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ENUMERATIVE COMBINATORICS ON DETERMINANTS AND SIGNED BIGRASSMANNIAN POLYNOMIALS

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Abstract. As an application of linear algebra for enumerative combinatorics, we introduce two new ideas, signed bigrassmannian polynomials and bigrassmannian determinant. First, a signed bigrassmannian polynomial is a variant of the statistic given by the number of bigrassmannian permutations below a permutation in Bruhat order as Reading suggested (2002) and afterward the author developed (2011). Second, bigrassmannian determinant is a q-analog of the determinant with respect to our statistic. It plays a key role for a determinantal expression of those polynomials. We further show that bigrassmannian determinant satisfies weighted condensation as a generalization of Dodgson, Jacobi-Desnanot and Robbins-Rumsey (1986).

1. Introduction

The purpose of this article is to introduce two new ideas, *signed bigrassmannian polynomials* and *bigrassmannian determinant* as an application of linear algebra for enumerative combinatorics. We begin with explaining our motivation.

1.1. **Reading's problem (2002): bigrassmannian statistic.** Permutation statistics has been of great importance in enumerative combinatorics; in particular, Mahonian and Eulerian statistics, such as inversions and descent numbers, are fundamental in the theory. Here what we deal with is a certain new statistic β, which we call *bigrassmannian statistic*. Reading [11] suggested the following problem:

Problem 1.1. *Let $\beta(w)$ be the number of join-irreducible (equivalently, bigrassmannian) permutations weakly below a permutation w in Bruhat order. Find its generating function $\sum_{w\in S_n} q^{\beta(w)}$.*

He gave examples of such generating functions for smaller n's:

$$1+q \qquad (S_2)$$
$$1+2q+2q^3+q^4 \qquad (S_3)$$
$$1+3q+q^2+4q^3+2q^4+2q^5+2q^6+4q^7+q^8+3q^9+q^{10} \qquad (S_4)$$

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Unfortunately, we failed to find any patterns of these coefficients nor factors of such polynomials. Instead, in this article, we study the following *signed* statistic (as signed Mahonian or signed Eulerian statistics):

$$B_n(q) = \sum_{w \in S_n} (-1)^{\ell(w)} q^{\beta(w)}$$

where $\ell(w)$ is the number of inversions; let us call $\{B_n(q) \mid n = 1, 2, 3, \dots\}$ *signed bigrassmannian polynomials*. Fortunately, we could find satisfactory descriptions of such polynomials. It turned out that it is also worthwhile to study these polynomials with a connection to *tournaments* and *Vandermonde determinant*. Since each $B_n(q)$ is a signed sum over the symmetric group, it is natural to come to this idea:

Main idea. *Use the determinant to find* $B_n(q)$.

The determinant is usually a function which outputs a scalar. For our purpose to find $B_n(q)$, we introduce its q-analog (Section 4); we call it *bigrassmannian determinant*.
As main results, we will prove three theorems:

- Theorem 3.11: a factorization of $B_n(q)$.
- Theorem 4.4: a determinantal expression of $B_n(q)$.
- Theorem 4.6: weighted condensation for bigrassmannian determinant.

In addition, we observe a corollary after each of these theorems.

1.2. **Overview.** In Section 2, we review some classic results on tournaments and Vandermonde determinant as mentioned above. These facts will play a fundamental role in the sequel. In Section 3, we introduce β-statistic for tournaments as well as permutations. Then, we find factors of $B_n(q)$ using weighted Vandermonde determinant. Section 4 continues to study $B_n(q)$ (from a little different aspect); we give a definition of bigrassmannian determinant for square matrices as a q-analog of the original one. This new idea leads to a determinantal expression of $B_n(q)$ as we shall see. Further, we prove that bigrassmannian determinant satisfies weighted condensation. It slightly generalizes the construction of Robbins-Rumsey's λ-determinants [13]. We end with some comments for future work in Section 5.

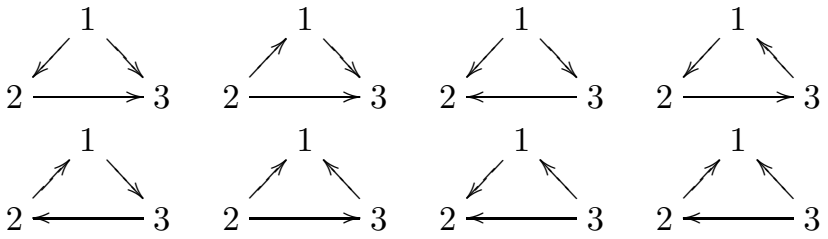
2. Tournaments and Vandermonde determinant

We begin with combinatorics of tournaments and Vandermonde determinant.

