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A LIMIT TRANSITION FROM THE HECKMAN-OPDAM HYPERGEOMETRIC FUNCTIONS TO THE WHITTAKER FUNCTIONS ASSOCIATED WITH ROOT SYSTEMS

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Abstract. We prove that the radial part of the class one Whittaker function on a split semisimple Lie group can be obtained as an appropriate limit of the Heckman-Opdam hypergeometric function.

Introduction

Among quantum integrable systems associated with root systems, there are three classes containing well-behaved joint eigenfunctions closely related with Lie theory. They are the trigonometric Calogero-Moser model, the rational CM model, and the Toda model. For the CM model, the eigenfunction is the Heckman-Opdam hypergeometric function and the Bessel function corresponding to trigonometric and rational cases respectively (cf. [12, 19, 20]). Among other eigenfunctions, they are up to constant multiples unique globally defined analytic functions. For special parameter they are the radial part of the spherical functions on a Riemannian symmetric space of the non-compact type and the Euclidean type respectively (cf. [14]). For the non-periodic Toda model, the eigenfunction is the class one Whittaker function defined by the Jacquet integral on a Riemannian symmetric space of the non-compact type (cf. [17, 11]). Among other eigenfunctions, it is up to a constant multiple unique eigenfunction of moderate growth.

On the other hand, there are two limit transitions between the Hamiltonians. One is from the trigonometric CM model to the rational CM model, and the other is from the trigonometric CM model to the Toda model (cf. [4, 16]). In the rank one case, corresponding limit transitions are one from the Gauss hypergeometric function to the Bessel function, and the other is from the Gauss hypergeometric function to the Macdonald function. In general, a limit transition for eigenfunctions in the former case was established by Ben Saïd-Ørsted [1] and de Jeu [4]. In this paper we establish a limit transition in the latter case. Namely we prove that a limit of the Heckman-Opdam hypergeometric function is the radial part of the Whittaker function on a split semisimple Lie group (Theorem 5). Similar result for functions on $Sp(2, \mathbb{R})$ was proved by Hirano-Ishii-Oda [15], which motivated the study of this paper.

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1. Preliminaries

1.1. **The Heckman and Opdam hypergeometric function.** In this subsection, we review on the Heckman-Opdam hypergeometric function associated with a root system. See [12] and [20] for details.

Let $\mathfrak{a}$ be a Euclidean space of dimension n equipped with an inner product $(\ ,\ )$. We identify $\mathfrak{a}^*$ with $\mathfrak{a}$ as usual. For $\alpha \in \mathfrak{a}^* \setminus \{0\}$ define

$$\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)}.$$

Let R denote a reduced root system in $\mathfrak{a}^*$. Choose a positive system $R_+ \subset R$ and let B denote the set of the simple roots. Let W denote the Weyl group for R. For $\alpha \in R$ let k_α be a non-negative number such that $k_{w\alpha} = k_\alpha$ for all $w \in W$. We call $k : \alpha \mapsto k_\alpha$ a multiplicity function. We put

$$\rho(k) = \frac{1}{2} \sum_{\alpha \in R_+} k_\alpha \alpha.$$

Let $A = \exp \mathfrak{a}$ and

$$A_+ = \{a \in A \,:\, \alpha(\log a) > 0 \text{ for all } \alpha \in R_+\}.$$

Let $\{\xi_1, \ldots, \xi_n\}$ be an orthonormal basis of $\mathfrak{a}$. Define

$$L(k) = \sum_{i=1}^{n} \partial_{\xi_i}^2 + \sum_{\alpha \in R_+} k_\alpha \frac{1 + e^{-\alpha}}{1 - e^{-\alpha}} \partial_\alpha. \tag{1}$$

There exist a commutative algebra $\mathbb{D}(k)$ of differential operators containing $L(k)$ and we have an isomorphism $\gamma : \mathbb{D}(k) \to S(\mathfrak{a}_\mathbb{C})^W$, the set of W-invariant elements of the symmetric algebra of $\mathfrak{a}_\mathbb{C} = \mathfrak{a} \otimes_\mathbb{R} \mathbb{C}$.

