous subsection, we can summarize the above results in the following proposition.

Proposition 4. (The existence of a period-three saddle cycle)

Fix β such that $|\beta - \beta_c|$ is sufficiently small, but positive. Then, for (ϵ, λ) sufficiently near $(0, 1)$, the map given by (5) has a period-three saddle cycle.

4.5.3 Another look at the appearance of the period-three cycles

We have seen two types of period-three cycles in Sections 4.5.1 and 4.5.2. Here, we will explain the mechanism through which these period-three cycles occur.

Fig.5 (a) displays the graphs of $\varphi_1(x; \beta)$ for the parameter values $\beta = 2.6$ and $\beta = 2.78$. Fig.5 (b) shows the bifurcation diagram of $\varphi_1(x; \beta)$ with respect to β . This implies that there is some parameter value β^* at which the map $\varphi_1(x; \beta)$ undergoes a *fold bifurcation*, giving rise to two fixed points $\bar{x}$ and x_L , which correspond to a period-three attracting cycle and a period-three saddle cycle of the map (5), respectively. The fixed point x_L approaches $x^* = 1/3$ as β increases to β_c , and coincides with x^* at $\beta = \beta_c$. When β exceeds β_c , the fixed point $x_R > 1/3$ occurs.

4.5.4 Coexistence of cycles

Finally, we will briefly review and summarize the propositions on the existence of cycles investigated in the previous sections.

For all parameter values, the Nash equilibrium $(\frac{1}{3}, \frac{1}{3})$ is a fixed point. Moreover, the fixed point is an attractor when $\beta < \beta_c$ and a repeller when $\beta > \beta_c$. Proposition 2 ensures the existence of an attracting closed invariant curve for $(\epsilon, \lambda) \neq z$ and $\beta > \beta_c$. From Propositions 3 and 4, we have a period-three attracting cycle and a period-three saddle cycle for some values of $(\epsilon, \beta, \lambda)$ sufficiently close to $(0, 3, 1)$.

We can summarize Propositions 1, 2, 3, and 4 in the following proposition.

Proposition 5. (The coexistence of cycles)

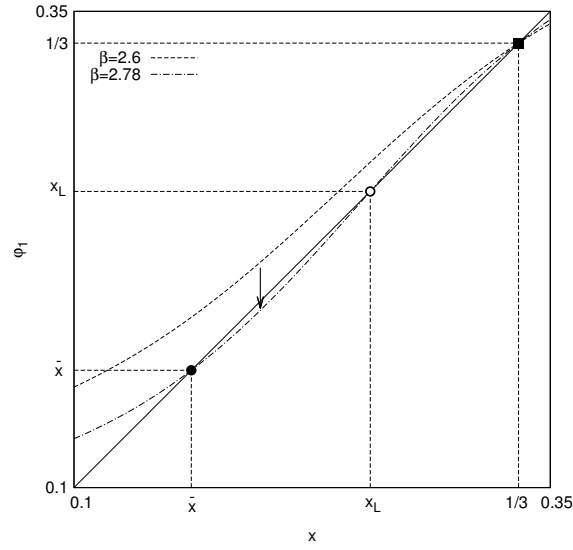
For $(\epsilon, \lambda) \neq (0, 1)$ and sufficiently near $(0, 1)$, the map given by (5) has the following cycles:

Case I : $\beta < \beta_c$;

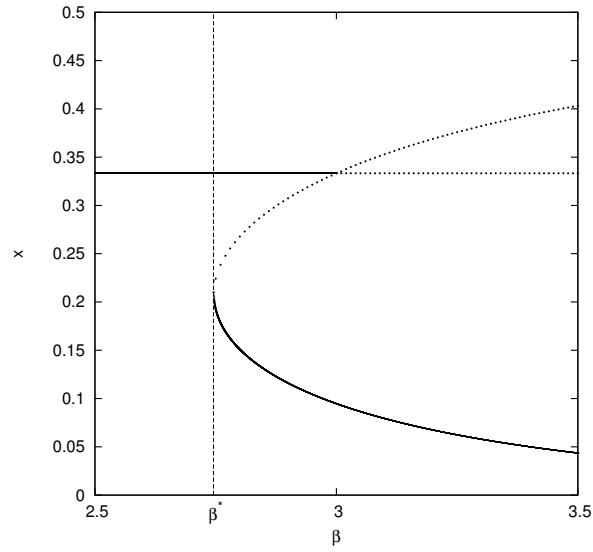
An attracting fixed point, a period-three attracting cycle, and a period-three saddle cycle.

Case II : $\beta > \beta_c$;

A repelling fixed point, an attracting closed invariant curve, a period-three attracting cycle, and a period-three saddle cycle.



(a) The graphs of map $\varphi_i(x; \beta_c)$ for parameter values $\beta = 2.6$ and $\beta = 2.78$.



(b) Multiple periodic cycles for $\beta < \beta_c$.

Figure 5: At parameter value $\beta^* \approx 2.7455 < 3 = \beta_c$, the map $\varphi_i(x; \beta_c)$ undergoes a fold bifurcation, giving rise to two fixed points. In panel (b), the solid curves correspond to attractors (an attracting fixed point for the horizontal line and a period-three attracting cycle for the parabolic line) and the dotted curves correspond to non-attractors (a repelling fixed point for the horizontal line and a period-three saddle cycle for the parabolic line).

