Qualitative behaviors of solutions in reaction-diffusion systems

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Contents

Chapter 1

Introduction

My field of interests are partial differential equations of parabolic type. In particular, I study reaction-diffusion equations arising in phase transition, biology and combustion problems.

1.1 Discrete Allen-Cahn problem

The first subject I have approached during my PhD is the discrete Allen-Cahn problem on a line, especially the bifurcation diagram of positive stationary solutions. Moreover, dynamics of any positive solution is also discussed.

In order to get an initial idea, we start the typical model equation

$$u_t = u_{xx} + \mu u - u^3, \quad x \in (0, \pi)$$

with $u(0,t) = u(1,t) = 0$. In order to understand the dynamics of the solution near the zero stationary solution, we analyze the linearization at $u \equiv 0$, i.e.,

$$u_t = u_{xx} + \mu u,$$

with $u(0,t) = u(1,t) = 0$. Let us substitute $u(x,t) = \sum_{n\in\mathbb{Z}} \hat{u}_n \sin n\pi x$, then we get

$$\frac{d}{dt}\hat{u}_n = (\mu - \pi^2 n^2)\hat{u}_n.$$

Since we are considering positive solutions, the first Fourier mode must be dominant, which implies that $\mu = \pi^2$ is a bifurcating point. Also if $\mu < \pi^2$, then the solution of the above approximated problem

$$\hat{u}_n = e^{-(\pi^2 n^2 - \mu)t}\hat{u}_n$$

implies that the solution converges to zero as $t \to \infty$. On the other hand, if $\mu > \pi^2$, the solution can depart from the origin as time passes, which implies the existence of a positive stationary solution. This is a heuristic explanation of the structure of local bifurcation. See [36, 42] for the rigorous analysis.

In Chapter 2, we have investigated bifurcation diagram and asymptotic behavior of any positive solution to the discrete reaction diffusion equation

$$\begin{cases} \dfrac{u_j^{n+1}-u_j^n}{\tau} = \dfrac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2} + f(u_j^n), \quad 1 \le j \le N-1, \quad n \ge 0, \\ u_0^n = u_N^n = 0, \quad n > 0 \end{cases}$$

for a given initial value $\{u_j^0\}_{1\le j\le N-1}$ satisfying $u_j^0 \ge 0$ for all $1 \le j \le N-1$, and $f \in C^1[0,\infty)$ is a function satisfying the following conditions

(1) $f(u)/u$ is strictly monotone decreasing in $u > 0$.

(2) $f(0) = 0,\ f'(0) = \mu$ for a constant $\mu > 0$.

(3) $f(m) = 0$ for a constant $m > 0$.

Roughly speaking, assume that the discretized parameters τ/h^2 and τ are sufficiently small, then $\mu = \frac{4}{h^2}\sin^2(\pi h/2)$ is the bifurcation point and which determines the asymptotic behavior as continuous model. Note that $\frac{4}{h^2}\sin^2(\pi h/2)$ converges to π^2 as $h \to 0$.

We also consider the corresponding problem for the initial value problem for a semi-discrete difference equation of the form $\frac{du_j}{dt}(t) = \frac{u_{j+1}(t)-2u_j(t)+u_{j-1}(t)}{h^2} + f(u_j(t))$.

We write the paper in a self-contained manner. For example, we established fundamental techniques such as the discrete maximum principle, which was can not be found in other papers, an error estimate about Euler scheme and we also explain discrete Fourier transform. By the combination of these tools, all the results are obtained for both of the discrete models.

1.2 Center problem of reaction diffusion problem

The second subject I have approached during my PhD is the center problem of reaction diffusion problem.

Many models of natural phenomena use systems of differential equations in the plane and the qualitative theory of differential equations, introduced by Poincaré, can be used to describe the behavior of such systems in most cases. One of the problems here is to distinguish a focus and a center, which is called the center problem. The resolution of this problem requires the computation of the so-called Poincaré map or constructing a first integral. In fact, the center problem and the problem of finding the first integral are equivalent when the singularity has associated non-vanishing complex conjugated eigenvalues.

When we first encounter ordinary differential equations, they are presented with a variety of special techniques designed to solve certain particular types of equations, such as separable, homogeneous or exact. Indeed, this was the state of the art around the middle of the nineteenth century, when Sofus Lie discover that these special methods were special cases of a general integration procedure based on the invariance of the differential equation under a continuous group of symmetries. In particular, Lie symmetry gives us integrating factor, which is necessary to construct a first integral. As a result, to solve the center problem of ordinary differential equation at the non-degenerate linear center is reduced characterized by existence of symmetric vector fields.

On the other hand, finding the first integral is one of the strong technique not only ordinary differential equation but also many types of partial differential equation problem. Recently, Latos–Suzuki–Yamada [24] considered Lotka-Volterra reaction diffusion system and find *eventually spatially homogeneous oscillation*. More precisely, the solution of reaction diffusion system becomes spatially homogeneous as $t \to \infty$ and converges to one solution of the corresponding ordinary differential equation solution. Another example is given by Karali–Suzuki–Yamada [19], in which eventually spatially homogeneous oscillation Gierer-Meinhardt system is analyzed. Their method is based on the transformation that transforms the corresponding ordinary differential equation to a Hamilton system and by applying that Hamiltonian, they construct a Lyapunov functional. Some technical assumption on space dimension is assumed to get a priori bound of the solution. A similar phenomenon is recently considered by Guo-Shimojo [17]. They consider singular predator -prey model and apply algebraic Darbouxian integrable theory to construct a first integral of ordinary differential equation part. By integrating that first integral in space they construct a Lyapunov functional to get the eventually spatially homogeneous oscillation. Our aim of Chapter 3 is generalizing these results to completely general nonlinearity, from the context of Lie group. As a result, our theorem can be applied to all reaction diffusion problem which produce eventually spatially homogeneous oscillation, that all we will encounter in the future.

In Chapter 3, I first briefly explain that Lie symmetry gives us integrating factor and construct the first integral of ordinary differential equation problem through it. Secondly, by integrating the ordinary differential equation first integral on space, we construct a local Lyapunov functional of the corresponding reaction diffusion problem. Finally, by the standard invariant domain theory and the general theory of infinite dimensional dynamical system, we can prove that the solution of reaction diffusion equation approaches to one of the solution of ordinary differential equation. Our result in [1] do not any restriction of space dimension, but diffusion coefficient of both components are equal.

1.3 Blow-up problem

The third subject I have approached during my PhD is the problem of blow-up of solutions that occurs at space infinity. I focus on solutions that blow up throughout $\mathbb{R}^N$.

Let us define some related technical terms about the following a semilinear heat equation of the form

$$\begin{cases} u_t = \Delta u + u^p, & x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \mathbb{R}^N, \end{cases}$$

where $p > 1$. We say that the solution u blows up *only at space infinity* if the following holds

(a) u blows up in finite time;

(b) $\limsup_{t \nearrow T(u_0)} \|u(\cdot,t)\|_{L^\infty(K)} < \infty$ holds for any compact set $K \subset \mathbb{R}^N$.

In this case, the *global blow-up profile* $u(x,T) := \lim_{t \to T} u(x,t)$ is defined for every $x \in \mathbb{R}^N$. In the following, by "blow-up at space infinity", we always mean that (a), (b) hold.

Let us recall known results on blow-up at space infinity. Lacey [21] considered a one-dimensional problem on the half-line and constructed examples of solutions that blow up (only) at space infinity. He also obtained results of the blow-up profile. Giga and Umeda [15] considered the equation $u_t = \Delta u + u^p$ on $\mathbb{R}^N$ and derived sufficient conditions for blow-up at space infinity. Roughly speaking, they showed that blow-up at space infinity occurs if the initial data u_0 satisfies $0 \le u_0 \le M$, $u_0 \to M$ as $|x| \to \infty$ for some constant $M > 0$. Moreover, Giga and Umeda [16] extended the result of [15] to a more general equation of the form $u_t = \Delta u + f(u)$, where the nonlinearity $f(u)$ satisfies certain growth conditions; it also gives a result on "blow-up direction". Later, Shimojo discusses the blow-up of solutions that occurs only at space infinity and give sufficient conditions for such phenomena, and also studies the precise shape global profile of solutions at the blow-up time. Among other things, he establishes a nearly optimal upper bound for the blow-up profile, which shows that the profile $u(x,T)$ cannot grow too fast as $|x| \to \infty$. It is also proved that such blow-up is always complete [39] . This means that the solutions can not be extended beyond the blow-up time as a mild solution.

We consider a Cauchy problem of quasilinear equation $u_t = \Delta u^m + u^p$ on $\mathbb{R}^N$, where $p > 1$ and $m > 0$. We assume that the initaial data u_0 satisfies $0 < u_0 \le M$, $u_0 \not\equiv M$ and $\lim_{|x| \to \infty} u_0 = M$ for some constant $M > 0$. Let us recall the results on this problem. For the case $0 < m < 1$, the heat conductivity mu^{m-1} becomes small as u increases. Hence we can see that diffusion is very slow where u is large. Thus, the blow-up at space infinity must occur. This is rigorously proved by Seki [37], which generalized the results of [16], for more general equations and weak solutions. On the other hand, if $m > 1$, diffusion is very fast where u is so large. Hence the speed of heat propagation, from the space infinity to the origin near the blow-up time, becomes large compared with the semilinear problem. Thus a natural question is that "If $m \in (1, \infty)$ is sufficiently large, whether blow-up at space infinity fails or not ?" A partial answer of this problem was obtained by Seki-Suzuki-Umeda [38]. Their result implies that if $0 < m < p$ and $p > 1$, only blow-up at space infinity occurs. Motivated by these results, we consider the case $m > p > 1$ in Chapter 4. Our result is that the *total blow-up*, which means that $B(u_0) = \mathbb{R}^N$ occurs. More precisely, our result in [2] is the following:

> Let $m > p > 1$. Then the Cauchy problem of $u_t = \Delta u^m + u^p$ has a total blow-up solution with the initial value $u_0 \in C(\mathbb{R}^N)$ satisfying $0 \le u_0 \le M$ and $\lim_{|x| \to \infty} u_0 = M$ for some constant $M > 0$.

Chapter 2

Discrete Allen–Cahn equation

2.1 Introduction

In this chapter, we consider the following initial value problem for a difference equation given by

$$\begin{cases} \dfrac{u_j^{n+1} - u_j^n}{\tau} = \dfrac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2} + f(u_j^n), \quad 1 \leq j \leq N-1, \quad n \geq 0, \\ u_0^n = u_N^n = 0, \quad n > 0 \end{cases} \tag{2.1}$$

for a given initial value $\{u_j^0\}_{1 \leq j \leq N-1}$ satisfying $u_j^0 \geq 0$ for all $1 \leq j \leq N-1$, and $f \in C^1[0, \infty)$ is a function $f(u) = u(\mu - g(u))$. Now N is a positive integer. Here we assume that g is strictly monotone increasing in $u > 0$ and satisfies $g(0) = g'(0) = 0$. We also assume that there exists a real number $m > 0$ such that $g(m) = \mu$. In other words, f satisfies the following conditions

(1) $f(u)/u$ is strictly monotone decreasing in $u > 0$.

(2) $f(0) = 0,\ f'(0) = \mu$ for a constant $\mu > 0$.

(3) $f(m) = 0$ for a constant $m > 0$.

Note that constant states $u_j^n \equiv 0$ and $u_j^n \equiv m$ satisfy the difference equation. We also consider initial value problem for a difference equation given by

$$\begin{cases} \dfrac{du_j}{dt}(t) = \dfrac{u_{j+1}(t) - 2u_j(t) + u_{j-1}(t)}{h^2} + f(u_j(t)), \quad 1 \leq j \leq N-1,\ t > 0, \\ u_0(t) = u_N(t) = 0, \quad t > 0, \\ u_j(0) = u_j^0, \quad 0 \leq j \leq N. \end{cases} \tag{2.2}$$

This difference equation is obtained from the space discretization of the Allen–Cahn equation. Interesting questions about traveling fronts, for instance, are about the existence of traveling waves, their monotonicity for space, stability and its convergence rate to a traveling wave. About these problems, for the continuous Allen–Cahn model, we refer the reader to [4, 12, 27, 40, 6, 36], for example. The lattice system (2.2) arises in chemical reaction theory [11, 18, 23] and biology [5, 20]. A similar model appears for example in [10] in material science and in [30] in image processing. More precisely, [11] introduced coupled Nagumo equations and [18] considered cellular automaton models. The authors in [23] use computers to find propagation failure phenomenon of traveling wave. The lattice system on $\mathbb{Z}$ when zero is a solution for these systems is discussed in [5] and they focused on conditions forcing non-convergence to zero of solutions as time approaches infinity. For a lattice system, propagation and its failure are considered in [20] .