2.1. **Tournaments.**

Definition 2.1. A *tournament* is a complete digraph with vertices labeled by $1, 2, \ldots, n$. We denote by T_n the set of all tournaments.

Example 2.2. Here are eight elements in T_3:



Since there are two choices of direction for each pair (i, j) such that $1 \leq i < j \leq n$, we have $|T_n| = 2^{n(n-1)/2}$ in total.

In what follows, the letter G means an element of T_n unless otherwise specified (G is for Graph).

Definition 2.3. An *inversion* of G is a directed edge $j \to i$ with $j > i$. The *length* $\ell(G)$ is the number of inversions of G. An *upset* of an inversion $j \to i$ is the vertex j. Define $\omega_G(j)$ to be the outdegree of j.

2.2. **Cycle and transitivity.** Below, we just say a "cycle" to mean a 3-cycle (which is the only kind of cycles we treat).

Definition 2.4. Let (i, j, k) be a triple such that $i < j < k$. Suppose i, j, k form a cycle in G. Say the cycle is *positive* if $k \to j \to i \to k$; it is *negative* if $i \to j \to k \to i$. Besides, say G is *transitive* if it does not contain any cycles.

Observe that precisely six tournaments in Example 2.2 are transitive.

2.3. **Permutations.** By S_n we mean the symmetric group on $[n] = \{1, 2, \ldots, n\}$. The set of *inversions* of $w \in S_n$ is

$$N(w) = \{(i, j) \in [n] \times [n] \mid i < j \text{ and } w^{-1}(i) > w^{-1}(j)\}.$$

Define the *length* $\ell(w)$ to be $|N(w)|$. Let $G(w)$ be the tournament such that $j \to i$ is an inversion of $G(w) \iff (i, j) \in N(w)$. Say the tournament $G(w)$ is *induced* from a permutation $w \in S_n$. Let us make sure the following:

Fact 2.5. Bressoud [2, Exercise 2.4.2] There is a bijection between S_n and transitive tournaments in T_n.

Thanks to this result, we naturally view $S_n \subseteq T_n$ in what follows. In particular, $\ell(G(w)) = \ell(w)$.

2.4. **Vandermonde determinant.**

Definition 2.6. Let $x_1, \ldots, x_n$ and λ be commutative variables. The *n-th Vandermond λ-determinant* is

$$V_n(\boldsymbol{x}, \lambda) = \prod_{1 \le i < j \le n} (x_i + \lambda x_j).$$

This is a polynomial in x_i's (the bold letter $\boldsymbol{x}$ means such variables for short) and λ. We must explain why we used the word "determinant": Following Robbins-Rumsey [13], we recursively define a determinant-like function $|\ \ |_\lambda$ for square matrices as follows. First, we formally define $|\ \ |_\lambda$ for the 0 by 0 matrix to be 1 and for a 1 by 1 matrix (a_{11}) to be a_{11} itself. Now let A be an n by n matrix for $n \ge 2$. Let A_i^j denote the matrix that remains when we delete the i-th row and j-th column of A. If we wish to delete more than one row (column), the numbers of the deleted rows (columns) are listed as subscripts (superscripts). The *λ-determinant* of A is

$$|A|_\lambda = \frac{|A_1^1|_\lambda |A_n^n|_\lambda + \lambda |A_n^1|_\lambda |A_1^n|_\lambda}{|A_{1n}^{1n}|_\lambda} \quad \text{(a rational function of } \lambda)$$

provided $|\ \ |_\lambda$ of all minors of A are nonzero. In particular, $\lambda = -1$ recovers the original determinant (going back to Dodgson and Desnanot-Jacobi). From this point of view, we can understand $V_n(\boldsymbol{x}, \lambda)$ as the λ-determinant of the Vandermonde matrix: $V_n(\boldsymbol{x}, \lambda) = |x_j^{n-i}|_\lambda$.

Definition 2.7. The *Vandermonde monomial* for G is $\rho(G) = \lambda^{\ell(G)} \prod_{j \in [n]} x_j^{\omega(j)}$.