Let Q be the $\mathbb{Z}$-span of R and let Q_+ be the $\mathbb{Z}_+$-span of R_+. There exists a solution $\phi(a) = \Phi(\lambda, k; a)$ for

$$D\varphi = \gamma(D)(\lambda)\varphi \tag{2}$$

of the form

$$\Phi(\lambda, k; a) = \sum_{\mu \in Q_+} \Gamma_\mu(\lambda, k) e^{(\lambda - \rho(k) - \mu)(\log a)}, \quad \Gamma_0(\lambda, k) = 0. \tag{3}$$

The coefficients $\Gamma_\mu(\lambda, k)$ are determined by recurrence relations coming from $L(k)$.

If $\lambda \in \mathfrak{a}_\mathbb{C}^* = \mathfrak{a}^* \otimes_\mathbb{R} \mathbb{C}$ satisfies the condition

$$(2\lambda + \mu, \mu) \neq 0 \ \text{ for all } \ \mu \in Q \setminus \{0\}, \tag{4}$$

then $\{\Phi(w\lambda, k; a) \,:\, w \in W\}$ forms a basis of the solution space of (2) on A_+.

Define
$$\tilde{c}(\lambda, k) = \prod_{\alpha \in R_+} \frac{\Gamma((\lambda, \alpha^\vee))}{\Gamma((\lambda, \alpha^\vee) + k_\alpha)},$$
where $\Gamma(\,)$ is the gamma function and
$$c(\lambda, k) = \frac{\tilde{c}(\lambda, k)}{\tilde{c}(\rho(k), k)}.$$
Define
$$F(\lambda, k; a) = \sum_{w \in W} c(w\lambda, k)\Phi(w\lambda, k; a). \tag{5}$$
The function F is called the Heckman-Opdam hypergeometric function for the root system R. It is well behaved compared to Φ.

Theorem 1 (Heckman-Opdam). *$F(\lambda, k; a)$ is a unique W-invariant solution for (2) that is analytic in $a \in A$, is holomorphic in $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, and satisfies*
$$F(w\lambda, k; a) = F(\lambda, k; a) \quad (w \in W),$$
$$F(\lambda, k; wa) = F(\lambda, k; a) \quad (w \in W).$$

1.2. **Notation on Lie groups.** Let G be a normal real form of a connected complex semisimple Lie group and K a maximal compact subgroup. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K respectively. Let θ denote the corresponding Cartan involution of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the decomposition into ± 1 eigenspaces of θ. Equip an inner product $(\ ,\ )$ on $\mathfrak{g}$ induced by the Killing form on $\mathfrak{g}$.

Fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. Let $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ denote the set of the restricted roots. Fix a positive system Σ_+ and let Π denote the set of the simple roots. Notice that each root space has dimension 1, because we assume that G is split. Put $\rho = \frac{1}{2}\sum_{\alpha \in \Sigma_+} \alpha$. Let W denote the Weyl group of Σ. It is isomorphic to $N_K(\mathfrak{a})/Z_K(\mathfrak{a})$, where $N_K(\mathfrak{a})$ (resp. $Z_K(\mathfrak{a})$) is the normalizer (resp. centralizer) of $\mathfrak{a}$ in K.

Let $\mathfrak{n}$ be the sum of root spaces for the positive roots. Put $N = \exp \mathfrak{n}$ and $A = \exp \mathfrak{a}$. Then we have the Iwasawa decomposition $G = NAK = KAN$.

Remark. In the previous section, we adopt notation of Heckman and Opdam. Relations between notation in the previous section and this section are given by
$$R = 2\Sigma, \quad R_+ = 2\Sigma_+, \quad B = 2\Pi, \quad k_{2\beta} = \frac{1}{2}m_\beta = \frac{1}{2} \ \ (\beta \in \Sigma). \tag{6}$$
The operator $L(k)$ in (1) is the radial part of the Laplace-Beltrami operator $L_{G/K}$ on G/K, the algebra $\mathbb{D}(k)$ consists of the radial parts of invariant differential operators on G/K with respect to the Cartan decomposition

$G = KAK$, $c(\lambda, k)$ is Harish-Chandra's c-function, and the hypergeometric function $F(\lambda, k; a)$ is the radial part of the zonal spherical function on G/K.