Let $\{v_j\}_{1\le j\le N-1}$ be the stationary solution of the problem for (2.2). In other words, it is a solution of the difference equation

$$\begin{cases} \dfrac{v_{j+1}-2v_j+v_{j-1}}{h^2}+v_j(\mu-g(v_j))=0, \quad 1\le j\le N-1, \\ v_0=v_N=0. \end{cases} \tag{2.3}$$

The main assertion in this section is as follows.

Theorem 2.1. *Let $h=1/N$ for $N\in\mathbb{N}$ and let $\{u_j^n\}$ be the solution of* (2.1) *with the initial value $\{u_j^0\}_{1\le j\le N-1}$ with $u_j^0\ge 0 \quad (1\le j\le N-1)$. Let $K=1+\max_{1\le s\le m}|f'(s)|$ and $\theta=\tau/h^2$. Assume that $\tau>0$ is small enough to satisfy*

$$0<2e^{K\tau}\theta\le\frac{1}{2}$$

and

$$\frac{e^{K\tau}-1}{\tau}+e^{K\tau}\min_{0\le s\le m}f'(s)>0.$$

If $\mu\le\frac{4}{h^2}\sin^2\left(\frac{\pi h}{2}\right)$, $v_j=0$ $(0\le j\le N)$ is the only stationary solution of (2.1) *with $v_j\ge 0 \quad (1\le j\le N-1)$. Moreover, one has $\lim_{n\to\infty}\max_{1\le j\le N-1}|u_j^n|=0$. If $\mu>\frac{4}{h^2}\sin^2\left(\frac{\pi h}{2}\right)$, there exists a unique stationary solution $\{v_j\}_{1\le j\le N-1}$ with $v_j>0$ $(1\le j\le N-1)$. Assume $0<u_j^0<m$ $(1\le j\le N-1)$, then one has $\lim_{n\to\infty}\max_{1\le j\le N-1}|u_j^n-v_j|=0$.*

We also get the results for a semi-discrete equation (2.2).

Proposition 2.1. *Let $h=1/N$ for $N\in\mathbb{N}$ and let $\{u_j(t)\}$ be the solution of* (2.2) *with the initial value $\{u_j^0\}_{1\le j\le N-1}$ with $u_j^0\ge 0 \quad (1\le j\le N-1)$.*

If $\mu\le\frac{4}{h^2}\sin^2\left(\frac{\pi h}{2}\right)$, the zero solution $v_j=0$ $(1\le j\le N-1)$ is the only solution of (2.3) *with $v_j\ge 0 \quad (1\le j\le N-1)$. Moreover, one has $\lim_{n\to\infty}\max_{1\le j\le N-1}|u_j^n|=0$. If $\mu>\frac{4}{h^2}\sin^2\left(\frac{\pi h}{2}\right)$, there exists a unique positive stationary solution $\{v_j\}_{1\le j\le N-1}$ with $v_j>0$ $(1\le j\le N-1)$. Assume $0<u_j^0<m$ $(1\le j\le N-1)$, then one has $\lim_{t\to\infty}\max_{1\le j\le N-1}|u_j(t)-v_j|=0$.*

An analogous result for the continuous model can be found in [42], this is the discrete version of their claim.

The remainder of this chapter is organized as follows. Section 2.2 is devoted to discuss the comparison principles of (2.1). We discuss the relation between the problems (2.1) and (2.2) in Section 2.3, which is useful in the proof of the following sections. In Section 2.4, we establish the comparison principles for the problem (2.2). We recall the fundamental eigenvalue problem for the discrete Laplacian in Section 2.5. We show Theorem 2.1 in Section 2.6. The proof of Theorem 2.1 is given in Section 2.7

2.2 Comparison principles for the space and time discrete model

In this section, we consider the following initial value problem for a difference equation (2.1). A basic comparison principle for the problem (2.1) is the following proposition. See Proposition 2.1 of [8] for related work.

Proposition 2.2. *Assume that*

$$0 < 2e^{K_1\tau}\theta \le \frac{1}{2}, \quad \text{where} \quad \theta = \frac{\tau}{h^2}, \tag{2.4}$$

$K_1 = 1 + \sup_{1\le j\le N, n\ge 0} |g_j^n|$ *and*

$$g_j^n = \int_0^1 f'(\vartheta u_j^n + (1-\vartheta)v_j^n)\, d\vartheta.$$

Moreover, suppose

$$\frac{e^{K_1\tau}-1}{\tau} + e^{K_1\tau}g_j^n > 0 \tag{2.5}$$

for all $n \ge 0$ *and* $1 \le j \le N$. *Let* $\{v_j^n\}$ *and* $\{u_j^n\}$ *satisfy*

$$\frac{u_j^{n+1}-u_j^n}{\tau} \ge \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2} + f(u_j^n), \quad 1 \le j \le N-1$$

and

$$\frac{v_j^{n+1}-v_j^n}{\tau} \le \frac{v_{j+1}^n - 2v_j^n + v_{j-1}^n}{h^2} + f(v_j^n), \quad 1 \le j \le N-1$$

for $n = 0, 1, 2\cdots$. *Assume that*

$$0 \le v_j^0 \le u_j^0 \le m \quad \text{for all} \quad j \in \mathbb{Z}. \tag{2.6}$$

Then

$$0 \le v_j^n \le u_j^n \le m \quad \text{for all} \quad j \in \mathbb{Z}, \quad n = 0, 1, 2, \cdots. \tag{2.7}$$

We remark that the condition (2.5) is automatically satisfied when $\tau \to 0$, because this condition can be reduced to $K_1 + g_j^n > 0$. By a simple calculation, we can check that $w_j^n = u_j^n - v_j^n$ satisfies

$$\frac{w_j^{n+1} - w_j^n}{\tau} \geq \frac{w_{j+1}^n - 2w_j^n + w_{j-1}^n}{h^2} + g_j^n w_j^n.$$

Hence Proposition 2.2 can be reduced to the following lemma.

Lemma 2.1. *Let* $\{g_j^n\}$ *satisfy* $\sup_{n\geq 0, 1\leq j\leq N-1} |g_j^n| < \infty$ *and let* $\{w_j^n\}$ *satisfy*

$$\begin{cases} \frac{w_j^{n+1}-w_j^n}{\tau} \geq \frac{w_{j+1}^n - 2w_j^n + w_{j-1}^n}{h^2} + g_j^n w_j^n, & 1 \leq j \leq N-1,\ n \geq 0 \\ w_j^0 \geq 0 \quad 1 \leq j \leq N-1. \end{cases}$$

Assume that (2.4) *and* (2.5)*, where* $K_1 = 1 + \sup_{1\leq j\leq N, n\geq 0} |g_j^n|$. *Then one has* $w_j^n \geq 0$ *for all* $1 \leq j \leq N-1$ *and* $n \geq 0$.

Proof. Define $W_j^n := e^{K_1 n\tau} w_j^n$. Then by a calculation, we can check that

$$\frac{W_j^{n+1} - W_j^n}{\tau} = e^{K_1\tau} \frac{W_{j+1}^n - 2W_j^n + W_{j-1}^n}{h^2} + e^{K_1\tau} g_j^n W_j^n + \frac{e^{K_1\tau} - 1}{\tau} W_j^n.$$

By solving this equation, we get

$$W_j^{n+1} = \theta e^{K_1\tau} W_{j+1}^n + (1 - 2e^{K_1\tau}\theta) W_j^n + \theta e^{K_1\tau} W_{j-1}^n + \tau\Big(\frac{e^{K_1\tau} - 1}{\tau} + e^{K_1\tau} g_j^n\Big) W_j^n.$$

The right hand side is non-negative, hence by the induction argument, we conclude that W_j^n for all $n \geq 0$ and $0 \leq j \leq N$. □

Next we shall prove that the monotonicity of solutions in time is guaranteed.

Lemma 2.2. *Suppose the same condition as Proposition 2.2 for the functions*

$$\bar{g}_j^n = \int_0^1 f'(\phi u_j^{n+1} + (1-\phi) u_j^n)\, d\phi.$$

instead of $\{g_j^n\}$. *Assume*

$$0 \leq u_j^0 \leq m \quad \text{for all} \quad j \in \mathbb{Z}.$$

and

$$u_j^1 \geq u_j^0 \quad \text{for all} \quad j \in \mathbb{Z}.$$

Then

$$u_j^{n+1} \geq u_j^n \quad \text{for all} \quad j \in \mathbb{Z}, \quad n = 0, 1, 2, \cdots. \tag{2.8}$$

Proof. We define $\bar{w}_j^n = \frac{u_j^{n+1} - u_j^n}{\tau}$, then the lemma follows from Lemma 2.1. □

2.3 Relations between the two discrete Allen–Cahn equations

In this section we recall the standard Euler method to estimate difference between the solution of the (2.1) and that of (2.2). Let $T > 0$ be positive constant and consider an initial value problem for an ordinary differential equation system

$$\boldsymbol{y}'(t) = \boldsymbol{F}(\boldsymbol{y}(t)), \quad 0 < t \le T, \qquad \boldsymbol{y}(0) = \boldsymbol{y}_0,$$

where $\boldsymbol{y}(t) = (y_1(t), \ldots, y_{N-1}(t)) \in \mathbb{R}^{N-1}$ is a vector valued functions and $\boldsymbol{F} : \mathbb{R}^{N-1} \to \mathbb{R}^{N-1}$ is a locally Lipschitz vector valued map with constant L. More precisely, for any positive constant $\rho > 0$ there exists $L > 0$ such that

$$\|\boldsymbol{F}(\boldsymbol{z}) - \boldsymbol{F}(\boldsymbol{y})\| \le L\|\boldsymbol{z} - \boldsymbol{y}\| \qquad \text{if} \qquad \|\boldsymbol{z} - \boldsymbol{y}_0\|, \|\boldsymbol{y} - \boldsymbol{y}_0\| \le \rho, \tag{2.9}$$

where $\|\cdot\|$ is the standard Euclidean norm. We also define

$$M = \sup_{\|\boldsymbol{y}-\boldsymbol{y}_0\| \le \rho} \|\boldsymbol{F}(\boldsymbol{y})\| < \infty. \tag{2.10}$$

We also choose small $T > 0$ such that $MT \le \rho$. We consider time variables

$$t_n = n\tau, \qquad 0 \le n \le \left[\frac{T}{\tau}\right].$$

Here $[T/\tau]$ is the largest integer that is less than or equals T/τ. The Euler method is a scheme for obtaining an approximated value $\boldsymbol{Y}^{n+1}$ for $\boldsymbol{y}(t_{n+1})$ using only the approximation $\{\boldsymbol{Y}^n\}_{0 \le n \le [T/\tau]}$ for $\boldsymbol{y}(t_n)$ and the vector function $\mathbf{F}$, namely

$$\begin{cases} \boldsymbol{Y}^{n+1} = \boldsymbol{Y}^n + \tau \boldsymbol{F}(\boldsymbol{Y}^n), \quad 0 \le n \le \left[\frac{T}{\tau}\right] \\ \boldsymbol{Y}^0 = \boldsymbol{y}^0. \end{cases} \tag{2.11}$$

We define the global truncation error at step n by

$$\boldsymbol{r}^n = \boldsymbol{Y}^n - \boldsymbol{y}(t_n). \tag{2.12}$$

$\{\boldsymbol{Y}^n\}_{0 \le n \le [T/\tau]}$ is called the Euler approximation.