To develop intuition regarding the mechanism of the coexistence of cycles described in Proposition 5, we will provide a graphical explanation. Recall that the Jacobian matrix given by (6) has non-real, conjugate complex eigenvalues, denoted as μ_1 and μ_2 ,

$$\mu_{1,2}(\beta) = (1 - \lambda) - \frac{1 - \epsilon}{6}\lambda\beta \pm i\frac{1 + \epsilon}{2\sqrt{3}}\lambda\beta.$$

Fig.6 depicts the unit circle on the complex plane and the neighborhood of the point $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$. Now, suppose that (ϵ, λ) is sufficiently near $(0, 1)$.

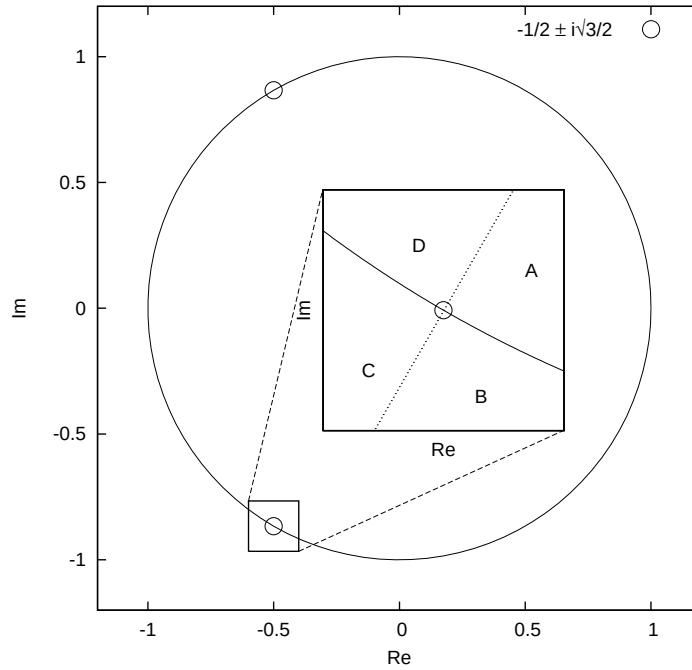


Figure 6: For (ϵ, λ) near but not on $(0, 1)$ and $\beta < \beta_c$, eigenvalue $\mu_2(\beta)$ lies on area A, which is case I of Proposition 5. For (ϵ, λ) near but not $(0, 1)$ and $\beta > \beta_c$, $\mu_2(\beta)$ lies on area B, which is case II of Proposition 5.

We can summarize the possible positions of the eigenvalue $\mu_2(\beta)$ ⁴ on the complex plane as follows:

⁴Because of conjugacy of eigenvalues $\mu_{1,2}(\beta)$, it suffices to examine one of them.

1. For $(\epsilon, \lambda) \neq (0, 1)$ and $\beta < \beta_c$, $\mu_2(\beta)$ lies on area A.
2. For $(\epsilon, \lambda) \neq (0, 1)$ and $\beta > \beta_c$, $\mu_2(\beta)$ lies on area B.
3. For $(\epsilon, \lambda) \neq (0, 1)$ and $\beta = \beta_c$, $\mu_2(\beta_c)$ lies on the unit circle between areas A and B.
4. For $(\epsilon, \lambda) = (0, 1)$ and $\beta \neq \beta_c$, $\mu_2(\beta_c)$ lies on the dotted line.
5. For $(\epsilon, \lambda) = (0, 1)$ and $\beta = \beta_c$, $\mu_2(\beta_c)$ coincides with the point $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$.

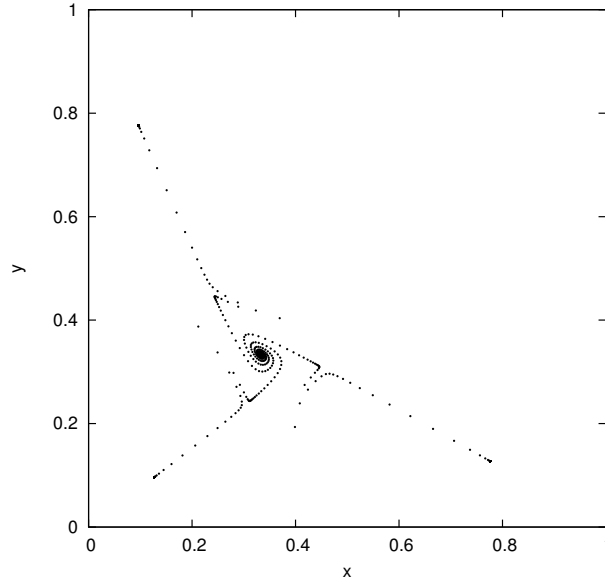
Note that, for the parametric restriction that $\epsilon \geq 0$, $\mu_2(\beta)$ never lies on areas C and D. Proposition 5 therefore describes cases (i) and (ii). Due to the smooth $(\epsilon, \lambda, \beta)$ -dependence of the eigenvalues, the result of Proposition 5 is robust⁵ against perturbation of the parameters $(\epsilon, \lambda, \beta)$.