1.3. **The Whittaker function on a semisimple Lie group.** In this subsection, we review the class one Whittaker functions on a split semisimple Lie group following Hashizume [11].

Let ψ be a unitary character of N. We denote the differential character of $\mathfrak{n}$ to $\sqrt{-1}\mathbb{R}$ by the same letter ψ. Let $C^\infty_\psi(G/K)$ denote the space of C^∞-functions on G satisfying $u(ngk) = \psi(n)u(g)$ for all $n \in N$, $g \in G$, and $k \in K$. By the Iwasawa decomposition $G = NAK$, the values of $u \in C^\infty_\psi(G/K)$ are completely determined by $u|_A$. Let $\mathbb{D}(G/K)$ denote the commutative algebra of left G-invariant differential operators on G/K and $\chi_\lambda : \mathbb{D}(G/K) \to \mathbb{C}$ the Harish-Chandra homomorphism. Let $\mathcal{A}_\psi(G/K, \mathcal{M}_\lambda)$ be the subspace of $C^\infty_\psi(G/K)$ defined by

$$\mathcal{A}_\psi(G/K, \mathcal{M}_\lambda) = \{u \in C^\infty_\psi(G/K) \,:\, Du = \chi_\lambda(D)u \ \text{ for all } D \in \mathbb{D}(G/K)\}.$$

Notice that $C^\infty_\psi(G/K, \mathcal{M}_\lambda)$ consists of real analytic functions, because $L_{G/K}$ is an elliptic differential operator.

For $\beta \in \Pi$ let $X_\beta \in \mathfrak{g}^\beta$ be a unit root vector. For $\alpha \in B = 2\Pi$ put $l_\alpha = -\sqrt{-1}\psi(X_{\alpha/2})$. For $u \in \mathcal{A}_\psi(G/K, \mathcal{M}_\lambda)$, $\varphi = e^{-\rho}u|_A$ satisfies

$$\Big(\sum_{i=1}^n \partial^2_{\xi_i} - 2\sum_{\alpha\in B} l^2_\alpha e^\alpha\Big)\varphi = (\lambda, \lambda)\varphi. \tag{7}$$

There exists a solution $\Psi_{\mathrm{T}}(\lambda, \psi; a)$ of the equation (7) on A_+ of the form

$$\Psi_{\mathrm{T}}(\lambda, \psi; a) = a^\lambda \sum_{\mu\in Q_+} b_\mu(\lambda)a^\mu, \quad b_0(\lambda) = 0. \tag{8}$$

Moreover, extending function $u(a) = e^\rho\, \Psi_{\mathrm{T}}(\lambda, \psi; a)$ on A to G so that $u \in C^\infty_\psi(G/K)$ is also a joint eigenfunction of $\mathbb{D}(G/K)$ and belongs to $\mathcal{A}_\psi(G/K, \mathcal{M}_\lambda)$. If $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ satisfies the condition (4), then $\{e^\rho\, \Psi_{\mathrm{T}}(w\lambda, \psi; a) : w \in W\}$ forms a basis of $\mathcal{A}_\psi(G/K, \mathcal{M}_\lambda)|_A$ (cf. [11, Corollary 5.3, Theorem 5.4]).

For $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ define function 1_λ on G by

$$1_\lambda(nak) = a^{\lambda+\rho} \quad (n \in N,\, a \in A,\, k \in K).$$

For $g \in G$ let $H(g)$ denote the element of $\mathfrak{a}$ defined by $g \in K \exp H(g)N$. We normalize the Haar measure dn and $d\bar{n}$ on N and $\bar{N} = \theta N$ by

$$\theta(dn) = d\bar{n}, \qquad \int_{\bar{N}} e^{-2\rho(H(\bar{n}))} d\bar{n} = 1$$

(cf. [14, Ch. IV, §6]). Define the class one Jacquet integral

$$W(\lambda,\psi;g) = \int_N 1_\lambda(\bar{w}_0^{-1} ng)\psi(n)^{-1} dn. \tag{9}$$

Here $\bar{w}_0$ is a representative in $N_K(\mathfrak{a})$ of the longest element $w_0 \in W$. For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ with $\operatorname{Re}(\lambda,\alpha) > 0$ for all $\alpha \in \Sigma_+$, the class one Jacquet integral $W(\lambda,\psi;g)$ converges absolutely and uniformly and belongs to $\mathcal{A}_\psi(G/K,\mathcal{M}_\lambda)$. Moreover $W(\lambda,\psi;g)$ is continued to a meromorphic function of $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ as an element of $\mathcal{A}_\psi(G/K,\mathcal{M}_\lambda)$ (cf. [11, Theorem 6.6]). We call this meromorphic continuation of $W(\lambda,\psi;a)$ the Whittaker function.