Proposition 2.3. *Let $\{\boldsymbol{Y}^n\}_{0 \le n \le [T/\tau]}$ be given by (2.11). Define $L > 0$ and $M > 0$ by (2.9) and (2.10), respectively. Suppose that $MT < \rho$, then $\boldsymbol{r}^n$ satisfies $\|\boldsymbol{r}^n\| \le \frac{M\tau}{2} e^{TL}$ for $0 \le n \le [T/\tau]$.*

Proof. First we shall show that

$$|\boldsymbol{Y}^n - \boldsymbol{y}^0| \le \rho \quad 0 \le n \le \left[\frac{T}{\tau}\right]. \tag{2.13}$$

For the case $n = 0$ is trivial. Assume that it is holds true until $n-1$. Then

$$\boldsymbol{Y}^n = \sum_{i=1}^{n} (\boldsymbol{Y}^i - \boldsymbol{Y}^{i-1}) + \boldsymbol{Y}^0 = \tau \sum_{i=1}^{n} \boldsymbol{F}(\boldsymbol{Y}^{i-1}) + \boldsymbol{y}^0$$

and

$$\| \boldsymbol{Y}^n - \boldsymbol{y}^0 \| \leq \tau \sum_{i=1}^{n} \| \boldsymbol{F}(\boldsymbol{Y}^{i-1}) \| \leq M t_n \leq MT \leq \rho.$$

Thus (2.13) holds for all $0 \leq n \leq \frac{T}{\tau}$. Then one has

$$\boldsymbol{y}(t_{n+1}) - \boldsymbol{y}(t_n) = \int_{t_n}^{t_{n+1}} \boldsymbol{y}'(t) dt = \int_{t_n}^{t_{n+1}} \boldsymbol{F}(\boldsymbol{y}(t)) dt = \tau \int_0^1 \boldsymbol{F}(\boldsymbol{y}(t_n + \tau s)) ds.$$

Combining this with (2.11), we conclude

$$\boldsymbol{r}^{n+1} = \boldsymbol{r}^n - \tau \int_0^1 (\boldsymbol{F}(\boldsymbol{y}(t_n + \tau s)) - \boldsymbol{F}(\boldsymbol{Y}^j)) \, ds. \tag{2.14}$$

Here one has

$$\| \boldsymbol{y}(t_n + \tau s) - \boldsymbol{y}(t_n) \| = \left\| \int_{t_n}^{t_n + \tau s} \boldsymbol{y}'(\sigma) d\sigma \right\| = \left\| \int_{t_n}^{t_n + \tau s} \boldsymbol{F}(\boldsymbol{y}(\sigma)) d\sigma \right\| \leq M \tau s$$

Combine this with (2.14), one has

$$\| \boldsymbol{r}^{n+1} - \boldsymbol{r}^n + \boldsymbol{F}(\boldsymbol{y}(t_n))\tau - \boldsymbol{F}(\boldsymbol{Y}^n)\tau \| \leq \tau \int_0^1 LM\tau s \, ds = \frac{1}{2} LM\tau^2$$

Here we apply an inequality

$$|\boldsymbol{F}(\boldsymbol{y}(t_n)) - \boldsymbol{F}(\boldsymbol{Y}^n)| \leq L \| \boldsymbol{y}(t_n) - \boldsymbol{Y}^n \| \leq L \| \boldsymbol{r}^n \|$$

together with the triangle inequality to conclude that

$$\| \boldsymbol{r}^{n+1} \| \leq \| \boldsymbol{r}^n \| + \| \boldsymbol{F}(\boldsymbol{y}(t_n)) - \boldsymbol{F}(\boldsymbol{Y}^n) \| \tau + \frac{1}{2} LM\tau^2 \leq (1 + \tau L) \| \boldsymbol{r}^n \| + \frac{1}{2} LM\tau^2$$

for all $0 \leq n \leq K$. By the induction argument starting from $\| \boldsymbol{r}^0 \| = 0$, this inequality yields

$$\| \boldsymbol{r}^n \| \leq \frac{LM\tau^2}{2} \sum_{k=1}^{n-1} (1 + \tau L)^k = \frac{M\tau}{2} \{ (1 + \tau L)^n - 1 \} \leq \frac{M\tau}{2} e^{TL}.$$

We complete the proof. □

2.4 Comparison principles for the space discrete model

First we prove the comparison principle for the discrete reaction-diffusion equation. See Lemma 1 of [5] and Lemma 3.4 of [7] for related work.

Lemma 2.3. *Let $g_j(t)$ be functions satisfying $\sup_{0\le j\le N, 0\le t\le T} g_j(t) < \infty$. Suppose that functions $w_j(t)$ satisfy*

$$\begin{cases} \frac{d}{dt}w_j \ge \frac{w_{j+1}-2w_j+w_{j-1}}{h^2} + g_j(t)w_j, & 1\le j\le N-1, t>0 \\ w_0 = w_N = 0, \quad w_j(0)\ge 0, & 1\le j\le N-1. \end{cases}$$

Then $w_j(t)\ge 0$ for all $(j,t)\in\{0,1,,\dots,N\}\times(0,T)$.

Proof. We discrete the time $\theta = \tau/h^2 \in (0,1/2)$ and $t_n = n\tau$ for some small $\tau>0$ and denote the approximate value of $w_j(t_n)$ by w_j^n. Choosing $\tau>0$ sufficiently close to 0, we can assume without loss of generality (2.4) and (2.5) holds. From Lemma 2.1, we conclude that $w_j^n \ge 0$ for all $1\le j\le N-1$ and $n\ge 0$. Finally, by taking a limit $\tau\to 0$ and by applying Proposition 2.3, we conclude that $w_j(t)\ge 0$ for all $t\ge 0$. □

Proposition 2.4. *Let $T>0$ and suppose that real-value functions $u_j, v_j : [0,T)\to\mathbb{R}$ are differentiable in $t\in(0,T)$ for each $j\in\{1,2,\dots N-1\}$ and satisfy*

$$\frac{d}{dt}u_j - \frac{u_{j+1}-2u_j+u_{j-1}}{h^2} - f(u_j) \tag{2.15}$$
$$\ge \frac{d}{dt}v_j - \frac{v_{j+1}-2v_j+v_{j-1}}{h^2} - f(v_j), \quad 1\le j\le N-1, t\in(0,T)$$
$$u_0 = u_N = 0 = v_0 = v_N = 0, \qquad u_j(0)\ge v_j(0), \quad 1\le j\le N-1. \tag{2.16}$$

Then $u_j(t)\ge v_j(t)$ for all $(j,t)\in\{0,1,\dots N\}\times(0,T)$.

Proof. Let us define $w_j(t) = u_j(t) - v_j(t)$ for all $1\le j\le N-1$ and $t\in(0,T)$. Then w_j satisfies

$$\frac{d}{dt}w_j - \frac{w_{j+1}-2w_j+w_{j-1}}{h^2} \ge f(u_j(t)) - f(v_j(t))$$

Hence we obtain

$$\frac{d}{dt}w_j(t) \ge \frac{w_{j+1}(t)-2w_j(t)+w_{j-1}(t)}{h^2} + g_j(t)w_j(t),$$

where

$$g_j(t) = \int_0^1 f'(\vartheta u_j(t) + (1-\vartheta)v_j(t))\,d\vartheta.$$

Set $K = 1+\sup_{0\le s\le m}|f'(s)|$, and apply Lemma 2.3 to get the desired result. □

Next we establish the strong comparison principle. See Lemma 3.5 of [7] for related work.

Proposition 2.5. *Let $T > 0$ and suppose that real-value functions $u_j(t), v_j(t) : [0,T) \to \mathbb{R}$ are differentiable in $t \in (0,T)$ for each $j \in \{1, 2, \dots N-1\}$ and satisfy* (2.15)-(2.16). *Moreover there exists $1 \le J \le N-1$ such that $u_J(0) > v_J(0)$. Then $u_j(t) > v_j(t)$ for all $(j,t) \in \{1, \dots N-1\} \times (0,T)$.*

Proof. We define the function $w_j(t)$ as in the proof of Proposition 2.4. Set $K = 1 + \sup_{0\le s \le m} |f'(s)|$, and define $w_j(t) := e^{-Kt} W_j(t)$, then we have

$$\frac{d}{dt} W_j(t) \ge \frac{W_{j+1}(t) - 2W_j(t) + W_{j-1}(t)}{h^2} + \{K + g_j(t)\} W_j(t). \tag{2.17}$$

By Proposition 2.4, all we need to prove is that the solution of (2.17) starting from the initial data

$$W_J(0) > 0, \quad W_j(0) = 0, \quad \text{for} \quad j \ne J$$

satisfies

$$W_j(t) > 0 \quad \text{for all} \quad 1 \le j \le N-1.$$

If $t_1 \in (0,T)$ is sufficiently small then $W_J(t_1) > 0$. Moreover, we have

$$W_{J-1}(t_1) > 0, \quad W_{J+1}(t_1) > 0$$

since the right hand side of (2.17) is positive at time $t = 0$ on $j = J \pm 1$. Proposition 2.4 implies that we can assume that $W_j(t_1) = 0$ for $|j - J| \ge 2$ without loss of generality. Then by a similar argument again, if $t_2 \in (0,T)$ is sufficiently small, we get

$$W_J(t + t_2) > 0 \quad \text{for all} \quad J-2 \le j \le J+2.$$

Continuing the same argument, we have

$$W_j(t) > 0, \qquad \text{for all} \quad 0 \le j \le N.$$

The proof is complete. □

We give a result which guarantee the monotonicity of solutions in time.

Proposition 2.6. *Let $T > 0$ and suppose that real-value functions $u_j(t) : [0,T) \to \mathbb{R}$ are differentiable in $t \in (0,T)$ for each $j \in \{1, 2, \dots N-1\}$ and satisfy*

$$\frac{u_{j+1}(0) - 2u_j(0) + u_{j-1}(0)}{h^2} - f(u_j(0)) \ge 0, \quad 1 \le j \le N-1, \quad u_0 = u_N = 0,$$

Then $\frac{d}{dt} u_j(t) \ge 0$ for all $(j,t) \in \{0, 1, \dots N\} \times (0,T)$.

Proof. The function $U_j(t) = \frac{d}{dt} u_j(t)$ satisfies

$$\frac{dU_j}{dt} = \frac{U_{j+1}(t) - U_j(t) + U_{j-1}(t)}{h^2} + f'(u_j(t)) U_j(t).$$

Hence we can apply Proposition 2.4, since f' is smooth, and we conclude that $U_j(t) \ge 0$ for all $t \ge 0$ and $1 \le j \le N$. □

2.5 Eigenvalues and eigenfunctions

Let us introduce notations

$$\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{N-2} \\ v_{N-1} \end{pmatrix}, \quad A = \frac{1}{h^2} \begin{pmatrix} 2 & -1 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}$$

and

$$\|\boldsymbol{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_{N-1}^2}$$

We also denote the standard inner products of two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ by $(\boldsymbol{v}, \boldsymbol{w})$. For the discrete Laplacian on a line it is well known that

$$\boldsymbol{p}_i = \begin{pmatrix} \sin\theta_i \\ \sin 2\theta_i \\ \vdots \\ \sin(N-2)\theta_i \\ \sin(N-1)\theta_i \end{pmatrix}, \quad \text{where} \quad \theta_i = \frac{i\pi}{N}$$

for $1 \leq i \leq N-1$ and

$$\lambda_i = \frac{4}{h^2} \sin^2\left(\frac{\theta_i}{2}\right) \tag{2.18}$$

These values are characterized by the min-max principle, and the next property about the eigenvalue problem of the discrete Laplacian is useful in the following argument.