In any evolutionary game dynamics, the occurrence of a period three cycle in the RPS game is not surprising. Indeed, the RPS game in discrete-time best response dynamics exhibits a period-three cycle. It is new to the literature, however, that an attracting period-three cycle is shown to coexist with an attracting Nash equilibrium or attracting closed invariant curve in the RPS game in the discrete-time logit setting, which is not observed in discrete-time best response dynamics. Figs.7 (a) and (b) display the coexistence of attractors on the (x, y) -plane. The coexistence of attractors implies that there are multiple long-run outcomes that are typical in the sense that each long-run state attracts trajectories from a large set of initial conditions in the state space. In other words, each long-run outcome can be reached with a positive probability when the initial conditions are randomly chosen. Interestingly, our results show that the long-run outcomes are *qualitatively* different in terms of the dynamic properties. For example, one trajectory converges to a steady state and the other one settles down to a cyclical behavior.

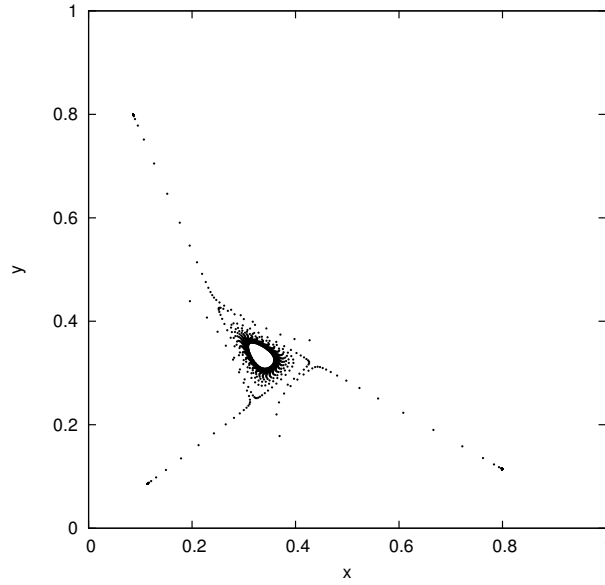
One might be interested in the basins of attraction of attractors and their boundaries. Fig.8 represents the basins of attraction of the attracting period-three cycle and the attracting closed invariant curve. The latter corresponds to Fig.7 (b). Fig.8 and Fig.7 (b) reveal that the stable manifolds of the period-three saddle cycle form the boundary of the basins of the attractors.

Finally, we will give some remarks on the goodness or badness of the game. In continuous-time (simplest possible) replicator dynamics, the RPS game is known to be good if and only if the Nash equilibrium is a global attractor. In continuous-time logit dynamics, the limit cycle around the Nash equilibrium appears only if the game is bad, as Hommes and Ochea (2012) show. Based on our findings, the rich dynamic features investigated in this section occur when the payoff value of losing is close to zero, which is the payoff value in the tie. In short, we have found that at least in such limiting good games, the interesting dynamic phenomena summarized in Proposition 5 can be observed.

⁵However, as we have seen so far and will see below, most results given in Proposition 5 such as the existence of the attracting Nash equilibrium, an attracting closed invariant curve, or the coexistence of such attractors will disappear as noise vanishes, i.e., as β goes to infinity. In contrast, the attracting period-three cycle will persist for large β .



(a) Two trajectories from different initial conditions, for $\beta < \beta_c$.



(b) Two trajectories from different initial conditions, for $\beta > \beta_c$.

Figure 7: Trajectories of the RPS game for fixed parameters $\epsilon = 0.1$, and $\lambda = 1$. (a) $\beta = 2.8 < \beta_c$. One trajectory converges to the attracting period-three cycle, and the other one converges to the attracting Nash equilibrium. (b) $\beta = 2.85 > \beta_c$. One trajectory converges to the attracting period-three cycle, and the other one converges to the attracting closed invariant curve surrounding the repelling Nash equilibrium.

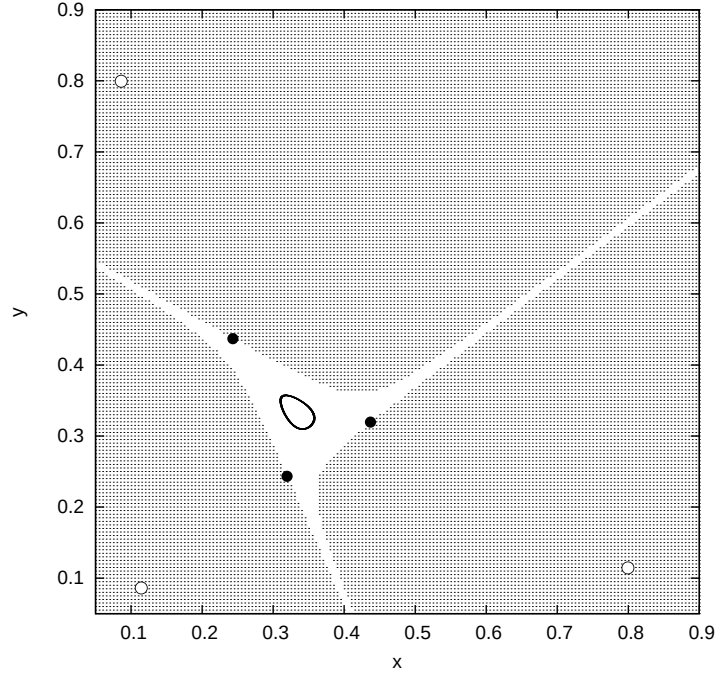


Figure 8: Basins of attraction of the attracting period-three cycle and the attracting closed invariant curve. A trajectory starting from the shaded regions converges to the attracting period-three cycle represented by open circles. In contrast, a trajectory starting from the white region converges to the attracting closed invariant curve surrounding the Nash equilibrium $(1/3, 1/3)$. The solid circles represent the period-three saddle cycle. $\epsilon = 0.1, \lambda = 1$ and $\beta = 2.85 > \beta_c$.