Proposition 2.8. *We have*

$$V_n(\boldsymbol{x}, \lambda) = \sum_{G \in T_n} \rho(G).$$

Proof. To a tournament G, assign a monomial with the choices of x_i or λx_j from each factor of $\prod_{i<j}(x_i + \lambda x_j)$. Then λ in the monomial counts inversions and x_j records the outdegree. □

Now, split the sum into two parts, transitive or not:

$$\sum_{G \in T_n} \rho(G) = \sum_{G \in S_n} \rho(G) + \sum_{G \in T_n \setminus S_n} \rho(G).$$

Fact 2.9.

$$\sum_{G \in T_n \setminus S_n} \rho(G) \Bigg|_{\lambda = -1} = 0.$$

Proof. See Bressoud [2, Exercise 2.4.4]. □

3. Signed bigrassmannian statistic

3.1. **Bigrassmannian statistic.**

Definition 3.1. Define the *bigrassmannian statistic* for a tournament G as $\beta(G) = \sum_{\substack{j \to i \\ j>i}} (j-i)$.

Table 1 shows this (and inversion) statistic over S_4.

Remark. This statistic is named after the bigrassmannian permutations; say $w \in S_n$ is *bigrassmannian* if there exists a unique pair $(i,j) \in [n-1] \times [n-1]$ such that $w^{-1}(i) > w^{-1}(i+1)$ and $w(j) > w(j+1)$. We refer to Lascoux-Schützenberger [10], Geck-Kim [7], Reading [11] and the author [9] for combinatorics of these permutations.

Define *Bruhat order* $\leq$ on S_n as the transitive closure of the following binary relation: $v \to w$ meaning $w = vt_{ij}$, for some $i < j$, t_{ij} a transposition and $\ell(v) < \ell(w)$. Let $B(w) = \{u \text{ bigrassmannian} \mid u \leq w\}$ and set $\beta(w) = |B(w)|$. The author [9] showed that

$$\beta(w) = \sum_{(i,j) \in N(w)} (j-i).$$

Thus, we can compute β simply as weighted enumeration of inversions:

$$\beta(3412) = (3-1) + (3-2) + (4-1) + (4-2) = 8,$$

for example (Figure 1). From this point of view, our definition above is a natural extension of β for tournaments. This statistic implicitly appeared also in the Gessel-Viennot's lattice path counting context [2, Theorem 3.7] as the quantity $\sum_{i=1}^{n} i(i-w(i))$:

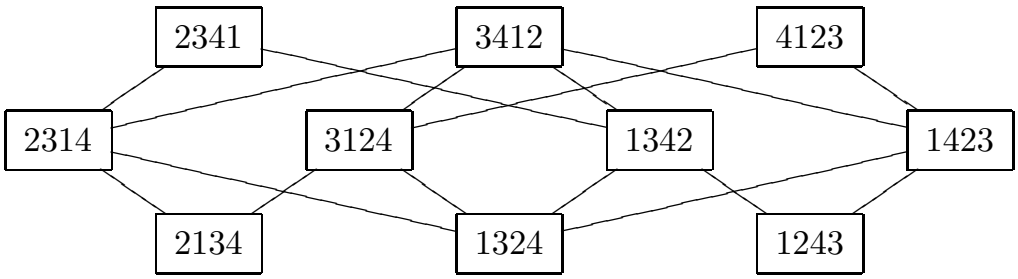
Proposition 3.2. *For each $w \in S_n$, we have*

(1) $\beta(w) = \sum_{i=1}^{n} (i-w(i))^2/2 = \sum_{i=1}^{n} i(i-w(i))$.
(2) $\beta(w) = \beta(w^{-1})$.

Table 1. statistics of ℓ and β over S_4

	ℓ	β		ℓ	β		ℓ	β		ℓ	β
1234	0	0	2134	1	1	3124	2	3	4123	3	6
1243	1	1	2143	2	2	3142	3	5	4132	4	7
1324	1	1	2314	2	3	3214	3	4	4213	4	7
1342	2	3	2341	3	6	3241	4	7	4231	5	9
1423	2	3	2413	3	5	3412	4	8	4312	5	9
1432	3	4	2431	4	7	3421	5	9	4321	6	10

FIGURE 1. Bruhat order of bigrassmannian permutation in S_4



Proof. (1) See [9] for the first equality. It follows that

$$\beta(w) = \frac{1}{2}\sum_{i=1}^{n}(i - w(i))^2 = \frac{1}{2}\sum_{i=1}^{n}(i^2 - 2iw(i) + w(i)^2) = \sum_{i=1}^{n} i(i - w(i)).$$