Lemma 2.4. *There exists a vector* $\hat{\boldsymbol{p}}_1$ *whose components are all positive such that*

$$A\hat{\boldsymbol{p}}_1 = \lambda_1 \hat{\boldsymbol{p}}_1 \tag{2.19}$$

and $\|\hat{\boldsymbol{p}}_1\| = 1$. *Moreover, the first eigenvalue is given by*

$$\lambda_1 = \min_{\boldsymbol{v} \neq 0} \frac{(A\boldsymbol{v}, \boldsymbol{v})}{(\boldsymbol{v}, \boldsymbol{v})} \tag{2.20}$$

and the maximum eigenvalue is represented as

$$\lambda_{N-1} = \max_{\boldsymbol{v} \neq 0} \frac{(A\boldsymbol{v}, \boldsymbol{v})}{(\boldsymbol{v}, \boldsymbol{v})}$$

2.6 Proof of Proposition 2.1

We prove the non-existence result for a positive stationary solution. Now we multiply $\boldsymbol{v}$ to both hand sides of (2.3) to obtain

$$-(A\boldsymbol{v},\boldsymbol{v})+\sum_{j=1}^{N-1}v_j^2(\mu-g(v_j))=0.$$

By (2.20), we get

$$-\lambda_1\|\boldsymbol{v}\|^2+\sum_{j=1}^{N-1}v_j^2(\mu-g(v_j))\geq 0.$$

and

$$(\lambda_1-\mu)\|\boldsymbol{v}\|^2+\sum_{j=1}^{N-1}v_j^2 g(v_j)\leq 0.$$

Hence we conclude that $v_j=0$ for all $1\leq j\leq N-1$ provided that

$$\frac{4}{h^2}\sin^2\left(\frac{\pi h}{2}\right)=\lambda_1\geq\mu.$$

Here we also apply (2.18). Now we prove the convergence to the zero vector from any solution of (2.2). Let us multiply the equation (2.2) by $u_j(t)$ and summing for $1\leq j\leq N-1$, we obtain a Lyapunov functional:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}(t)\|^2=\sum_{j=1}^{N-1}u_j(t)u_j'(t)&=-(A\boldsymbol{u}(t),\boldsymbol{u}(t))+\mu\|\boldsymbol{u}(t)\|^2-\sum_{j=1}^{N-1}u_j(t)^2g(u_j(t))\\&\leq-(\lambda_1-\mu)\|\boldsymbol{u}(t)\|^2-\sum_{j=1}^{N-1}u_j(t)^2g(u_j(t)).\end{aligned}$$

Note that the right hand side is equal or less than zero for all $t\geq 0$, hence the trajectory is bounded for all $t\geq 0$. From a general theory of dynamical system, Lyapunov functional is constant on the omega limit set, that means the right-hand side must be zero on the omega limit set. Since each term of the right-hand side is nonpositive and g is strict increasing and $g(0)=0$, the omega limit set consists of only zero sequence $\{v_j\}_{1\leq j\leq N-1}=\{0\}_{1\leq j\leq N-1}$. This means that the omega limit set of any solution consists of this single point set, hence any solution $\{u_j(t)\}_{1\leq j\leq N-1}$ converges to zero as $t\to\infty$.

Next, we prove the unique existence of a positive stationary solution for the case

$$\mu>\frac{4}{h^2}\sin^2\left(\frac{\pi h}{2}\right)=\lambda_1 \tag{2.21}$$

In order to prove it, we shall construct a supersolution and a subsolution and consider time evolution from those initial data. Let us introduce a supersolution, which is given by a constant vector

$$\bar{\boldsymbol{v}} = \begin{pmatrix} \bar{v}_1 \\ \bar{v}_2 \\ \vdots \\ \bar{v}_{N-2} \\ \bar{v}_{N-1} \end{pmatrix} = \begin{pmatrix} m \\ m \\ \vdots \\ m \\ m \end{pmatrix}, \tag{2.22}$$

where $m > 0$ is a real number satisfying $g(m) = \mu$. By a calculation, it is easy to check

$$\frac{\bar{v}_{j+1} - 2\bar{v}_j + \bar{v}_{j-1}}{h^2} + \bar{v}_j(\mu - g(\bar{v}_j)) = \begin{cases} 0, & 2 \le j \le N-2 \\ -m/h^2 < 0, & j = 1, N-1. \end{cases} \tag{2.23}$$

Thus the above constant vector is a supersolution. Next we shall introduce a subsolution

$$\underline{\boldsymbol{v}} = \begin{pmatrix} \underline{v}_1 \\ \underline{v}_2 \\ \vdots \\ \underline{v}_{N-2} \\ \underline{v}_{N-1} \end{pmatrix} = \varepsilon \boldsymbol{p}_1 = \varepsilon \begin{pmatrix} \sin\theta_1 \\ \sin 2\theta_1 \\ \vdots \\ \sin{(N-2)\theta_1} \\ \sin{(N-1)\theta_1} \end{pmatrix} \tag{2.24}$$

where $\varepsilon > 0$ is sufficiently small to be determined later. Then the assumption $g(0) = 0$ together with the continuity of g, $g(\underline{v}_j) \le \mu$ for all $1 \le j \le N-1$ if $\varepsilon \in (0, \mu)$ is sufficiently small. On the other hand, (2.19) yields $A\boldsymbol{p}_1 = \lambda_1 \boldsymbol{p}_1$. Thus all we need to check is

$$-\lambda_1 p_1^j + p_1^j(\mu - g(\varepsilon p_1^j)) \ge 0$$

for all $1 \le j \le N-1$, where p_1^j is the j-th component of the eigenvector $\boldsymbol{p}_1$. Recall that $g(0) = g'(0) = 0$, thus by taking $\varepsilon \in (0, \mu)$ sufficiently small, we get the desired inequality

$$-\lambda_1 + \mu - g(\varepsilon p_1^j) \ge 0.$$

thus $\underline{\boldsymbol{v}}$ is a subsolution. We denote the solution of this problem (2.2) starting from the initial vector $\boldsymbol{u}^0 = \{u_j^0\}_{0 \le j \le N}$ by $\boldsymbol{u}(t; \boldsymbol{u}^0)$. Define

$$\bar{\boldsymbol{u}}(t) = \boldsymbol{u}(t; \bar{\boldsymbol{v}}), \quad \underline{\boldsymbol{u}}(t) = \boldsymbol{u}(t; \underline{\boldsymbol{v}}).$$

By Proposition 2.6, each component of $\bar{\boldsymbol{u}}(t)$ is monotone decreasing in t, and each component of $\underline{\boldsymbol{u}}(t)$ is monotone increasing in t. Thus we can define a limit function

$$\boldsymbol{U} = \lim_{t \to \infty} \bar{\boldsymbol{u}}(t), \quad \boldsymbol{V} = \lim_{t \to \infty} \underline{\boldsymbol{u}}(t).$$

These vectors satisfy

$$\frac{U_{j+1} - 2U_j + U_{j-1}}{h^2} + U_j(\mu - g(U_j)) = 0$$

$$\frac{V_{j+1} - 2V_j + V_{j-1}}{h^2} + V_j(\mu - g(V_j)) = 0$$

Now we multiply the first equation by V_j and the second equation by U_j, calculate their difference and sum up together for j to get

$$\sum_{j=1}^{N-1} U_j V_j (g(U_j) - g(V_j)) = 0. \tag{2.25}$$

Here we used the symmetry relation $(A\boldsymbol{U}, \boldsymbol{V}) = (\boldsymbol{U}, A\boldsymbol{V})$ of the discrete Laplacian. By the monotonicity for time, $U_j, V_j > 0$ for all $1 \le j \le N-1$. The comparison principle yields $U_j \ge V_j$ for all $1 \le j \le N-1$, hence $g(U_j) \ge g(V_j)$. Thus $g(U_j) = g(V_j)$ must hold for all $1 \le j \le N-1$ from (2.25). Since g is strictly monotone increasing for $u > 0$ to conclude that $U_j = V_j$ for all $1 \le j \le N-1$. Proposition 2.4 yields the convergence result $\lim_{t\to\infty} \max_{1\le j\le N-1} |u_j(t) - v_j| = 0$ for any initial data satisfying the inequalities $0 < u_j^0 < m$ for all $1 \le j \le N-1$.

2.7 Proof of Theorem 2.1

The proof about the existence of the stationary problem has already done, since the stationary problem is the same between (2.1) and (2.2).

Let us define the solution of (2.1) starting from the initial vector $\boldsymbol{u}^0 = \{u_j^0\}_{0\le j\le N}$ by $\boldsymbol{u}^n(\boldsymbol{u}^0)$. Define $\bar{\boldsymbol{u}}^n := \boldsymbol{u}^n(\bar{\boldsymbol{v}})$, where $\bar{\boldsymbol{v}}$ is given in (2.22). Lemma 2.2 implies that each component of $\bar{\boldsymbol{u}}^n$ is nonnegative and monotone non-increasing for n. Hence we can define

$$\boldsymbol{U} := \lim_{n\to\infty} \bar{\boldsymbol{u}}^n$$

By taking a limit $n \to \infty$ in (2.1) and using the continuity of the function f, we obtain

$$\frac{U_{j+1} - 2U_j + U_{j-1}}{h^2} + f(U_j) = 0, \quad 1 \le j \le N-1.$$

First we consider the case,

$$\frac{4}{h^2}\sin^2\left(\frac{\pi h}{2}\right) = \lambda_1 \ge \mu.$$

Under this assumption the only nonnegative stationary solution is zero vector, which implies that $U_j = 0$ for all $1 \le j \le N-1$. Note that we can check the assumption (2.4)-(2.5), and we can apply Proposition 2.2 and the comparison principle. Hence $0 \le u_j^n \le \bar{u}_j^n$ for all $n \ge 0$ and $0 \le j \le N$. By taking a limit $n \to 0$, we prove the desired result.

Next we consider the case

$$\frac{4}{h^2}\sin^2\left(\frac{\pi h}{2}\right) = \lambda_1 < \mu$$

and prove the convergence to the positive stationary solution. Define

$$\underline{\boldsymbol{u}}^n := \boldsymbol{u}^n(\underline{\boldsymbol{v}}),$$

where $\underline{\boldsymbol{v}}$ is given in (2.24). This time by applying Lemma 2.2, we conclude that each component of $\bar{\boldsymbol{u}}^n$ is monotone non-increasing for n and each component of $\underline{\boldsymbol{u}}^n$ is monotone non-decreasing for n. Also all components of these vectors are bounded from above and below. Hence there exists

$$\boldsymbol{U} = \lim_{n \to \infty} \bar{\boldsymbol{u}}^n, \quad \boldsymbol{V} = \lim_{n \to \infty} \underline{\boldsymbol{u}}^n.$$

These vectors are solutions to the same stationary problem as discussed in Section 2.6, and its proof is completely the same as that of Section 2.6. Now we apply Proposition 2.2 to conclude that the solution u_j^n satisfies $\underline{u}_j^n \leq u_j^n \leq \bar{u}_j^n$ for all $n \geq 0$ and $0 \leq j \leq N$. Finally, by taking a limit $n \to 0$, we can prove the desired result.