4.6 Dynamics near the resonance: Global dynamics

In this section, we will use the codimension-two bifurcation theory to derive the bifurcation set near the resonance, providing some implications of the global bifurcation in our model. Specifically, we will exploit the flow-approximation technique, which transforms the original map into a unit-time shift along the orbits of certain autonomous ordinary differential equations at resonance parameter values. This allows us to determine information about the global dynamics, which is not available when studying only the original map. See Arnold (1977), Arrowsmith and Place (1994), Kuznetsov (1995), and Whitley (1983) for more details.

As in the previous section, we assume $\lambda = 1$, for simplicity of calculations. Moreover, let

$$\alpha = (\epsilon, \beta) \quad \text{and} \quad \alpha_c = (0, 3).$$

It is known that the third iteration of the map with the resonance condition generally

allows for flow approximation. First, we will derive the differential equation that approximates the original map. By applying the lemmas in Kuznetsov (1995) to our model, straightforward but tedious calculations give the following lemmas:

Lemma 5. *For all α sufficiently near α_c , the map given by (5) can be transformed into the complex form*

$$\zeta \mapsto T_\alpha(\zeta) = \mu(\alpha)\zeta + B(\alpha)\bar{\zeta}^2 + C(\alpha)\zeta|\zeta|^2 + O(|\zeta|^4), \quad (10)$$

where

$$B(\alpha) = -\frac{\beta^2}{12} \left[\sqrt{3}(\epsilon^2 - 1) - (\epsilon^2 + 4\epsilon + 1)i \right],$$

and

$$C(\alpha) = -\frac{\beta^3}{12} \left[(\epsilon^3 - 1) + \sqrt{3}(\epsilon^3 + 2\epsilon^2 + 2\epsilon + 1)i \right].$$

Lemma 6. *The third iteration of the map given by (10) can be represented for all α sufficiently near α_c in the form*

$$T_\alpha^3(\zeta) = \rho_\alpha^1(\zeta) + O(|\zeta|^4),$$

where ρ_α^t is the flow of the planar system

$$\dot{\zeta} = \nu(\alpha)\zeta + B_1(\alpha)\bar{\zeta}^2 + C_1(\alpha)\zeta|\zeta|^2, \quad (11)$$

where ν , B_1 , and C_1 are smooth complex-valued functions of α , $\nu(\alpha_c) = 0$,

$$\begin{aligned} B_1(\alpha_c) &= 3\bar{\mu}(\alpha_c)B(\alpha_c) = -\frac{9}{2}i \quad \text{and} \\ C_1(\alpha_c) &= -3|B(\alpha_c)|^2 + 3\mu^2(\alpha_c)C(\alpha_c) = -\frac{81}{4}. \end{aligned}$$

Let $\phi(\alpha)$ and $\psi(\alpha)$ denote the real and imaginary parts of $\nu(\alpha)$, respectively. Then, we have

$$\begin{aligned} \phi(\alpha) &= 3\log|\mu(\alpha)| \quad \text{and} \\ \psi(\alpha) &= 3\arg\mu(\alpha) \pmod{2\pi}. \end{aligned}$$

See, for example, Kuznetsov (1995) for more detail on this point. Therefore, in our model, we can represent $\phi(\alpha)$ and $\psi(\alpha)$ in the forms

$$\phi(\alpha) = 3\log\frac{\beta\sqrt{\epsilon^2 + \epsilon + 1}}{3} \quad \text{and} \quad (12)$$

$$\psi(\alpha) = \left[\pi + 3\arctan\frac{\sqrt{3}(\epsilon + 1)}{\epsilon - 1} \right]. \quad (13)$$

Moreover, let

$$\zeta = \kappa(\alpha)\eta,$$

with

$$\kappa(\alpha) = \frac{1}{|B_1(\alpha)|} \exp\left(i \frac{\arg B_1(\alpha)}{3}\right).$$

Then the system in (11) becomes

$$\dot{\eta} = [\phi(\alpha) + i\psi(\alpha)]\eta + \bar{\eta}^2 + \frac{C_1(\alpha)}{|B_1(\alpha)|^2} \eta |\eta|^2. \quad (14)$$

Taking $\eta = v + iw$, we obtain

$$\begin{cases} \dot{v} = \phi(\alpha)v - \psi(\alpha)w + v^2 - w^2 + [b_1(\alpha)v - c_1(\alpha)w](v^2 + w^2) \\ \dot{w} = \psi(\alpha)v + \phi(\alpha)w - 2vw + [c_1(\alpha)v + b_1(\alpha)w](v^2 + w^2), \end{cases} \quad (15)$$

where

$$\begin{aligned} b_1(\alpha) &= \operatorname{Re} \left[\frac{C_1(\alpha)}{|B_1(\alpha)|^2} \right] \quad \text{and} \\ c_1(\alpha) &= \operatorname{Im} \left[\frac{C_1(\alpha)}{|B_1(\alpha)|^2} \right]. \end{aligned}$$