The Whittaker function $W(\lambda,\psi;g)$ is up to a constant multiple a unique element of $\mathcal{A}_\psi(G/K,\mathcal{M}_\lambda)$ that is of moderate growth (cf. [3, Theorem 9.1]).

The analytic properties of the integral (9) were studied by Jacquet [17], Schiffmann [21], Goodman-Wallach [7], and Hashizume [10, 11], etc. The function $W(\lambda,\psi;g)$ satisfies the following functional equation.

$$W(\lambda,\psi;g) = M(w,\lambda,\psi)W(w\lambda,\psi;g) \quad (w \in W). \tag{10}$$

Here the factor $M(w,\lambda,\psi)$ is independent of g and is given by the product formula

$$M(ww',\lambda,\psi) = M(w',\lambda,\psi)M(w,w'\lambda,\psi) \quad (w,\, w' \in W), \tag{11}$$

$$M(s_\alpha,\lambda,\psi) = \left(\frac{2l_\alpha^2}{(\alpha,\alpha)}\right)^{(\lambda,\alpha^\vee)} \frac{\Gamma(-(\lambda,\alpha^\vee)+1/2)}{\Gamma((\lambda,\alpha^\vee)+1/2)} \quad (\alpha \in B), \tag{12}$$

where s_α denote the simple reflection corresponding to $\alpha \in B$. (cf. [11, (7.5)–(7.7)]. Notice again that $R = 2\Sigma$ and G is split.)

Let $\boldsymbol{c}(\lambda)$ denote the Harish-Chandra c-function for the split Lie group G, which is given by $\boldsymbol{c}(\lambda) = c(\lambda,k)$ with $k_\alpha = 1/2$ for all $\alpha \in R$. That is

$$\boldsymbol{c}(\lambda) = \frac{\tilde{\boldsymbol{c}}(\lambda)}{\tilde{\boldsymbol{c}}(\rho)},$$

$$\tilde{\boldsymbol{c}}(\lambda) = \prod_{\alpha \in R_+} \frac{\Gamma((\lambda,\alpha^\vee))}{\Gamma((\lambda,\alpha^\vee)+\frac{1}{2})}.$$

Hashizume [11, Theorem 7.8] expressed the Whittaker function $W(\lambda,\psi;a)$ as a linear combination of $\Psi_{\mathrm{T}}(w\lambda,\psi;a)$ explicitly.

Theorem 2 (Hashizume)**.** *Let ψ be a non-degenerate character of N and assume that $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ satisfies (4). Then*

$$W(\lambda,\psi;a) = a^\rho \sum_{w \in W} M(w_0 w,\lambda,\psi)\boldsymbol{c}(w_0 w\lambda)\Psi_{\mathrm{T}}(w\lambda,\psi;a). \tag{13}$$

2. Limit transition from the Heckman-Opdam hypergeometric function to the Whittaker function

2.1. Limit transition from the Calogero-Moser Hamiltonian to the Toda Hamiltonian. In this subsection, we review the limit transition from the quantum trigonometric Calogero-Moser model to the Toda model.

Define a function $\delta(k) = \delta(k; a)$ by

$$\delta(k)^{1/2} = \prod_{\alpha \in R_+} (e^{\frac{1}{2}\alpha} - e^{-\frac{1}{2}\alpha})^{k_\alpha}.$$

We have

$$\begin{aligned} \delta(k)^{1/2} \circ \{L(k) + (\rho(k), \rho(k))\} \circ \delta(k)^{-1/2} \\ = \sum_{i=1}^{n} \partial_{\xi_i}^2 + \sum_{\alpha \in R_+} \frac{k_\alpha(1-k_\alpha)(\alpha,\alpha)}{4\sinh^2 \frac{1}{2}\alpha} \end{aligned} \tag{14}$$

We denote the right hand side of (14) by $H_{\mathrm{CM}}(k)$. It is the Hamiltonian for the trigonometric Calogero-Moser model.