Chapter 3

Center problem of reaction diffusion system

3.1 Introduction

Let us consider a reaction diffusion system of the form:

$$u_t = d_u \Delta u - v + P(u,v), \quad v_t = d_v \Delta v + u + Q(u,v), \quad x \in \Omega,\ t > 0, \tag{3.1}$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t > 0, \tag{3.2}$$

with

$$u(\cdot,0) = u_0, \qquad v(\cdot,0) = v_0, \quad x \in \Omega, \tag{3.3}$$

where d_u and d_v are positive constants, P and Q are analytic functions on $\mathbb{R}^2$ starting in at least second order terms, i.e. such that $P(0,0) = Q(0,0) = 0$ and $\partial_u P(0,0) = \partial_v P(0,0) = \partial_u Q(0,0) = \partial_v Q(0,0) = 0$. Moreover, Ω is a bounded smooth domain in $\mathbb{R}^N$ with $N \geq 1$, and ν is the outward unit normal vector. We assume that the vector field

$$\mathcal{X} = \{-V + P(U,V)\}\partial_U + \{U + Q(U,V)\}\partial_V$$

has a commutating analytic vector field of the form

$$\mathcal{Y} = \{U + R(U,V)\}\partial_U + \{V + S(U,V)\}\partial_V.$$

Here commuting means $[\mathcal{X},\mathcal{Y}] \equiv 0$, where the bracket used here is the Lie bracket. Under this assumption, the ordinary differential equation system

$$\frac{dU}{dt} = -V + P(U,V), \quad \frac{dV}{dt} = U + Q(U,V) \tag{3.4}$$

has an isochronous center at the origin, i.e. there exists a neighborhood $\mathcal{U} \subset \mathbb{R}^2$ of the origin $(U,V) = (0,0)$ such that every orbit in a punctual neighborhood $\mathcal{U} \setminus \{(0,0)\}$ is a cycle surrounding $(0,0)$, and the period of all such curves are constant 2π. It is well known that the wedge product $\mathcal{X} \wedge \mathcal{Y} = \det\{\mathcal{X}, \mathcal{Y}\}$ is an inverse integrating factor of both $\mathcal{X}$ and $\mathcal{Y}$. Hence the rescaled vector $\mathcal{X}/(\mathcal{X} \wedge \mathcal{Y})$ and $\mathcal{Y}/(\mathcal{X} \wedge \mathcal{Y})$ are hamiltonian ones except for the zero set of $\mathcal{X} \wedge \mathcal{Y}$. Therefore, a first integral H of $\mathcal{X}$ can be computed from well-known integral

$$H(U, V) = \int \frac{\{U + Q(U, V)\}\, dU - \{-V + P(U, V)\}\, dV}{\mathcal{X} \wedge \mathcal{Y}}. \tag{3.5}$$

Similarly, a first integral I of $\mathcal{Y}$ can be computed as

$$I(U, V) = \int \frac{\{V + S(U, V)\}\, dU - \{U + R(U, V)\}\, dV}{\mathcal{X} \wedge \mathcal{Y}}.$$

Moreover, a near-identity change of variables, $\xi = U + o(U, V), \eta = V + o(U, V)$, analytic in $\mathcal{U}$ that linearize $\mathcal{X}$ is obtained as follows:

$$\xi = \frac{\sqrt{2f(H)}}{\sqrt{1 + g^2(I)}}, \quad \eta = \frac{\sqrt{2f(H)}g(I)}{\sqrt{1 + g^2(I)}},$$

where f and g are two functions such that

$$f(H(U, V)) = \frac{U^2 + V^2}{2} + o(U^2 + V^2), \quad g(I(U, V)) = \frac{V + o(U, V)}{U + o(U, V)}. \tag{3.6}$$

In other words, ξ and η satisfy $d\xi/dt = -\eta$, $d\eta/dt = \xi$. By (3.5), we have

$$\partial_U f(H(U, V)) : \partial_V f(H(U, V)) = \partial_U H(U, V) : \partial_V H(U, V) = U + Q : V - P \tag{3.7}$$

as long as $\mathcal{X} \wedge \mathcal{Y} \neq 0$. For these fundamental results, see [34] for example. In the following, we write $F(U, V) = f(H(U, V))$ to shorten the notation, and consider a set

$$\mathcal{U}_c := \{(U, V) \mid 0 \leq F(U, V) \leq c\} \subset \mathcal{U}$$

for sufficiently small $c > 0$.

We consider the initial value problem with a smooth initial data (u_0, v_0), then the problem (3.1)-(3.3) has a unique solution $(u, v) \in \big(C([0, T); L^\infty(\Omega))\big)^2$, where $T = T(u_0, v_0) \in (0, \infty]$ denotes the maximal existence time of the solution. Moreover, we have either $T = \infty$, or

$$T < \infty \quad \text{and} \quad \limsup_{t \to T} \{\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)}\} = \infty.$$

In the former case we say that the solution is *global*, while in the latter case we say that the solution *blows up* in a finite time. Also, T is called the *blow-up time* of the solution if $T < \infty$. The blow-up problem for the parabolic system has been studied extensively by many authors, see [25, 28, 31, 33, 41] and also a recent monograph by Quittner and Souplet [32].

Our first result about the global solution is the following.

Proposition 3.1. *Let* $(u(x,t), v(x,t))$ *be a global solution of the problem* (3.1)-(3.3) *and let* $c > 0$ *be sufficiently small. Suppose that*

$$(u(x,t), v(x,t)) \in \mathcal{U}_c \tag{3.8}$$

for all $x \in \Omega$ *and* $t \geq 0$*. Then there exists some orbit* $\hat{O}$ *of* (3.4) *such that*

$$\lim_{t\to\infty} \mathrm{dist}_{C^2}((u(\cdot,t), v(\cdot,t)); \hat{O}) = 0. \tag{3.9}$$

Moreover, if this $\hat{O}$ *is not the origin,*

$$\lim_{t\to\infty} \|(u(\cdot,t+2\pi), u(\cdot,t+2\pi)) - (u(\cdot,t), u(\cdot,t))\|_{C^2(\Omega)} = 0. \tag{3.10}$$

For this kind of eventually homogeneous periodic behavior is also discussed in [24, 19] for prey-predator system and the Gierer-Meinhardt system. Our results can be applied to all the reaction diffusion systems having non-degenerate isochronous center (See [34]).

By combining the standard invariant region theory, we obtain the following result.

Theorem 3.1. *Let* $d_u = d_v$*, and* $(u(x,t), v(x,t))$ *be a solution of the problem* (3.1)-(3.3)*. Let* $c > 0$ *be a sufficiently small constant. Suppose that*

$$(u_0(x), v_0(x)) \in \mathcal{U}_c \tag{3.11}$$

for all $x \in \Omega$*. Then the solution* $(u(x,t), v(x,t))$ *exists globally in time, and there exists some orbit* $\hat{O}$ *of* (3.4) *satisfying* (3.9)*. Moreover, if this* $\hat{O}$ *is not the origin,* (3.10) *holds.*

Here we remark that the period of the asymptotically periodic behavior is uniquely determined as 2π, because of the isochronous property. Compare our result with the main result of [24], in which the period is some unknown number.

One of the simplest equations whose reaction term has the origin as an isochronous center is

$$u_t = d_u\Delta u - v + u^2, \quad v_t = d_v\Delta v + u + uv, \quad x \in \Omega,\ t > 0, \tag{3.12}$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t > 0. \tag{3.13}$$

By a simple calculation, we can check that the problem (3.12) can be formulated as

$$u_t = d_u\Delta u - (1+v)^3\partial_v F, \quad v_t = d_v\Delta v + (1+v)^3\partial_u F, \tag{3.14}$$

where $F(u,v) = F_L(u,v) := \frac{U^2+V^2}{2(1+V)^2}$. The corresponding ordinary equation to (3.12) is given by

$$U' = -V + U^2, \quad V' = U + UV. \tag{3.15}$$

This equation has the origin $(0,0)$ as a center, and all the solution of (3.15) rotate around the origin and their angular speed is constant. In this chapter, we shall call this type of center as *uniform isochronous center*. The equation (3.4) having the origin as an uniform isochronous center is classified by [9, 34], and polynomial systems with the uniform isochronous center can always written in the form

$$U' = -V + U\bar{P}(U,V), \quad V' = U + V\bar{P}(U,V),$$

where $\bar{P}$ is polynomial. It is known that if $\bar{P}$ is homogeneous and even degree, then the origin is an isochronous center. For the case of homogeneous and odd degree, $\int_0^{2\pi} \bar{P}(\cos\varphi, \sin\varphi)\, d\varphi = 0$ is needed. A polynomial $\bar{P}(U,V) = U$ gives us the simplest system (3.15), which is called the second Loud system, having $(0,0)$ as uniform isochronous center. This can be easily checked, since the polar coordinate $U = r\cos\theta$, $V = r\sin\theta$ can bring the (3.15) to the form

$$r' = r^2\cos\theta, \quad \theta' = 1.$$

By taking θ as independent variable, we get

$$\frac{dr}{d\theta} = r^2\cos\theta.$$

An easy computation shows that the periodic solution $r(\theta, r_0)$ surrounding the center $r = 0$ such that $r(\theta = 0) = r_0 \in (0,1)$ is

$$r = \frac{r_0}{r_0\sin\theta + 1}.$$

This system is not Hamiltonian but it has a rational first integral

$$F_L(U,V) = \frac{U^2 + V^2}{2(1+V)^2},$$

and a reciprocal integrating factor $(1+V)^3$. It is also easy to check that the transformation $\chi = U/(1+V), \eta = V/(1+V)$ changes the equation to (3.15) to that of the harmonic oscillator $\chi' = -\eta, \eta' = \chi$. This means that $(\chi^2 + \eta^2)/2 = F_L(U,V)$ is the first integral of (3.15). By a simple calculation, we can check that the transverse commuting vector field is $\mathcal{Y} = \{U + UV\}\partial_U + \{V + V^2\}\partial_V$. Note that $F_L = c$ is a family of ellipse including the origin if $c < 1$, and it is a family of hyperbola if $c > 1$ and $F_L = 1$ is parabola. Thus as a corollary we conclude the following.

Corollary 3.1. *Let $d_u = d_v$, and $(u(x,t), v(x,t))$ be a solution of the problem* (3.12)-(3.13) *with initial data (u_0, v_0). Let $c > 0$ be a sufficiently small constant. Suppose that*

$$F_L(u_0(x), v_0(x)) \le c \tag{3.16}$$

for all $x \in \Omega$. Then the solution $(u(x,t), v(x,t))$ exists globally in time, and there exists some orbit $\hat{O}$ of (3.4) *satisfying* (3.9). *Moreover, if this $\hat{O}$ is not the origin,* (3.10) *holds.*

Next we consider blowing up solutions. We can easily see that this system has a trivial finite time blow-up solution by substituting $v \equiv -1$ into (3.12), since $v \equiv -1$ yields $u_t = d_u \Delta u + u^2 + 1$. This is a single equation and it is easy to show the blow-up of the component $u(x,t)$ by the comparison principle. We can easily prove that if the second component of the initial data satisfies $v_0(x) \leq -1$ the solution blows up in a finite time. In fact, the set $\{(U,V) \mid U \in \mathbb{R}, V \leq -1\}$ is the invariant domain of (3.15), by the standard invariant region theorem of [44], the solution to (3.12)-(3.13) also satisfies $v(x,t) \leq -1$ for all $x \in \Omega$ as long as the solution exists. In particular, we have a differential inequality, $u_t \geq d_u \Delta u + u^2 + 1$ and the solution blows up in a finite time. The next theorem gives us a nontrivial sufficient condition to initial data which produces blow-up solutions.

Theorem 3.2. *Let $d_u = d_v$ and $c > 1$. Assume that*

$$(u_0(x), v_0(x)) \in \Gamma_c := \left\{ (U,V) \mid -\frac{U}{\sqrt{c-1}} - \frac{c}{c-1} \leq V \leq \frac{U}{\sqrt{c-1}} - \frac{c}{c-1},\ U \geq \frac{1}{\sqrt{c-1}} \right\}.$$

Then the solution of the problem (3.13)-(3.15) *blows up.*

This chapter is organized as follows. In Section 3.2, we consider global behavior of the solution to prove Proposition 3.1 and Theorem 3.1. In Section 3.3, we show Theorem 3.2 and discuss blow-up phenomenon.

3.2 Proof of eventually periodic motion

The work of [24, 19] considers nonlinearity formulated as the Hamiltonian system. Our theorem is some kind of generalization of them, since the ordinary differential equation system with an isochronous nonlinearity is Darboux integrable (See [34]), which has a property (3.5) and (3.7). Except this difference, our proof is essentially the same as that of [19], but we write the argument for the reader's convenience.