Next, (2) follows from the facts that (a) $u \mapsto u^{-1}$ is an order-preserving automorphism in Bruhat order on S_n, (b) u is bigrassmannian $\iff$ so is u^{-1}; we do not go into details here because the proof is not so important for our discussions below. □

Definition 3.3. Let $x_1, \ldots, x_n$, λ and q be commutative variables. The *weighted Vandermonde monomial* for G is $\chi(G) = \lambda^{\ell(G)} q^{\beta(w)} \prod_{j\in[n]} x_j^{\omega(j)}$.

Definition 3.4. The n-th *weighted Vandermonde determinant* is

$$V_n(\boldsymbol{x}, \lambda, q) = \prod_{1\le i<j\le n} (x_i + \lambda q^{j-i} x_j).$$

Example 3.5.

$$\begin{aligned} V_3(x, \lambda, q) &= (x_1 + \lambda q x_2)(x_1 + \lambda q^2 x_3)(x_2 + \lambda q x_3) \\ &= x_1^2 x_2 + \lambda q x_1 x_2^2 + \lambda q x_1^2 x_3 + (\lambda^2 + \lambda) q^2 x_1 x_2 x_3 \\ &\quad + \lambda^2 q^3 x_2^2 x_3 + \lambda^2 q^3 x_1 x_3^2 + \lambda^3 q^4 x_2 x_3^2. \end{aligned}$$

Proposition 3.6. *We have*

$$V_n(\boldsymbol{x}, \lambda, q) = \sum_{G\in T_n} \chi(G).$$

Proof. The idea is similar to Proposition 2.8. □

Lemma 3.7.

$$\sum_{G\in T_n\setminus S_n} \chi(G)\Big|_{x_1=\cdots=x_n=1,\lambda=-1} = 0.$$

To prove this lemma, we need a further definition and proposition.

Definition 3.8. For $i < j < k$, define a map $C_{ijk} : T_n \to T_n$ as follows: if i, j, k form a cycle in G, then $C_{ijk}(G)$ is the tournament with all three edges in the cycle reversed and all other edges unchanged. If i, j, k do not form a cycle in G, then simply set $C_{ijk}(G) := G$.

Observe that C_{ijk} is an involution.

Proposition 3.9. *Let $i < j < k$. If i, j, k form a cycle in G, then $\ell(C_{ijk}(G)) \in \{\ell(G) - 1, \ell(G) + 1\}$ and $\beta(C_{ijk}(G)) = \beta(G)$.*

Proof. A positive cycle contains two inversions whereas a negative cycle contains one. The map C_{ijk} interchanges these so that lengths differ by one. However, β is invariant because of the equality $k - i = (k - j) + (j - i)$. □

Proof of Lemma 3.7. Consider the lexicographic order on $\{(i, j, k) \in [n]^3 \mid i < j < k\}$. We will construct a perfect matching on the set $T_n \setminus S_n$. First, choose all tournaments G from T_n such that $(1, 2, 3)$ is a cycle in G. It is either positive or negative; hence $G \leftrightarrow C_{123}(G)$ gives a matching. Next, choose all tournaments H from the remaining tournaments such that $(1, 2, 4)$ is a cycle in H. Again, $H \leftrightarrow C_{124}(H)$ gives a matching. Continue this procedure up to $(n-2, n-1, n)$. We certainly exhausted all tournaments in $T_n \setminus S_n$ with the perfect matching constructed. As shown above, each pair has lengths of opposite parity and the same β. Thus $x_1 = \cdots = x_n = 1$ and $\lambda = -1$ yield zero. □

3.2. **Signed bigrassmannian polynomials.**

Definition 3.10. Let n be a positive integer. The n-th *signed bigrassmannian polynomial* is

$$B_n(q) = \sum_{w \in S_n} (-1)^{\ell(w)} q^{\beta(w)}.$$

Theorem 3.11. *For all $n \geq 1$, we have*

$$B_n(q) = \prod_{k=1}^{n-1} (1 - q^k)^{n-k}.$$

Proof. As before, split $V_n(\boldsymbol{x}, \lambda, q)$ into two parts:

$$\prod_{1 \leq i < j \leq n} (x_i + \lambda q^{j-i} x_j) = V_n(\boldsymbol{x}, \lambda, q) = \sum_{G \in S_n} \chi(G) + \sum_{G \in T_n \setminus S_n} \chi(G).$$