First, we will examine the local dynamics of the system in (15). Clearly, the system in (15) always has a trivial equilibrium at the origin. Let $J_\alpha(x, y)$ denote the Jacobian matrix of the system. Then, J_α , evaluated at the trivial equilibrium, is given by

$$J_\alpha(0, 0) = \begin{pmatrix} \phi(\alpha) & -\psi(\alpha) \\ \psi(\alpha) & \phi(\alpha) \end{pmatrix}, \quad (16)$$

with eigenvalues $\phi(\alpha) \pm i\psi(\alpha)$. The equilibrium is an attractor if $\phi(\alpha) < 0$ and a repeller if $\phi(\alpha) > 0$. Moreover, $b_1(\alpha_c) = -1$, by calculation, and the non-degeneracy condition

$$b_1(\alpha_c) \neq 0$$

is satisfied. This implies that the trivial equilibrium undergoes a supercritical Hopf bifurcation at the bifurcation curve $\phi(\alpha) = 0$, which can be explicitly represented by

$$\beta = \frac{3}{\sqrt{\epsilon^2 + \epsilon + 1}}.$$

Note that for $\alpha = \alpha_c$, no limit cycle appears because $\psi(\alpha_c) = 0$. In addition to the trivial equilibrium, it is known that for $\alpha \neq \alpha_c$, there are three nontrivial equilibria, which are always saddles. Needless to say, the trivial equilibrium of the system in (15) corresponds to the Nash equilibrium, and the Hopf curve corresponds to the Neimark-Sacker curve.

Moreover, the saddle equilibria correspond to the period-three saddle cycles mentioned in Section 4.5.2.

Next, we will discuss the global dynamics of the system in (15), which are the main focus of this section. It is known that the system in (15) has the bifurcation curve

$$H = \left\{ (\phi, \psi) : \phi = -\frac{b_1}{2}\psi^2 + o(\psi^2) \right\},$$

at which the limit cycle disappears as a result of a *heteroclinic bifurcation*. That is, for parameter values on the curve H , stable and unstable manifolds of saddles coincide simultaneously, forming a heteroclinic cycle. See, for example, Arrowsmith and Place (1994) for more details. Using formulae (12) and (13) and $b_1(\alpha_c) = -1$ to solve $\phi = -\frac{b_1}{2}\psi^2$, we can obtain the *approximated* heteroclinic curve in the following form:

$$\beta = \frac{3 \exp \left[\frac{1}{6} \left(\pi + 3 \arctan \frac{\sqrt{3}(\epsilon+1)}{\epsilon-1} \right)^2 \right]}{\sqrt{\epsilon^2 + \epsilon + 1}}. \quad (17)$$

Fig.9 displays the bifurcation diagram of the system in (15).

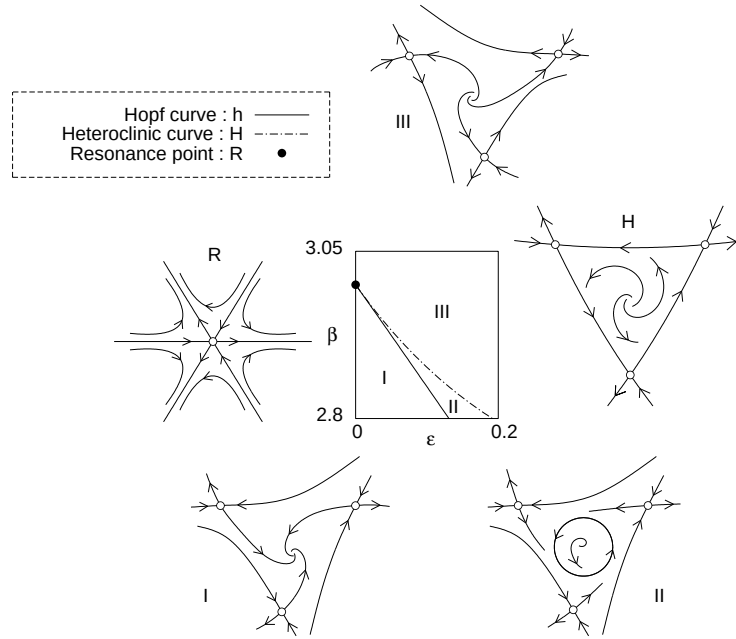
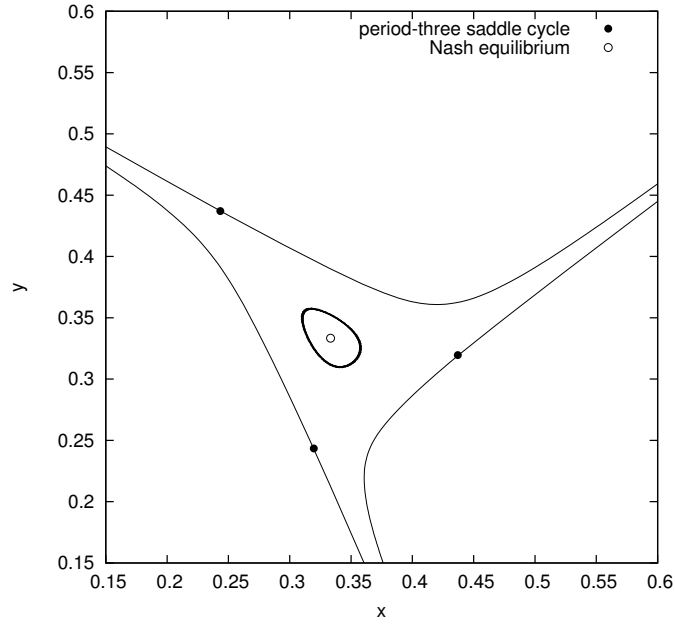