Proof of Proposition 3.1. By a simple calculation,

$$\begin{aligned} \frac{d}{dt}\int_\Omega F(u(t),v(t))\,dx &= \int_\omega F_u u_t + F_v v_t\,dx = \int_\omega F_u\{d_u\Delta u - v + P\} + F_v\{d_v \Delta v + u + Q\}\,dx \\ &= -\int_\Omega d_u F_{uu}|\nabla u|^2 + (d_u + d_v)F_{uv}\nabla u \cdot \nabla v + d_v F_{vv}|\nabla v|^2\,dx \quad (3.17) \end{aligned}$$

for the solution of (3.1)-(3.2), since (3.7) holds. Note that (3.6) implies $F_{uu} = F_{vv} = 1 + o(1)$ and $F_{uv} = o(1)$ thus there exists a small neighborhood

$$\mathcal{U}_c = \{(U,V) \mid 0 \leq F(U,V) \leq c\} \subset \mathcal{U} \quad c > 0$$

near the origin $(0,0)$ such that the right hand side of (3.17) is non-positive if the solution $(u(x,t), v(x,t))$ of (3.1) belongs to $\mathcal{U}_c$ for all $x \in \Omega$. Under this situation, by the standard parabolic regularity theory, the orbit

$$O = \{(u(\cdot,t), v(\cdot,t))\}_{t \geq 0}$$

is compact in $C^2(\bar{\Omega}) \times C^2(\bar{\Omega})$. Hence the theory of dynamical system implies that ω limit set of the above O defined by

$$\omega(u_0, v_0) = \{(u_*, v_*) \mid \exists t_j \to \infty \quad \text{s.t.} \quad \|(u(t_j), v(t_j)) - (u_*, v_*)\|_{C^2} = 0\}$$

is compact, connected and invariant set in $C^2(\bar{\Omega}) \times C^2(\bar{\Omega})$. Let us consider the solution $(\tilde{u}, \tilde{v})$ of the problem (3.1)-(3.2) with the initial data

$$\tilde{u}|_{t=0} = u_*, \quad \tilde{v}|_{t=0} = v_* \quad x \in \bar{\Omega},$$

This solution satisfies

$$\frac{d}{dt} \int_\Omega F(\tilde{u}(\cdot, t), \tilde{v}(\cdot, t))\, dx = 0, \quad t > 0. \tag{3.18}$$

The reason is that the Lyapunov functional is constant on the ω limit set $\omega(u_0, v_0)$, which is an invariant set. Since (3.17) is non-positive for all $t \geq 0$, the value

$$F_\infty = \lim_{t\to\infty} \int_\Omega F(u(x, t), v(x, t))\, dx$$

exists. Furthermore, all $(u_*, v_*) \in \omega(u_0, v_0)$ are independent of the space variable x. The invariance of $\omega(u_0, v_0)$ under the flow implies that (u_*, v_*) lies on one of the ordinary differential equation orbit. On the other hand, the orbit of the system (3.4) is determined by solving $F = C$ for a constant C. Thus by solving $F = F_\infty$, we conclude that $\omega(u_0, v_0)$ is contained in some ordinary differential equation orbit $\hat{O}$, and we conclude

$$\lim_{t\to\infty} d_{C^2}((u(\cdot, t), v(\cdot, t)); \hat{O}) = 0,$$

which means that the solution of the problem (3.1)-(3.2) converges to the origin or eventually converges to one periodic solution.

Next, we prove (3.10). By the parabolic regularity and Ascori-Arzela theorem, there exists a subsequence $\{t'_j\}$ of the sequence $\{t_j\}$ such that

$$\lim_{j\to\infty} \sup_{t\in[-4\pi,4\pi]} \|(u(\cdot, t + t'_j), v(\cdot, t + t'_j)) - (U(t), V(t))\|_{C^2} = 0,$$

where $(U(t), V(t)) \in \hat{\mathcal{O}}$ is a solution of (3.4), which has a period 2π from the isochronous assumption. Therefore,

$$\begin{aligned}
\limsup_{j\to\infty} &\|(u(\cdot, t + t'_j + 2\pi), v(\cdot, t + t'_j + 2\pi)) - (u(\cdot, t + t'_j), v(\cdot, t + t'_j))\|_{C^2} \\
&\leq \lim_{j\to\infty} \|(u(\cdot, t + t'_j + 2\pi), v(\cdot, t + t'_j + 2\pi)) - (U(t + 2\pi), V(t + 2\pi))\|_{C^2} \\
&\quad + \lim_{j\to\infty} \|(u(\cdot, t + t'_j), v(\cdot, t + t'_j)) - (U(t), V(t))\|_{C^2} = 0
\end{aligned}$$

From these argument, we conclude

$$\lim_{\tau\to\infty} \|(u(\cdot, t + \tau + 2\pi), v(\cdot, t + \tau + 2\pi)) - (u(\cdot, t + \tau), v(\cdot, t + \tau)\|_{C^2} = 0.$$

By substituting $t = 0$, we get the result. □

Proof of Theorem 3.1. If $c > 0$ is sufficiently small, the set $\mathcal{U}_c$ is a closed set enclosed by an ellipse because of (3.6), and any solution of (3.4) starting from this set is trapped in $\mathcal{U}_c$ for all $t \geq 0$. From the assumption (3.16), $(u_0(x), v_0(x))$ lies in $\mathcal{U}_c$ for all $x \in \Omega$. By applying theorem of [44], we conclude that $\mathcal{U}_c$ is invariant also for (3.1), since the function $F : \mathcal{U}_c \to \mathbb{R}$ is convex, if we choose $c > 0$ sufficiently small necessary. Now using (3.17) and repeating the same argument as above, we obtain the desired result. □

Remark 3.1. *If $\mathcal{Y}$ is a normalizer of $\mathcal{X}$, i.e., $[\mathcal{X}, \mathcal{Y}] = \mu\mathcal{X}$ for some function μ, then the wedge product $\mathcal{X} \wedge \mathcal{Y}$ is an inverse integrating factor of $\mathcal{X}$. In particular, the origin becomes a center for* (3.4)*. Under the existence of normalizer the proof of Proposition 3.1 work, since* (3.7) *holds. Hence we get a similar result as Proposition 3.1, but the period in* (3.10) *depends on initial data as [19, 24].*

3.3 Isochronous nonlinearity and Loud system

If $c > 1$, the set $\{(U, V) \mid F_L(U, V) = c\}$ is hyperbola, whose asymptotes are given by

$$V = \frac{\pm 1}{\sqrt{c-1}} U - \frac{c}{c-1}.$$

We shall construct a convex invariant region which gives us blow-up solutions for the system (3.12)-(3.13).

Lemma 3.1. *Let $c > 1$. Then the set*

$$\Gamma_c := \left\{ (U, V) \mid -\frac{U}{\sqrt{c-1}} - \frac{c}{c-1} \leq V \leq \frac{U}{\sqrt{c-1}} - \frac{c}{c-1},\ U \geq \frac{1}{\sqrt{c-1}} \right\} \tag{3.19}$$

is an invariant for (3.15).

Proof. The vector fields associated with the (3.15) is

$$\mathcal{X} = (-V + U^2)\partial_U + (U + UV)\partial_V.$$

First we shall show that the slope of the above vector filed is smaller than $1/\sqrt{c-1}$ on the upper boundary of Γ_c, which is a line given by $V = \frac{U}{\sqrt{c-1}} - \frac{c}{c-1}$. More precisely,

$$\frac{dV}{dU} = \frac{U(1+V)}{-V+U^2} = \frac{U\left(\frac{U}{\sqrt{c-1}} - \frac{1}{c-1}\right)}{U\left(U - \frac{1}{\sqrt{c-1}}\right) + \frac{c}{c-1}} \leq \frac{1}{\sqrt{c-1}} \quad \text{on} \quad V = \frac{U}{\sqrt{c-1}} - \frac{c}{c-1}$$

if $U \geq 1/\sqrt{c-1}$. Next we shall show that the slope of the vector filed $\mathcal{X}$ is larger than $-1/\sqrt{c-1}$ on the lower boundary of Γ_c, which is a line given by $V = -\frac{U}{\sqrt{c-1}} - \frac{c}{c-1}$. This condition holds if $U \geq 1/\sqrt{c-1}$, since

$$\frac{dV}{dU} = \frac{U(1+V)}{-V+U^2} = \frac{U\left(-\frac{U}{\sqrt{c-1}} - \frac{1}{c-1}\right)}{U\left(U + \frac{1}{\sqrt{c-1}}\right) + \frac{c}{c-1}} \geq -\frac{1}{\sqrt{c-1}} \quad \text{on} \quad V = -\frac{U}{\sqrt{c-1}} - \frac{c}{c-1}.$$

Furthermore,

$$-V+U^2 \geq U^2 - \frac{U}{\sqrt{c-1}} + \frac{c}{c-1} = \frac{c}{c-1} > 0 \quad \text{on} \quad \Big\{U = \frac{1}{\sqrt{c-1}}\Big\} \cap \Gamma_c$$

Therefore, Γ_c defined by (3.19) is an invariant region of (3.15). □

Now we prove the blow-up results for (3.12)-(3.13).

Proof of Theorem 3.2. Assume on the contrary that the solution exists globally in time. Since the set Γ_c constructed in Lemma 3.1 is convex, we can apply the standard theorem about the invariant region of [44]. Hence, by the assumption of initial data, $d_u = d_v$ and Lemma 3.1, we conclude that $(u(x,t), v(x,t)) \in \Gamma_c$ for all $x \in \Omega$ and $t \geq 0$. This implies

$$u_t = d_u \Delta u - v + u^2 \geq d_u \Delta u + \Big(u^2 - \frac{u}{\sqrt{c-1}} + \frac{c}{c-1}\Big).$$

since $u(x,t) \geq 1/\sqrt{c-1}$, there exists $\delta > 0$ sufficiently small such that

$$(1-\delta)U^2 \geq \frac{U}{\sqrt{c-1}} \quad \text{for all} \quad U \geq \frac{1}{\sqrt{c-1}}.$$

Thus we get the inequality

$$u_t \geq d_u \Delta u + \delta u^2 + \frac{c}{c-1}.$$

By the comparison principle, the solution blows up in a finite time, since $u \geq 1/\sqrt{c-1}$ for all $x \in \Omega$ and $t \geq 0$. This is a contradiction. □

In fact, we do not need to use the form of the nonlinearity in the proof above. It can be proved also from the following general proposition.

Proposition 3.2. *Let $d_u = d_v$. Suppose that the origin is the uniform isochronous center of (3.4). Let $(u(\cdot,t), v(\cdot,t))$ be the solution of (3.1)-(3.2). Assume that there exists a convex closed connected invariant region D satisfying the following properties*

(H1) The origin $(U,V) = (0,0)$ is outside of the domain D.

(H2) The domain D is unbounded and sandwiched by two asymptotes of the boundary.

Then the solution $(u(\cdot,t), v(\cdot,t))$ with initial data satisfying $(u_0(x), v_0(x)) \in D$ for all $x \in \Omega$ blows up.

Proof. Assume the solution is global in time to get a contradiction. We only consider the case that the asymptotic of the domain D is given by the form $V = \pm(\tan\alpha)\, U + \beta$ for $\alpha \in (0, \pi/2)$ and $\beta \in \mathbb{R}$, because the proof of the other cases are similar. Since D is connected, $D \subset \{-(\tan\alpha)\, U + \beta \leq V \leq (\tan\alpha)\, U + \beta\}$. By the assumption (H1), there exists $m \in \mathbb{R}$ such that $D \cap \{(U,V) \mid (\tan m)U = V\} \neq \emptyset$ and $(\tan m) u_0(x) \leq v_0(x)$ for all $x \in \Omega$. Let us solve the ODE (3.4) along the line $V = \tan(m)U$. Then from the definition of uniform isochronousity, that is $\theta'(t) \equiv 1$, the solutions are lying on $V = \tan(m+t)U$. It is clear that the domain $D(t) = D \cap \{(U,V) \mid \tan(m+t)U \leq V\}$ is convex. Applying the theorem of [44], we conclude $(u(\cdot,t), v(\cdot,t)) \in D(t)$ for all $t \geq 0$. On the other hand, there exists $t_1 \geq 0$ such that $D(t) = \emptyset$ for all $t > t_1$, which is a contradiction. Hence the solution can not exists globally, which yields blow-up by the existence theorem. $\square$

Chapter 4

Total blow-up of non-decaying solution for quasi-linear problem

4.1 Introduction

We consider the nonlinear diffusion equation:

$$\begin{cases} u_t = \Delta u^m + u^p, & x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = u_0(x) > 0, & x \in \mathbb{R}^N \end{cases} \tag{4.1}$$

with $m > p > 1$ and $u_0 \in C(\mathbb{R}^N)$ for $N \geq 1$. This problem is known to admit a time local solution([22, 29]), but it may cease to exist in a finite time. We say that the solution of (4.1) *blows up* in finite time if there is some $T = T(u_0) < \infty$ such that

$$\limsup_{t \nearrow T} \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} = \infty \tag{4.2}$$

and $T(u_0)$ is called the *blow-up time* of the solution u with the initial value u_0. We define the *blow-up set* by

$$B(u_0) = \Big\{ a \in \mathbb{R}^N \Big| \limsup_{x \to a,\, t \nearrow T} |u(x,t)| = \infty \Big\}.$$

Each element of $B(u_0)$ is called a *blow-up point* of u. We say that the solution u of (4.1) blows up *only at space infinity* if, in addition to (4.2), $B(u_0) = \emptyset$. In this case, the *global blow-up profile* $u(x,T) := \lim_{t \to T} u(x,t)$ is defined for every $x \in \mathbb{R}^N$.