With $x_1 = \cdots = x_n = 1$ and $\lambda = -1$, the second sum vanishes as shown in Lemma 3.7. As a result, we obtain

$$\prod_{1 \le i < j \le n} (1 - q^{j-i}) = \sum_{w \in S_n} (-1)^{\ell(w)} q^{\beta(w)}$$

or

$$B_n(q) = \prod_{k=1}^{n-1} (1 - q^k)^{n-k}.$$

□

Corollary 3.12. *For $n \ge 3$, we have*

$$\sum_{w \in S_n} (-1)^{\ell(w)} \beta(w) = 0.$$

In other words, the β-statistic is sign-balanced.

Proof. Note that $B_n(q)$ has a factor $(1-q)^{n-1}$ with $n-1 \ge 2$. Differentiate it once and let $q = 1$. Then we get zero, as required. □

Example 3.13. (cf. Reading's examples in Introduction)

$$\begin{aligned} B_2(q) &= 1 - q, \\ B_3(q) &= (1-q)^2(1-q^2) = 1 - 2q + 2q^3 - q^4, \\ B_4(q) &= (1-q)^3(1-q^2)^2(1-q^3) \\ &= 1 - 3q + q^2 + 4q^3 - 2q^4 - 2q^5 - 2q^6 + 4q^7 + q^8 - 3q^9 + q^{10}. \end{aligned}$$

4. Bigrassmannian determinant

4.1. **Definition.** Next we want to understand $B_n(q)$ as a new sort of a determinant as mentioned in Introduction. From now on, we assume that $A = (a_{ij}) = (a_{ij}(q))$ is an n by n matrix with entries being complex rational functions in $q^{1/2}$ (i.e., elements of $\mathbb{C}(q^{1/2})$). The reason why we introduce $q^{1/2}$ and q^{-1} will be clearer in the next subsection.

Definition 4.1. The *bigrassmannian determinant* of A is

$$\mathrm{bdet}(A) = \sum_{w \in S_n} (-1)^{\ell(w)} q^{\beta(w)} \prod_{i=1}^{n} a_{iw(i)}.$$

We formally define bdet of the 0 by 0 matrix to be 1.

For example, $\mathrm{bdet}(a_{11}) = a_{11}$, $\mathrm{bdet}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - qa_{12}a_{21}$ and

$$\mathrm{bdet}\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}a_{22}a_{33} - qa_{12}a_{21}a_{33} - qa_{11}a_{23}a_{32} + q^3a_{12}a_{23}a_{31} + q^3a_{13}a_{21}a_{32} - q^4a_{13}a_{22}a_{31}.$$

4.2. **Matrix deformation.** We now give a more explicit description of the bigrassmannian determinant in terms of the original one. For this purpose, let us introduce a special term: a *deformation* of $A = (a_{ij})$ is a new matrix $\boldsymbol{f}A := (f_{ij}(q)a_{ij})$ for some indexed family of rational functions $\boldsymbol{f} = \{f_{ij}(q) \in \mathbb{C}(q^{1/2}) \mid (i,j) \in [n] \times [n]\}$. Note that the operation $a_{ij} \mapsto f_{ij}(q)a_{ij}$ may *not* be $\mathbb{C}(q^{1/2})$-linear in any rows nor columns. Hence it is in general difficult to predict how determinants change under such an operation. However, as seen below, there are some nice cases:

Definition 4.2. Let $\boldsymbol{b} = \{b_{ij}(q)\} = \{q^{(i-j)^2/2}\}$. The *bigrassmannian deformation* of A is $\boldsymbol{b}A$.

Proposition 4.3. $\det(\boldsymbol{b}A) = \mathrm{bdet}(A)$.