Recall that $R = 2\Sigma$ (Remark 1.2) and let $B = 2\Pi$ be the simple system of $R_+ = 2\Sigma_+$. We assume that $l_\alpha = 1$ for all $\alpha \in B$. This means we assume that ψ is a special non-degenerate unitary character of N. The left hand side of (7) gives the Hamiltonian for the quantum Toda model

$$H_{\mathrm{T}} = \sum_{i=1}^{n} \partial_{\xi_i}^2 - 2\sum_{\alpha \in B} e^{\alpha}. \tag{15}$$

Let M be a positive real number. Define a positive multiplicity function k_M by

$$k_M(\alpha)(k_M(\alpha) - 1)(\alpha, \alpha) = 2e^{2M}$$

and define $a_M \in A$ by

$$\log a_M = w_0 \log a + M\rho^\vee,$$

where w_0 is the longest element of W. Notice that

$$\rho^\vee = \frac{1}{2}\sum_{\beta \in \Sigma_+} \beta^\vee = \sum_{\alpha \in R_+} \alpha^\vee$$

is the Weyl vector of $\Sigma^\vee = 2R^\vee$ and $(\alpha, \rho^\vee) = 1$ for all $\alpha \in \Pi = \frac{1}{2}B$ (cf. Bourbaki [2, Ch VI Proposition 29]).

We shall consider a limit of the hypergeometric function when $M \to \infty$. Taking a limit of $H_{\mathrm{CM}}(k)$, we have the following lemma.

Lemma 3 (Inozemtsev)**.** *For any $\varphi \in C^\infty(A)$,*

$$\lim_{M \to \infty} H_{\mathrm{CM}}(k_M)\, \varphi(a_M) = H_{\mathrm{T}}\, \varphi(a). \tag{16}$$

This limit procedure was proved by Inozemtsev [16] (see also [5, Section 7] and [18]).

2.2. **Limit transition of eigenfunctions.** Define

$$\Psi_{\mathrm{CM}}(\lambda, k; a) = \delta(k;a)^{1/2}\,\Phi(\lambda, k; a).$$

By (3), $\varphi(a) = \Psi_{\mathrm{CM}}(\lambda, k; a)$ is of the form

$$\Psi_{\mathrm{CM}}(\lambda, k; a) = \sum_{\mu\in\Lambda} b_\mu(\lambda, k) e^{(\lambda-\mu)(\log a)}, \quad b_0(\lambda, k) = 1. \tag{17}$$

By (14), it is also a solution of

$$H_{\mathrm{CM}}(k)\,\varphi = (\lambda, \lambda)\,\varphi. \tag{18}$$

On the other hand, as we have seen in Subsection 1.3, there is a series solution $\varphi(a) = \Psi_{\mathrm{T}}(\lambda; a)$ of

$$H_{\mathrm{T}}\,\varphi = (\lambda, \lambda)\,\varphi.$$

Proposition 4. *If $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ satisfies the condition (4), then*

$$\lim_{M\to\infty} e^{-(\lambda,\rho^\vee)M}\Psi_{\mathrm{CM}}(\lambda, k_M; a_M) = \Psi_{\mathrm{T}}(w_0\lambda; a) \quad (a \in A_+). \tag{19}$$

The convergence is uniform on each subchamber

$$\{a \in A_+ \,:\, \alpha(\log a) > c > 0 \ (\alpha \in B)\},$$

where $c > 0$ is arbitrary.

Proof. The proof is an easy modification of the estimate of the Harish-Chandra series due to Gangolli [6] (see also Helgason [14, Ch IV §5]).