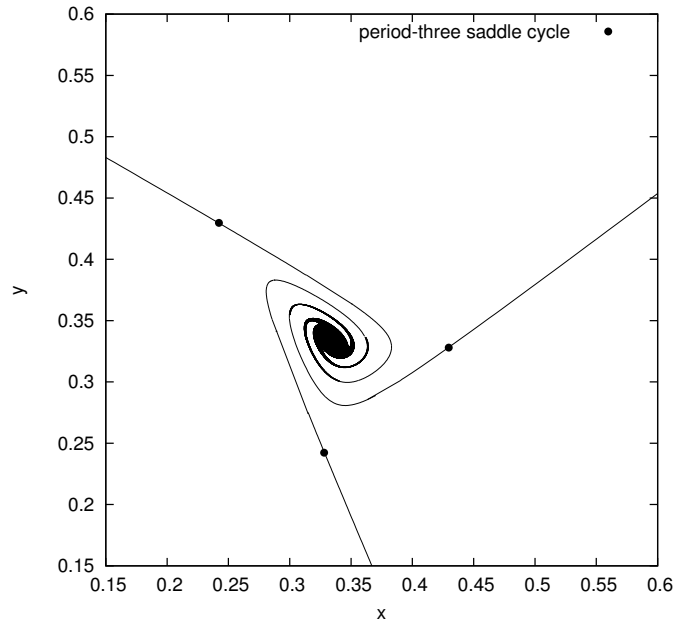
Figure 9: Bifurcation diagram of the system in (15). A stable limit cycle appears when we cross the Hopf curve h from below at a point that is not a resonance point R . Moreover, when we cross the heteroclinic curve H , the limit cycle disappears as a result of a heteroclinic bifurcation.

Now, we can translate the above results into the original map. In general, instead of the heteroclinic bifurcation curve H , the original map may possess a more complex bifurcation set. This is because the stable and unstable invariant manifolds of the saddle periodic points can intersect transversally, forming homoclinic or heteroclinic tangles. The existence of so-called Smale's horseshoes created by the homoclinic or heteroclinic tangles gives rise to chaotic invariant sets. See e.g. Whitley (1983) and Arrowsmith and Place (1994) for the occurrence of chaos for discrete two-dimensional dynamical systems. However, our discrete-time model seems to exhibit the dynamics similar to continuous-time counterpart in (15), owing to the symmetric property, and neither homoclinic nor heteroclinic tangles are formed. Fig.10 displays the stable manifolds, computed numerically, of the periodic saddle points of the map (5).

From Fig.10, we can expect that whenever the stable and unstable manifolds of the saddles intersect, they completely coincide and form saddle connections. Once saddle connections occur, the closed invariant curve generated by the Neimark-Sacker bifurcation disappears. In Fig.11, we can see that the closed invariant curve suddenly disappears at the parameter value $\beta_H \approx 2.87682$, which is computed numerically. On the other hand, using the formula for the heteroclinic curve given by (17), we can obtain the theoretical value of the heteroclinic bifurcation as $\beta \approx 2.87654$. Indeed, the calculated value β_H is very close to the theoretical value of heteroclinic bifurcation.



(a) For parameter values in area II of Fig.9.



(b) For parameter values in area III of Fig.9.

Figure 10: The stable manifolds of the saddle periodic points. (a) $\epsilon = 0.1, \beta = 2.85$. There is an attracting closed invariant curve around the Nash equilibrium. (b) $\epsilon = 0.1, \beta = 2.9$. The case where the attracting closed invariant curve disappears.

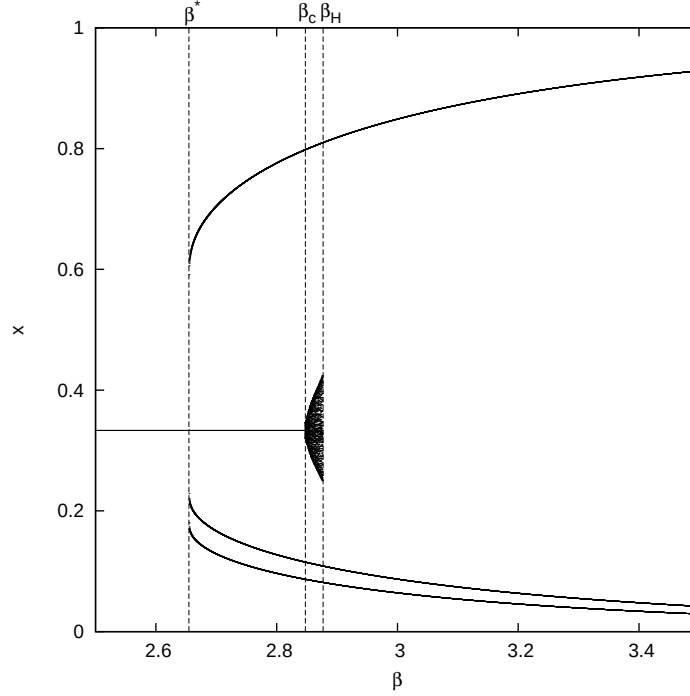


Figure 11: Bifurcation diagram near the resonance with respect to β . $\epsilon = 0.1, \lambda = 1$. When $\beta > \beta^* \approx 2.65474$ (fold bifurcation) and $\beta < \beta_c \approx 2.84747$ (Neimark-Sacker bifurcation), we can observe the coexistence of the attracting fixed point and the period-three attracting cycle. On the other hand, when $\beta > \beta_c \approx 2.84747$ and $\beta < \beta_H \approx 2.87682$ (heteroclinic bifurcation), we can observe the coexistence of the attracting closed invariant curve and the period-three attracting cycle.