Proof. By Proposition 3.2, we have

$$\begin{aligned} \det(\boldsymbol{b}A) &= \sum_{w \in S_n} (-1)^{\ell(w)} \prod_{i=1}^{n} q^{(i-w(i))^2/2} a_{iw(i)} \\ &= \sum_{w \in S_n} (-1)^{\ell(w)} q^{\beta(w)} \prod_{i=1}^{n} a_{iw(i)} \\ &= \mathrm{bdet}(A). \end{aligned}$$

□

Theorem 4.4 (a determinantal expression of $B_n(q)$)**.** *We have*

$$B_n(q) = \det(q^{(i-j)^2/2}).$$

Proof. $B_n(q) = \mathrm{bdet}(1)_{i,j=1}^n = \det(\boldsymbol{b}1) = \det(q^{(i-j)^2/2})$. □

Observe determinantal expressions of $B_3(q)$ and $B_4(q)$:

$$\det\begin{pmatrix} 1 & q^{1/2} & q^{4/2} \\ q^{1/2} & 1 & q^{1/2} \\ q^{4/2} & q^{1/2} & 1 \end{pmatrix} = (1-q)^2(1-q) \text{ and}$$

$$\det\begin{pmatrix} 1 & q^{1/2} & q^{4/2} & q^{9/2} \\ q^{1/2} & 1 & q^{1/2} & q^{4/2} \\ q^{4/2} & q^{1/2} & 1 & q^{1/2} \\ q^{9/2} & q^{4/2} & q^{1/2} & 1 \end{pmatrix} = (1-q)^3(1-q^2)^2(1-q^3).$$

We should now recognize that different deformations may give the same determinant: given a family $\boldsymbol{f}$, there possibly exists $\boldsymbol{g}$ such that $\boldsymbol{g} \neq \boldsymbol{f}$ and $\det(\boldsymbol{f}A) = \det(\boldsymbol{g}A)$ for *all* matrices A. In particular, this is the case for $\boldsymbol{b}$: Let $\boldsymbol{b}' = \{b'_{ij}\} = \{q^{i(i-j)}\}$ and $\boldsymbol{b}'' = \{b''_{ij}\} = \{q^{j(j-i)}\}$ (here we need q^{-1}). Then $\det(\boldsymbol{b}A) = \det(\boldsymbol{b}'A) = \det(\boldsymbol{b}''A)$ as shown just below; since we could not find any references mentioning this little invariance, we here record it as a Corollary.

Corollary 4.5. *(little invariance of the determinant)*

$$\det(q^{(i-j)^2/2}a_{ij}) = \det(q^{i(i-j)}a_{ij}) = \det(q^{j(j-i)}a_{ij}).$$

Proof. We only prove the first equality.

$$\begin{aligned}\det(q^{i(i-j)}a_{ij}) &= \sum_{w\in S_n}(-1)^{\ell(w)}\prod_{i=1}^{n} q^{i(i-w(i))}a_{i,w(i)} \\ &= \sum_{w\in S_n}(-1)^{\ell(w)}q^{\beta(w)}\prod_{i=1}^{n} a_{i,w(i)} = \det(q^{(i-j)^2/2}a_{ij}).\end{aligned}$$

□

Such "equivalent" deformations may be useful for evaluating and understanding combinatorial determinants (interpret $q^{(i-j)^2/2}$ as area of the triangle (i,i), (i,j) and (j,j) in $\mathbb{Z}^2$); see Bressoud [2, Section 3.3], Gessel-Viennot [8] and Stembridge [14], for details on Schur functions and nonintersecting lattice path counting by determinants. We will develop this idea in subsequent publications.

4.3. **Weighted condensation.** Our next task is to prove weighted condensation for bigrassmannian determinants; this is a natural idea as an analogy of the original determinant (and Robbins-Rumsey [13]). Let A be an n by n matrix with $n \geq 2$. Recall that A_i^j denotes the submatrix with the i-th row and j-th column deleted.

Theorem 4.6.

$$\mathrm{bdet}(A)\mathrm{bdet}(A_{1n}^{1n}) = \mathrm{bdet}(A_1^1)\mathrm{bdet}(A_n^n) - q^{n-1}\mathrm{bdet}(A_n^1)\mathrm{bdet}(A_1^n).$$

Some comments before the proof: Let $A = (a_{ij}), C = \boldsymbol{b}A = (c_{ij})$ and $c_{ij} = q^{(i-j)^2/2}a_{ij}$. For simplicity, we use $|\quad|$ for the original determinant.

We will confirm the following five statements.

(1) $|C_{1n}^{1n}| = \mathrm{bdet}(A_{1n}^{1n})$.
(2) $|C_1^1| = \mathrm{bdet}(A_1^1)$.
(3) $|C_n^n| = \mathrm{bdet}(A_n^n)$.
(4) $|C_n^1| = q^{(n-1)/2}\mathrm{bdet}(A_n^1)$.
(5) $|C_1^n| = q^{(n-1)/2}\mathrm{bdet}(A_1^n)$.