Let M be a positive number. Putting $k = k_M$, $a = a_M$, and writing $b_\mu(\lambda, k_M) = \tilde{b}_\mu(\lambda, M)$ in (17), we have

$$\Psi_{\mathrm{CM}}(\lambda, k_M; a_M) = e^{(\lambda,\rho^\vee)M} \sum_{\mu\in Q_+} \tilde{b}_\mu(\lambda, M) e^{(w_0\lambda+\mu)(\log a)}, \quad \tilde{b}_0(\lambda, M) = 1. \tag{20}$$

Then (18) becomes

$$\Big(\sum_{i=1}^n \partial_{\xi_i}^2 - 2\sum_{\alpha\in R_+} e^{2M}\sum_{j=1}^\infty j\, e^{-j((\alpha,\rho^\vee)M + w_0\alpha(\log a))}\Big)\Psi_{\mathrm{CM}}(\lambda, k_M; a_M) = (\lambda,\lambda)\Psi_{\mathrm{CM}}(\lambda, k_M; a_M) \tag{21}$$

by expanding the H_{CM} (the right hand side of (14)) into power series. Equations (20) and (21) give the recurrence relation for $\tilde{b}_\mu(\lambda, M)$ such as

$$(22)\qquad (2w_0\lambda{+}\mu,\mu)\tilde{b}_\mu(\lambda,M) = 2\sum_{\alpha\in R_+}\sum_{j\geq 1,\,\mu+jw_0\alpha\in Q_+} e^{(2-j(\alpha,\rho^\vee))M}\, j\,\tilde{b}_{\mu+jw_0\alpha}(\lambda,M).$$

Since $(\alpha,\rho^\vee) = 2$ for $\alpha \in B = 2\Pi$ and $(\alpha,\rho^\vee) \geq 4$ for $\alpha \in R_+ \setminus B$, the recurrence relation (22) converges to

$$(23)\qquad (2w_0\lambda+\mu,\mu)\tilde{b}_\mu(\lambda,\infty) = 2\sum_{\alpha\in B}\tilde{b}_{\mu-\alpha}(\lambda,\infty)$$

as $M \to \infty$. (23) is nothing but the recurrence relation for the coefficients in expansion (8) for $\Psi_{\mathrm{T}}(w_0\lambda; a)$ (cf. [11, §4]).

For $\mu \in Q_+$ we write $\mu = \sum_{\alpha\in B} n_\alpha \alpha$ and put $n(\mu) = \sum_{\alpha\in B} n_\alpha$. Choose a constant c such that

$$|2(w_0\lambda+\mu,\mu)| \geq c\,n(\mu)$$

for all $\mu \in Q_+$. By (22) we have

$$\begin{aligned}|\tilde{b}_\mu(\lambda,M)| &\leq 2c^{-1}\sum_{\alpha\in R_+}\left(\sum_{j\geq 1,\,\mu+jw_0\alpha\in Q_+} e^{(2-(j\alpha,\rho^\vee))M}\, j\,|\tilde{b}_{\mu+jw_0\alpha}(\lambda,M)|\right)\\ &\leq 2c^{-1}\sum_{\alpha\in R_+}\left(\sum_{j\geq 1,\,\mu+jw_0\alpha\in Q_+} j\,|\tilde{b}_{\mu+jw_0\alpha}(\lambda,M)|\right)\end{aligned}$$

for $M > 0$. We can prove in the same way as the proof of [14, Ch IV Lemma 5.3, Lemma 5.6] that there exists a constant $K_{a,l}$ such that

$$|\tilde{b}_\mu(\lambda,M)| \leq K_{\lambda,a}a^\mu$$

for all $\mu \in \Lambda$. This estimate shows the convergence of the series (20) and also guarantees the limit transition (19). $\square$

Now we state and prove our main result:

Theorem 5. *Assume that $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ satisfies (4). Then*

$$(24)\qquad \lim_{M\to\infty}\delta(k_M;a_M)^{1/2}\,\tilde{c}(\rho(k_M),k_M)\prod_{\alpha\in R_+}\Gamma(k_M(\alpha))\,F(\lambda,k_M;a_M) = \tilde{\boldsymbol{c}}(\rho)f(\lambda)\,a^{-\rho}\,W(\lambda,\psi;a),$$

where

$$(25)\qquad f(\lambda) = \prod_{\alpha\in R_+}\left(\frac{(\alpha,\alpha)}{2}\right)^{(\lambda,\alpha^\vee)/2}\Gamma((\lambda,\alpha^\vee)+\tfrac{1}{2})$$

and ψ is a unitary character of N defined by $l_\alpha = 1\,(\alpha \in B)$.