Let us recall known results on blow-up at space infinity. Lacey [21] considered a one-dimensional problem $u_t = \Delta u + f(u)$ on the half-line and constructed examples of solutions that blow up only at space infinity. He also obtained results of the global blow-up profile. Giga and Umeda [15] considered the equation $u_t = \Delta u + u^p$ on $\mathbb{R}^N$ and showed that blow-up at space infinity occurs if the initial data u_0 satisfies

$$0 < u_0 < M \quad \text{and} \quad \lim_{|x| \to \infty} u_0(x) = M$$

for some constant $M > 0$. Shimojo [39] considered semilinear heat equations on $\mathbb{R}^N$ and calculated the shape of global blow-up profile of solutions at the blow-up time. It is also proved that such blow-up is always complete, that means that the solution can not extend as a weak solution after blow-up time.

For the case $0 < m < 1$, the heat conductivity mu^{m-1} becomes small as u increases. Hence we can see that diffusion is very slow where u is large. Thus, the blow-up at space infinity must occur as the result for semilinear heat equation of [16]. This is proved by Seki [37] for $0 < m \leq 1 < p$. He also discusses the generalization of the nonlinearity of the form $u_t = \Delta u^m + f(u)$ including the case $0 < m \leq 1 < p$. On the other hand, if $m > 1$, diffusion is very fast where u is so large. Hence the speed of heat propagation, from the space infinity to the origin near the blow-up time, becomes much larger compared to the semilinear problem. Thus a natural question is that "If $m \in (1, \infty)$ is sufficiently large, whether blow-up only at space infinity fails or not ?" Partial answer of this problem was obtained by Seki-Suzuki-Umeda [38]. Their result implies that if $1 \leq m < p$, blow-up only at space infinity occurs. Motivated by these results, we consider the following problem:

> Can blow-up be confined to space infinity even if diffusion is so large such as $m > p > 1$?

In this chapter, we give a partial answer to this problem and show that the *total blow-up*, which means that $B(u_0) = \mathbb{R}^N$ occurs.

Theorem 4.1. *Let $p > 1$ and $m - p > 2(p-1)/N$. Then the problem (4.1) has a total blow-up solution with the initial value $u_0 \in C(\mathbb{R}^N)$ satisfying*

$$0 < u_0 < M \quad \text{and} \quad \lim_{|x| \to \infty} u_0(x) = M \tag{4.3}$$

for a certain positive constant M.

This chapter is organized as follows. In Section 4.2, we discuss the condition $m - p > 2(p-1)/N$ of Theorem 4.1 from the point of asymptotic expansion. The rigorous proof of Theorem 4.1 is given in Section 4.3 by constructing backward self-similar solution. .

Remark 4.1. For the problem (4.1) with nonnegative initial data satisfying the condition $\lim_{|x|\to\infty} u_0(x) = 0$, it is known that, if $p > m > 1$, the blow-up set reduces to finite number of points ([13, 43]). For $1 < p < m$, total blow-up occurs ([14]). There is also a third possibility, $B(u_0)$ is a bounded domain for $p = m$. See also Mochizuki and Suzuki [26] for higher dimensional problem. They consider the case that the support of the initial data is compact, and that the the support of the solution remains bounded if $p > m$ and it spreads out whole space if $p < m$ at the blow-up time. For the precise behavior of such solutions in one dimensional case is considered in the book [35].

4.2 Formal asymptotics

We shall explain why the condition $m - p > 2(p-1)/N$ yields total blow-up. By a formal asymptotic calculation. Let $f(u) = u^p$, then the solution of the following ordinary differential equation

$$U' = f(U), \quad U(0) = M \ (>0), \tag{4.4}$$

is written as $U(t) = \varphi\big(T(M) - t\big)$, where $\varphi(s) := \kappa s^{-\frac{1}{p-1}}$ and $\kappa := (p-1)^{-1/(p-1)}$. Here $T = T(M)$ is the blow-up time for the initial data $U(0) = M$. Substituting $t = 0$ gives $M = \varphi(T(M))$. Furthermore, by a simple calculation, we have

$$\varphi'(s) = -f(\varphi(s)), \qquad \lim_{s \to +0} \varphi(s) = +\infty. \tag{4.5}$$

Let us consider (4.1) with initial data $u_0(x) = M - \varepsilon q_0(x)$, where q is a positive function satisfying $\lim_{|x| \to \infty} q_0(x) = 0$ and $\varepsilon > 0$ is a small constant. The first approximation at space infinity must be the flat solution $\varphi(T - t)$. In order to calculate the second term, we shall consider a formal outer expansion

$$u(x,t) = \sum_{i=0}^{\infty} u^{(i)}(x,t) \varepsilon^i$$

and substitute this into $u_t = \Delta k(u) + f(u)$, where $k(u) = u^m$. Then

$$\begin{aligned} u_t^{(0)} &= \Delta k(u^{(0)}) + f(u^{(0)}), \\ u_t^{(1)} &= k'(u^{(0)}) \Delta u^{(1)} + f'(u^{(0)}) u^{(1)}. \end{aligned}$$

Observing the initial condition at space infinity, we assume $u^{(0)}(x,t) = \varphi(T - t)$ as the first approximation of the solution, hence

$$u_t^{(1)} = k'(\varphi(T-t)) \Delta u^{(1)} + f'(\varphi(T-t)) u^{(1)}. \tag{4.6}$$

Let $q(x,t) = e^{\Phi(t)\Delta} q_0$ be a solution of $q_t = k'(\varphi(T-t)) \Delta q$ with the initial condition $q(x,0) = q_0(x) \in L^1(\mathbb{R}^N)$. In other words,

$$q(x,t) = e^{\Phi(t)\Delta} q_0, \quad \Phi(t) = \int_0^t k'(\varphi(T-\tau))\, d\tau.$$

Here we employ the notation

$$(e^{s\Delta}\, q_0)(x) := \int_{\mathbb{R}^N} G(x-y, s) q_0(y)\, dy.$$

and G is the fundamental solution of the heat equation in $\mathbb{R}^N$:

$$G(x,s) := \frac{1}{(4\pi s)^{N/2}} \exp\Big(-\frac{|x|^2}{4s} \Big).$$

Then the solution of (4.6) is represented as $u^{(1)}(x,t) = -f(\varphi(T-t))q(x,t)$. This can be easily checked from the following calculation.

$$\begin{aligned}
u_t^{(1)} &= -f(\varphi(T-t))q_t - \frac{df(\varphi(T-t))}{dt}q \\
&= -f(\varphi(T-t))q_t + f'(\varphi(T-t))\varphi'(T-t)q \\
&= -f(\varphi(T-t))k'(\varphi(T-t))\Delta q - f'(\varphi(T-t))f(\varphi(T-t))q \\
&= k'(\varphi(T-t))\Delta u^{(1)} + f'(\varphi(T-t))u^{(1)},
\end{aligned}$$

where we applied (4.5) and substitute $s = T-t$. By a formal asymptotic expansion, together with $\varphi'(T-t) = -f(\varphi(T-t))$ again, we get

$$u(x,t) = \varphi(T-t) - \varepsilon f(\varphi(T-t))q(x,t) + O(\varepsilon^2) = \varphi(T-t+\varepsilon q(x,t))$$

provided that $|x|$ is sufficiently large such as $T-t \gg q(x,t)$. We shall discuss a sufficient condition for this ansatz. Note that $\Phi(t)$ is proportional to $(T-t)^{\frac{p-m}{p-1}} - T^{\frac{p-m}{p-1}}$, which implies $\Phi(T) = \infty$ if $m > p$. Assume, for simplicity, that the support of q_0 is compact. Then by applying the inequality

$$\sup_{x\in\mathbb{R}^N} |q(x,t)| \leq \frac{1}{(4\pi\Phi(t))^{N/2}} \int_{\mathbb{R}^N} q_0(x)\,dx,$$

we get the following sufficient condition for $T-t \gg q(x,t)$:

$$T-t \gg O\Big((T-t)^{\frac{N(m-p)}{2(p-1)}}\Big) = O(\Phi(t)^{-N/2}) \geq q(x,t).$$

Since we are interested in what happens as $t \to T_-$, we need the restriction below, which appeared in Theorem 4.1.

$$1 < \frac{N(m-p)}{2(p-1)} \qquad \Longleftrightarrow\ m-p > \frac{2}{N}(p-1).$$

Under this condition, we obtain the following approximation:

$$u(x,t) \approx \varphi(T-t+\varepsilon e^{\Phi(t)\Delta}q_0) \quad \text{if} \quad t \approx T$$

provided that $|x|$ is sufficiently large such as $T-t \gg q(x,t)$. Here $a \approx b$ means that there exist two constants $c_1, c_2 > 0$ such that $c_1 a \leq b \leq c_2 a$, where a and b are positive two functions. Taking a limit $t \to T$ and regarding $e^{\Phi(T)\Delta}q_0 \equiv 0$, we expect that total blow-up occurs, when $m-p > 2(p-1)/N$. On the other hand, the above formal calculation suggests that $m-p < 2(p-1)/N$ yields blow-up only at space infinity, and the global profile must be

$$u(x,T) \approx \varphi(\varepsilon e^{\Phi(T)\Delta}q_0) \quad \text{if} \quad t \approx T. \tag{4.7}$$

Note that $\Phi(T) < \infty$ if $m-p < 2(p-1)/N$. This conjecture (4.7) is proved rigorously in [39] for the semi-linear problem ($m=1$), by constructing suitable sub-super solutions.

4.3 Total blow-up for quasilinear equation

Our aim of this section is to construct a backward self-similar total blow-up solution of the problem (4.1) with the initial value $u_0 \in C(\mathbb{R}^N)$ satisfying (4.3).

Assume the solution u of (4.1) blows up in finite time T and let $T > 0$ be its blow-up time. We introduce a simple change of variable as described in Section 4.2:

$$u(x,t) = \varphi(T - t + h(x,t)). \tag{4.8}$$

From this and $\lim_{s\to 0}\varphi(s) = +\infty$, we can see that the blow-up of the solution $u(x,t)$ for (4.1) as $t \to T$ corresponds to an extinction of the solution $h(x,t)$ as $t \to T$. By a simple calculation together with (4.8) and (4.5),

$$\partial_t\varphi(T-t+h) = \varphi'(T-t+h)\{h_t - 1\}, \quad f(\varphi(T-t+h)) = -\varphi'(T-t+h).$$

By substituting (4.8) into $\Delta u^m = m(m-1)u^{m-2}|\nabla u|^2 + mu^{m-1}\Delta u$, we have

$$\begin{aligned}\Delta\varphi^m(T-t+h) &= m(m-1)\varphi^{m-2}(T-t+h)|\varphi'(T-t+h)\nabla h|^2 \\ &\quad + m\varphi^{m-1}(T-t+h)\{\varphi'(T-t+h)\Delta h + \varphi''(T-t+h)|\nabla h|^2\} \\ &= m(m-1)\varphi^{m-2}(T-t+h)|\varphi'(T-t+h)\nabla h|^2 \\ &\quad + m\varphi^{m-1}(T-t+h)\{\Delta h - f'(\varphi(T-t+h))|\nabla h|^2\}\varphi'(T-t+h).\end{aligned}$$

Here we apply the relation $\varphi''(s) = -f'(\varphi(s))\varphi'(s)$, which can be shown by differentiating (4.5). Substituting (4.8) into (4.1) and divide it by $\varphi'(T-t+h)$, we obtain

$$h_t = m\varphi^{m-1}(T-t+h)\Big[\Delta h + \Big\{(m-1)\frac{\varphi'(T-t+h)}{\varphi(T-t+h)} - f'(\varphi(T-t+h))\Big\}|\nabla h|^2\Big].$$

Applying $\varphi'(s)/\varphi(s) = -s^{-1}/(p-1)$ and $f'(\varphi(s)) = ps^{-1}/(p-1)$, we get the equation

$$h_t = \frac{m\kappa^{m-1}}{(T-t+h)^{\frac{m-1}{p-1}}}\Big\{\Delta h - \frac{(m+p-1)|\nabla h|^2}{(p-1)(T-t+h)}\Big\} \tag{4.9}$$

with the initial data $h(\cdot,0) = \varphi^{-1}(u_0) - T$.