4.7 Concluding remarks

We examined a discrete-time version of logit dynamics applied to the rock-paper-scissors game, where the level of noise is not negligibly small or the rationality of agents is not required to be extremely high.

First, we demonstrated that our model can exhibit non-stationary behaviors around the Nash equilibrium due to Neimark-Sacker bifurcation. More importantly, we showed that our model is capable of exhibiting other dynamic patterns near the strong resonance. As a result, we demonstrated that near the strong resonance, an attracting closed invariant curve generated by the Neimark-Sacker bifurcation coexists with periodic cycles of different stability types. To some extent, our findings, including the coexistence of attractors, may explain how different initial conditions make the long-run outcomes different in terms of a dynamic patterns, not only quantitatively but also qualitatively. Moreover, we specified the set of a global bifurcation of the approximated system, although the analytical computation of such a set is, in general, not an easy task. It is reasonably conjectured

that the set of the global bifurcation of the approximated system can be identified using that of the original discrete-time system.

Finally, our model can generate more complex dynamics than those demonstrated here. In fact, our computer simulations, not presented here, suggest that the model can exhibit chaotic dynamics for other sets of parameter values. It would be interesting to study possible complex dynamics, which we leave for future research.

4.8 Appendix

4.8.1 Proof of Proposition 2

Proof. Clearly, the map (5) satisfies condition (i) of Theorem 1. Moreover, we checked that conditions (ii) and (iii) are satisfied in Section 4.3. Since the imaginary part of the eigenvalue $\mu(\beta)$ is not zero for any parameter values, we need only check whether condition (iv) is satisfied at $k = 3, 4$. The primitive 3rd and 4th roots of unity are given by $-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ and $\pm i$, respectively. Recall

$$\mu_{1,2} = (1 - \lambda) - \frac{1 - \epsilon}{6}\lambda\beta \pm i\frac{1 + \epsilon}{2\sqrt{3}}\lambda\beta.$$

For the 3rd root, let

$$(1 - \lambda) - \frac{1 - \epsilon}{6}\lambda\beta_c = -\frac{1}{2} \quad \text{and} \quad (\text{A.1})$$

$$\frac{1 + \epsilon}{2\sqrt{3}}\lambda\beta_c = \frac{\sqrt{3}}{2}. \quad (\text{A.2})$$

Solving Eq. (A.1) for β_c and substituting it into Eq. (A.2), we obtain

$$(\lambda - 1) = (2 - \lambda)\epsilon.$$

This equation is satisfied if and only if

$$(\epsilon, \lambda) = z \quad (0, 1).$$

Similarly, we can show that the equations for the 4th root,

$$(1 - \lambda) - \frac{1 - \epsilon}{6}\lambda\beta_c = 0 \quad \text{and} \quad \frac{1 + \epsilon}{2\sqrt{3}}\lambda\beta_c = 1$$

are satisfied if and only if

$$(\epsilon, \lambda) \in \Omega \quad \{(\epsilon, \lambda) | 1 > \epsilon, \lambda = 1 - \sqrt{3}(1 - \epsilon)/3(1 + \epsilon)\}.$$

Next, in order to determine the stability of the invariant curve, we need to check the sign of $a(\beta_c)$. We can transform the map (5) into the following form:

$$\begin{pmatrix} X \\ Y \end{pmatrix} \mapsto \Lambda \begin{pmatrix} X \\ Y \end{pmatrix} + G(X, Y), \quad (\text{A.3})$$

where

$$\Lambda = \begin{pmatrix} (1 - \lambda) - \frac{1 - \epsilon}{6} \lambda \beta_c & -\frac{1 + \epsilon}{2\sqrt{3}} \lambda \beta_c \\ \frac{1 + \epsilon}{2\sqrt{3}} \lambda \beta_c & (1 - \lambda) - \frac{1 - \epsilon}{6} \lambda \beta_c \end{pmatrix}$$

and

$$G(X, Y) = \begin{pmatrix} \frac{\lambda}{2\sqrt{3}} g_1(X, Y) + \frac{\lambda}{\sqrt{3}} g_2(X, Y) + (1 - \lambda)X - \frac{\lambda}{2\sqrt{3}} \\ \frac{\lambda}{2} g_1(X, Y) + (1 - \lambda)Y - \frac{\lambda}{6} \end{pmatrix} - \Lambda \begin{pmatrix} X \\ Y \end{pmatrix}.$$

Moreover, $g_1(X, Y)$ and $g_2(X, Y)$ are of the following forms:

$$g_1(X, Y) = \frac{\tilde{v}_1(X, Y)}{\tilde{v}_1(X, Y) + \tilde{v}_2(X, Y) + \tilde{v}_3(X, Y)},$$

$$g_2(X, Y) = \frac{\tilde{v}_2(X, Y)}{\tilde{v}_1(X, Y) + \tilde{v}_2(X, Y) + \tilde{v}_3(X, Y)},$$

with

$$\tilde{v}_i(X, Y) = v_i \left(2Y + \frac{1}{3}, \sqrt{3}X - Y + \frac{1}{3}, -\sqrt{3}X - Y + \frac{1}{3} \right).$$