Proof. By the following formula for the Gamma function

$$\lim_{x\to\infty} \frac{\Gamma(\mu+x)}{\Gamma(x)\,x^{\mu}} = 1,$$

we have

$$\tilde{c}(\lambda, k_M) \sim f(\lambda)\tilde{\boldsymbol{c}}(\lambda) \prod_{\alpha\in R_+} \frac{e^{-(\lambda,\alpha^\vee)M}}{\Gamma(k_M(\alpha))} \tag{26}$$

as $M\to\infty$. By (5), Proposition 4, and (26), we have

$$\begin{aligned}\lim_{M\to\infty} \delta(k_M;a_M)^{1/2}\tilde{c}(\rho(k_M),k_M) \prod_{\alpha\in R_+} \Gamma(k_M(\alpha))\,F(\lambda,k_M;a_M)\\ = \sum_{w\in W} f(w\lambda)\tilde{\boldsymbol{c}}(w\lambda)\Psi_{\mathrm{T}}(w_0w\lambda;a).\end{aligned} \tag{27}$$

On the other hand, by Theorem 2, the right hand side of (24) is a linear combination of $\Psi_{\mathrm{T}}(w\lambda;a)$ $(w\in W)$, where the coefficient of $\Psi_{\mathrm{T}}(w\lambda;a)$ is given by

$$d(w,\lambda) := f(\lambda)M(w_0w,\lambda,\psi)\tilde{\boldsymbol{c}}(w_0w\lambda). \tag{28}$$

For $\beta\in R$, it follows from (11) and (12) that

$$\begin{aligned} d(w,s_\beta\lambda) &= f(s_\beta\lambda)M(w_0w,s_\beta\lambda,\psi)\tilde{\boldsymbol{c}}(w_0ws_\beta\lambda)\\ &= f(s_\beta\lambda)M(s_\beta,\lambda,\psi)^{-1}M(w_0ws_\beta,\lambda,\psi)\tilde{\boldsymbol{c}}(w_0ws_\beta\lambda)\\ &= d(ws_\beta,\lambda).\end{aligned}$$

The last equality follows from $s_\beta(B\setminus\{\beta\}) = B\setminus\{\beta\}$. Thus the right hand side of (24) is W-invariant with respect to λ. We have $d(w_0,\lambda) = f(\lambda)\tilde{\boldsymbol{c}}(\lambda)$ and it coincides with the coefficient of $\Psi_{\mathrm{T}}(\lambda,l;a)$ in the right hand side of (27), hence the result follows. □

Example. For R of type A_1

$$F(\lambda,k;a_t) = {}_2F_1(\tfrac{1}{2}(k-\lambda), \tfrac{1}{2}(k+\lambda)\,;\,k+\tfrac{1}{2}\,;\,-\sinh^2 t),$$

where ${}_2F_1(a,b,c;z)$ is the Gauss hypergeometric function. Then Theorem 5 reads

$$\lim_{k\to\infty} k^{-1/2}2^{-k}\sinh^k(-t+M)F(\lambda,k;a_{-t+M}) = \frac{1}{\sqrt{\pi}}K_\lambda(e^t/2),$$

where $K_\lambda(z)$ is the Macdonald function.

Remark. We restrict ourselves to split semisimple Lie groups, because the Hamiltonian (15) of the Toda lattice depends only on the reduced root system. The class one Whittaker function given by the Jacquet integral for a non-split semisimple Lie group is a constant multiple of the Whittaker function for the split Lie group of the same indivisible restricted roots.