Next we introduce new space and time variables and a function

$$w(y,\sigma) := \frac{h(x,t)}{T-t}, \quad y := (T-t)^{\beta}x, \quad \sigma = \log\Big(\frac{1}{T-t}\Big),$$

where $\beta := \dfrac{m-p}{2(p-1)}$ and h is the solution of (4.9). By the chain rule, together with

$$y_t(x,t) = -e^{\sigma}\beta y(x,t), \quad y_x(x,t) = e^{-\beta\sigma}, \quad \sigma_t(t) = e^{\sigma},$$

we obtain

$$h_t(x,t) = \partial_t\{(T-t)w(y,\sigma)\} = -\beta y \cdot \nabla w(y,\sigma) + w_\sigma(y,\sigma) - w(y,\sigma)$$

and

$$\nabla h(x,t) = e^{-(\beta+1)\sigma}\nabla w(y,\sigma), \quad \Delta h(x,t) = e^{-(2\beta+1)\sigma}\Delta w(y,\sigma).$$

Substituting these into (4.9), we have

$$\begin{aligned} &-\beta y\cdot\nabla w(y,\sigma) + w_\sigma(y,\sigma) - w(y,\sigma) \\ &= \frac{m\kappa^{m-1}}{(1+w(y,\sigma))^{\frac{m-1}{p-1}}}\Big\{\Delta w(y,\sigma) - \frac{m+p-1}{p-1}\frac{|\nabla w(y,\sigma)|^2}{1+w(y,\sigma)}\Big\}e^{\left(\frac{m-1}{p-1}-(2\beta+1)\right)\sigma}. \end{aligned}$$

Therefore, the function w satisfies the rescaled equation

$$w_\sigma = \frac{m\kappa^{m-1}}{(1+w)^{2\beta+1}}\Big\{\Delta w - \frac{m+p-1}{p-1}\frac{|\nabla w|^2}{1+w}\Big\} + (\beta y\cdot\nabla w + w) \tag{4.10}$$

for $y \in \mathbb{R}^N$ and $s > 0$. We can easily see that

$$\lim_{\sigma\to\infty}\|e^{-\sigma}w(\cdot,\sigma)\|_{L^\infty(\mathbb{R}^N)} = 0 \iff B(u_0) = \mathbb{R}^N. \tag{4.11}$$

The simplest example of solution for (4.10) is a constant $w \equiv 0$ which corresponds to a flat solution $u(x,t) = U(t)$ of the original problem (4.1). Here $U(t)$ is the solution of (4.4). Another typical example is the self-similar solution. In our case, it has the form $h(x,t) = (T-t)\,g((T-t)^\beta x)$, where $g = g(y)$ satisfies

$$\Delta g - \frac{m+p-1}{p-1}\frac{|\nabla g|^2}{1+g} + \frac{(1+g)^{2\beta+1}}{m\kappa^{m-1}}(\beta y\cdot\nabla g + g) = 0 \tag{4.12}$$

with $y = (T-t)^\beta x$. In other words, a solution h is self-similar if its rescaled function $w(y,\sigma)$ is independent of σ. If we assume that $g(y)$ is a radial function, $g = g(r)$ is the solution of the following ordinary differential equation:

$$g_{rr} + \frac{N-1}{r}g_r - \frac{m+p-1}{p-1}\frac{g_r^2}{1+g} + \frac{(1+g)^{2\beta+1}}{m\kappa^{m-1}}(\beta r g_r + g) = 0, \tag{4.13}$$

$$g(0) = \mu, \quad g_r(0) = 0, \tag{4.14}$$

where $r = |y|$ and $\mu > 0$ is a constant.

Let us note that the equation (4.13) has a trivial solution $g \equiv 0$, as well as the spatially homogeneous solution $g \equiv -1$. Let us also note that the problem (4.13)-(4.14) admits a solution $g(r)$ with asymptotic behavior:

$$g(r) = \mu - \frac{\mu(1+\mu)^{2\beta+1}}{2m\kappa^{m-1}N}r^2 + o(r^2) \quad \text{as} \quad r \to 0. \tag{4.15}$$

This asymptotics is obtained by solving an approximated ordinary differential equation:

$$g_{rr} + \frac{(1+\mu)^{2\beta+1}}{m\kappa^{m-1}}g \approx 0 \qquad \text{for} \qquad r \approx 0,$$

which comes from the even symmetric assumption $g_r(0) = 0$ and $g(0) = \mu$.

We must find a value μ with the corresponding solution of the above problem (4.13)-(4.14) that is non-negative and decreasing at space infinity.

Proposition 4.1. *Let $p > 1$ and $m-p > 2(p-1)/N$. Then the problem* (4.13)-(4.14) *has a strictly positive monotone solution satisfying $g(\infty) = 0$, if $\mu > 0$ is sufficiently small.*

If we assume this Proposition, by (4.8), the corresponding solution u of the problem (4.1) is written in the form:

$$u_s(x,t) = \varphi\Big((T-t)\,(1 + g((T-t)^{\beta}x))\Big), \qquad \beta > 0.$$

Combining this with $\varphi(0) = \infty$, we obtain $u_s(x,T) = \infty$ for any $x \in \mathbb{R}^N$. Thus $B(u_s(\cdot,0)) = \mathbb{R}^N$. Furthermore, the condition (4.3) of the initial value can be easily checked and our result is obtained. Now we shall prove the existence of strictly positive solution $g = g(r)$ for the problem (4.13) -(4.14).

Lemma 4.1. *Let $g = g(r)$ be the solution of the problem* (4.13)-(4.14) *If $g > 0$ on an interval $[0, R_0)$, then g is strictly decreasing on $[0, R_0)$.*

Proof. Define

$$r_0 = \sup\{r > 0 \mid g \text{ is strictly decreasing on } [0,r]\}$$

and assume $r_0 < R_0$. Then the definition of r_0 implies $g_r(r_0) = 0$ (both $g_r(r_0) > 0$ and $g_r(r_0) < 0$ easily lead to contradiction), and (4.13) implies $g_{rr}(r_0) < 0$. This in turn means that g is strictly decreasing on a right neighborhood of r_0, a contradiction with the definition of r_0. Hence $r_0 \geq R_0$. □

By Lemma 4.1, one can distinguish the following two cases two cases:

(a) $g > 0$ on $[0,\infty)$ and g is strictly decreasing on $[0,\infty)$.

(b) There exists $R \in (0,\infty)$ such that $g > 0$ on $[0,R)$ and $g(R) = 0$. This implies that g is strictly decreasing on $[0,R)$; thus, by continuity, it is strictly decreasing on $[0,R]$. In particular, $g_r(R) < 0$.

Now we exclude the second case (b) as the following lemma.

Lemma 4.2. *Assume that $\beta N > (1+\mu)^{2\beta+1}$. Let $g = g(r)$ be the solution of the problem* (4.13)-(4.14). *Then $g > 0$ on $[0,\infty)$.*

Proof. The decay rate of the solution is given by the solution of $\beta r\bar{g}_r + \bar{g} = 0$, which is the dominant term of the ordinary differential equation (4.13). Thus we introduce a function

$$v := -\frac{\beta r g_r}{g} \; : \quad [0,R) \to [0,\infty). \tag{4.16}$$

By the definition of R, the function v is a nonnegative function and well-defined. Assume that $R < \infty$. Then case (b) of Lemma 4.1 implies that $\lim_{r\to R} v(r) = \infty$. Differentiating

(4.16) and using (4.13), we get

$$\begin{aligned}
v_r &= -\frac{\beta r}{g}\Big(g_{rr} + \frac{1}{r}g_r\Big) + \beta r\Big(\frac{g_r}{g}\Big)^2 \\
&= \beta(N-2)\frac{g_r}{g} + \beta r\Big(\frac{g_r}{g}\Big)^2 - \frac{m+p-1}{p-1}\frac{\beta r g_r^2}{g(1+g)} + \frac{\beta r(1+g)^{2\beta+1}}{m\kappa^{m-1}}(1-v) \\
&= -(N-2)\frac{v}{r} + \frac{v^2}{\beta r} - \frac{m+p-1}{p-1}\frac{g}{1+g}\frac{v^2}{\beta r} + \frac{\beta r(1+g)^{2\beta+1}}{m\kappa^{m-1}}(1-v) \\
&= -(N-2)\frac{v}{r} + \Big(1 - \frac{m+p-1}{p-1}\frac{g}{1+g}\Big)\frac{v^2}{\beta r} + \frac{\beta r(1+g)^{2\beta+1}}{m\kappa^{m-1}}(1-v).
\end{aligned}$$

From (4.15) and (4.16), we see that

$$v(r) = \frac{\beta(1+\mu)^{2\beta+1}}{m\kappa^{m-1}N}r^2 + o(r^2) \quad \text{as} \quad r \to 0.$$

We will use this asymptotic in order to estimate the function v from above. Next we shall check that the function $\bar{v}(r) := \frac{\beta(1+\mu)^{2\beta+1}}{m\kappa^{m-1}N}r^2$ is a super-solution of the above ordinary differential equation, provided that

$$1 \le \beta N\frac{(1+g)^{2\beta+1}}{(1+\mu)^{2\beta+1}} + \frac{m+p-1}{p-1}\frac{g}{1+g} \tag{4.17}$$

for all $r \in [0, R)$. In fact, under the condition (4.17), we get

$$\begin{aligned}
&\bar{v}_r + (N-2)\frac{\bar{v}}{r} - \Big(1 - \frac{m+p-1}{p-1}\frac{g}{1+g}\Big)\frac{\bar{v}^2}{\beta r} - \frac{\beta r(1+g)^{2\beta+1}}{m\kappa^{m-1}}(1-\bar{v}) \\
&= \frac{N\bar{v}}{r}\Big\{1 - \frac{(1+g)^{2\beta+1}}{(1+\mu)^{2\beta+1}}\Big\} - \Big\{1 - \frac{m+p-1}{p-1}\frac{g}{1+g} - \beta N\frac{(1+g)^{2\beta+1}}{(1+\mu)^{2\beta+1}}\Big\}\frac{\bar{v}^2}{\beta r} \\
&\ge -\Big\{1 - \frac{m+p-1}{p-1}\frac{g}{1+g} - \beta N\frac{(1+g)^{2\beta+1}}{(1+\mu)^{2\beta+1}}\Big\}\frac{\bar{v}^2}{\beta r} \ge 0.
\end{aligned}$$

Here we used the relations $\bar{v}_r = 2\bar{v}/r$, together with

$$\frac{\beta r(1+g)^{2\beta+1}}{m\kappa^{m-1}} = \frac{N\bar{v}}{r}\frac{(1+g)^{2\beta+1}}{(1+\mu)^{2\beta+1}}$$

and an inequality $g(r) \le \mu$ for $r \in [0, R]$. The condition (4.17) is satisfied because the function g is nonnegative on $[0, R)$ and $\beta N > (1+\mu)^{2\beta+1}$. Therefore, by the comparison argument, $v \le \bar{v}$ for all $r \in [0, R)$ and $\lim_{r\to r_1} v(r) \le \bar{v}(R) < \infty$. This yields a contradiction. □

Proof of Proposition 4.1. Let $p > 1$ and $m-p > 2(p-1)/N$ then $\beta N > 1$. By Lemma 4.2, the problem (4.13)-(4.14) has a positive solution, if we choose $\mu > 0$ sufficiently small such that $\beta N > (1+\mu)^{2\beta+1}$. Lemma 4.1 implies that this solution is strictly decreasing. Furthermore, since there exists no positive spatially homogeneous solution of the equation (4.13), we obtain $g(\infty) = 0$. Hence we obtain the result. □

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