Using $G(X, Y)$ and formula (9), we can compute $a(\beta_c)$ to obtain

$$\begin{aligned} \text{sgn}[a(\beta_c)] &= \text{sgn}[\lambda \beta_c (\epsilon^4 + 2\epsilon^3 + 3\epsilon^2 + 2\epsilon + 1) - 3\text{Re}(\mu)(1 - \epsilon^3) \\ &\quad + 3\sqrt{3}\text{Im}(\mu_1)(\epsilon^3 + 2\epsilon^2 + 2\epsilon + 1)], \end{aligned} \quad (\text{A.4})$$

which is positive if $\epsilon \geq 1$. Furthermore, substituting

$$\begin{aligned} \text{Re}(\mu(\beta_c)) &= (1 - \lambda) - \frac{1 - \epsilon}{6} \lambda \beta_c, \\ \text{Im}(\mu_1(\beta_c)) &= \frac{1 + \epsilon}{2\sqrt{3}} \lambda \beta_c, \end{aligned}$$

and

$$\beta_c = \frac{3(1 - \lambda)(1 - \epsilon) + 3\sqrt{3\lambda(2 - \lambda)(1 + \epsilon)^2 + (1 - \epsilon)^2}}{2\lambda(1 + \epsilon + \epsilon^2)},$$

into Eq.(A.4), we obtain

$$\begin{aligned} \text{sgn}[a(\beta_c)] &= \text{sgn}[2(1 - \lambda)(1 - \epsilon)(1 + 2\epsilon + 3\epsilon^2 + 2\epsilon^3 + \epsilon^4) \\ &\quad + 3\sqrt{3\lambda(2 - \lambda)(1 + \epsilon)^2 + (1 - \epsilon)^2}\{(1 + \epsilon^2)^2 + \epsilon^3 + 2\epsilon + 1\}], \end{aligned}$$

which is positive if $\epsilon < 1$. Therefore, for any parameter values, $a(\beta_c)$ is positive. This completes the proof. $\square$

4.8.2 Proof of Lemma 2

Proof. A straightforward calculation shows that the Jacobian matrix of F_β evaluated at (x, x) has the following form:

$$\begin{pmatrix} \vartheta_\beta(x) & \varrho_\beta(x) \\ \varrho_\beta(x) & \vartheta_\beta(x) \end{pmatrix}. \quad (\text{B.1})$$

Therefore, (B.1) has the eigenvalues $\vartheta_\beta(x) + \varrho_\beta(x)$, with eigenvector ${}^t(1, 1)$, and $\vartheta_\beta(x) - \varrho_\beta(x)$ with eigenvector ${}^t(-1, 1)$. Note that $\varphi'(\bar{x}, \beta_c)$ corresponds to $\vartheta_{\beta_c}(\bar{x}) + \varrho_{\beta_c}(\bar{x})$. Moreover, from the argument in Lemma 1, we have $1 \geq \varphi'(\bar{x}, \beta_c) = \vartheta_{\beta_c}(\bar{x}) + \varrho_{\beta_c}(\bar{x})$.

First, we will show the hyperbolicity of the fixed point $\bar{x}$ of $\varphi(x, \beta_c)$. Suppose that $\vartheta_{\beta_c}(\bar{x}) + \varrho_{\beta_c}(\bar{x}) = 1$. Then, we have

$$4\exp(3\bar{x}) = \exp(3 - 6\bar{x}),$$

by calculation, and thus,

$$f_{\beta_c}(\bar{x}, \bar{x}) = \left(\frac{1}{6}, \frac{2}{3}\right) \neq (\bar{x}, -2\bar{x} + 1). \quad (\text{B.2})$$

However, this contradicts the fact that map f_{β_c} maps $(\bar{x}, \bar{x})$ to $(\bar{x}, -2\bar{x} + 1)$. Therefore, we have $\vartheta_{\beta_c}(\bar{x}) + \varrho_{\beta_c}(\bar{x}) \neq 1$.

Moreover, we can easily show that $\vartheta_{\beta_c}(x) - \varrho_{\beta_c}(x) > 0$, $\vartheta_{\beta_c}(x) > 0$, and $\varrho_{\beta_c}(x) > 0$, for $x < \frac{1}{3}$. These results imply that

$$1 > \vartheta_{\beta_c}(\bar{x}) + \varrho_{\beta_c}(\bar{x}) > \vartheta_{\beta_c}(\bar{x}) - \varrho_{\beta_c}(\bar{x}) > 0.$$

□

4.8.3 Proof of Lemma 4

Proof. Clearly, $\vartheta_\beta(x_L) + \varrho_\beta(x_L) > 1$ and $\vartheta_\beta(x_R) + \varrho_\beta(x_R) < 1$. Moreover, we can easily check that $\vartheta_\beta\left(\frac{1}{3}\right) - \varrho_\beta\left(\frac{1}{3}\right) \geq 1$, for $\beta \geq \beta_c$. Since $\vartheta_\beta(x) - \varrho_\beta(x)$ is positive and a monotone increasing function for x , we have

$$\begin{aligned} \vartheta_\beta(x_L) + \varrho_\beta(x_L) &> 1 > \vartheta_\beta(x_L) - \varrho_\beta(x_L) > 0 \quad \text{and} \\ \vartheta_\beta(x_R) - \varrho_\beta(x_R) &> 1 > \vartheta_\beta(x_R) + \varrho_\beta(x_R) > 0. \end{aligned}$$

□

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