We can change parameters $l_\alpha^2\,(\alpha \in B)$ in the left hand side of (7) by making a shift of variables, as it was pointed out by [18, §2.1]. Let $\varpi_\alpha\,(\alpha \in B)$ denote the fundamental weights corresponding to B. If we put

$$\log a = \log a' - \sum_{\alpha\in B,\ l_\alpha\neq 0} \frac{2}{(\alpha,\alpha)}\varpi_\alpha \log l_\alpha^2,$$

then

$$\alpha(\log a) = \begin{cases} \alpha(\log a') - \log l_\alpha^2 & (\alpha \in B,\, l_\alpha \neq 0) \\ \alpha(\log a') & (\alpha \in B,\, l_\alpha = 0). \end{cases}$$

Thus in the new coordinates $\sum_{i=1}^n \partial_{\xi_i}^2 - 2\sum_{\alpha\in B} l_\alpha^2\, e^\alpha$ becomes $\sum_{i=1}^n \partial_{\xi_i}^2 - 2\sum_{\alpha\in B,\ l_\alpha\neq 0} e^\alpha$. Moreover, if $l_\alpha = 0$ for some α, then the Whittaker function can be reduced to lower rank cases. This is the reason why we assume $l_\alpha = 1$ for all $\alpha \in B$.

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References

[1] S. Ben Saïd and B. Ørsted, *Bessel functions for root systems via the trigonometric setting*, Int. Math. Res. Not. **2005**, 551–585.

[2] N. Bourbaki, *Elements of Mathematics. Lie Groups and Lie Algebras: Chapters 4-6* (Elements of Mathematics). (English translation), Springer-Verlag, 2006.

[3] W. Casselman, H. Hecht, and D. Miličić, *Bruhat filtrations and Whittaker vectors for reeal groups*, The Mathematical Legacy of Harish-Chandra. Proceedings of Symposia in Pure Mathematics **68** (2000), 151–190.

[4] M. de Jeu, *Paley-Wiener theorems for the Dunkl transform*, Trans. Amer. Math. Soc. **358** (2006), 4225-4250.

[5] P. Etingof, *Whittaker functions on quantum groups and q-deformed Toda operators*, Differential topology, infinite-dimensional Lie algebras, and applications, 9–25, Amer. Math. Soc. Transl. Ser. 2, **194**, Amer. Math. Soc., 1999.

[6] R. Gangolli, *On the Plancherel formula and the Paley-Wiener theorem for spherical functions on semisimple Lie groups*, Ann. of Math. **93** (1971), 150–165.

[7] R. Goodman and N. Wallach, *Whittaker vectors and conical vectors*, J. Funct. Anal. **39** (1980), 199–279.

[8] R. Goodman and N. Wallach, *Classical and quantum mechanical systems of Toda-lattice type.* I, Commun. Math. Phys. **83** (1982), 355–386.

[9] R. Goodman and N. Wallach, *Classical and quantum mechanical systems of Toda-lattice type.* III, Commun. Math. Phys. **105** (1986), 473–509.

[10] M. Hashizume, *Whittaker models for real reductive groups*, Japan. J. Math. (N.S.) **5** (1979), 349–401.

[11] M. Hashizume, *Whittaker functions on semisimple Lie groups*, Hiroshima Math. J. **12** (1982), 259–293.

[12] G. Heckman and H. Schlichtkrull, *Harmonic Analysis and Special Functions on Symmetric Spaces*, Academic Press, 1994.

[13] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, 1978.

[14] S. Helgason, *Groups and Geometric Analysis*, Academic Press, 1984.

[15] M. Hirano, T. Ishii, and T. Oda, *Confluence from Siegel-Whittaker functions to Whittaker functions on $Sp(2,\mathbb{R})$*, Math. Proc. Cambridge Philos. Soc. **141** (2006), 15–31.

[16] V.I. Inozemtsev, *The finite Toda lattices*, Comm. Math. Phys. **121** (1989), 629-638.

[17] H. Jacquet, *Fonctions de Whittaker associées aux groupes de Chevalley*, Bull. Soc. Math. France, **95** (1967), 243–309.

[18] S.P. Khastgir, R. Sasaki, K. Takasaki, *Calogero-Moser Models* IV: *Limits to Toda theory*, Progress of theoretical physics **102** (1999), 749–776.

[19] E. Opdam, *Dunkl operators, Bessel functions and the discriminant of a finite Coxeter group*, Compositio Math. **85** (1993), 333-373.

[20] E.M. Opdam, *Lectures on Dunkl Operators for Real and Complex Reflection Groups*, MSJ Memoirs **8**, Mathematical Society of Japan, 2000.

[21] G. Schiffmann, *Intégrale d'entrelacement et fonctions de Whittaker*, Bull. Soc. Math. France, **99** (1971), 3–72.

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