Once we do this, then the conclusion follows from condensation for the original determinant:

$$|C||C_{1n}^{1n}| = |C_1^1||C_n^n| - |C_n^1||C_1^n|.$$

Proof. (1) $|C_{1n}^{1n}| = \mathrm{bdet}(A_{1n}^{1n})$: an (i,j)-entry of C_{1n}^{1n} is $c_{i+1,j+1}$.

$$\begin{aligned}|C_{1n}^{1n}| &= |c_{i+1,j+1}|_{i,j=1}^{n-2} = |q^{((i+1)-(j+1))^2/2}a_{i+1,j+1}| \\ &= |q^{(i-j)^2/2}a_{i+1,j+1}| = \mathrm{bdet}(A_{1n}^{1n}).\end{aligned}$$

(2) $|C_1^1| = \mathrm{bdet}(A_1^1)$: an (i,j)-entry of C_1^1 is $c_{i+1,j+1}$.

$$\begin{aligned}|C_1^1| &= |c_{i+1,j+1}|_{i,j=1}^{n-1} = |q^{((i+1)-(j+1))^2/2}a_{i+1,j+1}| \\ &= |q^{(i-j)^2/2}a_{i+1,j+1}| = \mathrm{bdet}(A_1^1).\end{aligned}$$

(3) $|C_n^n| = \mathrm{bdet}(A_n^n)$:this is similar to (2).
(4) $|C_n^1| = q^{(n-1)/2}\mathrm{bdet}(A_n^1)$: an (i,j)-entry of C_n^1 is $c_{i,j+1}$.

$$\begin{aligned}|C_n^1| &= |c_{i,j+1}|_{i,j=1}^{n-1} = \det(q^{(i-(j+1))^2/2}a_{i,j+1}) \\ &= |q^{((i-j)^2-2(i-j)+1)/2}a_{i,j+1}| = q^{-\sum i+\sum j}q^{(n-1)/2}|q^{(i-j)^2/2}a_{i,j+1}| \\ &= q^{(n-1)/2}\mathrm{bdet}(A_n^1).\end{aligned}$$

(5) $|C_1^n| = q^{(n-1)/2}\mathrm{bdet}(A_1^n)$: this is similar to (4).

□

Now we see an immediate consequence which is, however, not so obvious from the definition of $B_n(q)$.

Corollary 4.7. *Signed bigrassmannian polynomials can be defined recursively as follows:* $B_1(q) = 1, B_2(q) = 1-q$ *and* $B_n(q) = \dfrac{B_{n-1}(q)^2}{B_{n-2}(q)}(1-q^{n-1})$ *for* $n \geq 3$.

Proof. Apply the weighted condensation to $A = (1)_{i,j=1}^n$. All four determinants in the numerator are $B_{n-1}(q)$ while the denominator is $B_{n-2}(q)$. □

5. Concluding remarks

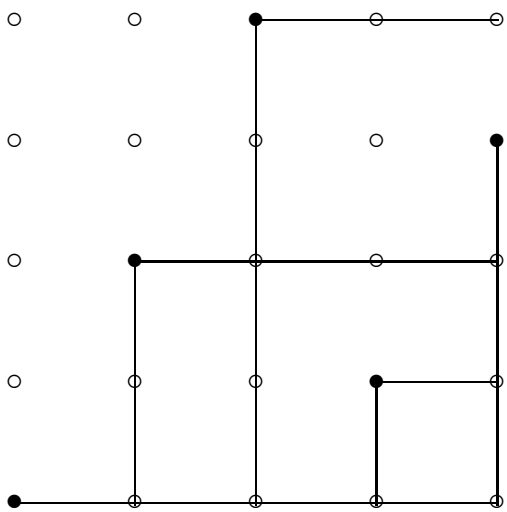
In this article, we introduced two new ideas, signed bigrassmannian polynomials and bigrassmannian determinant. We made use of tournaments as well as Vandermonde determinant to find $B_n(q)$. Then we introduced bdet as a q-analog of determinant as $q \to 1$ recovers the original one. Thanks to formulas of β-statistic, we obtained a determinantal expression of $B_n(q)$. Moreover, we established weighted condensation as an analogy of Robbins-Rumsey. After all, we did not find the unsigned statistic $\sum_{w \in S_n} q^{\beta(w)}$. Now an easy guess is to use the permanent instead. We leave this problem here for our future research.
We end with some more comments for subsequent work.

- What is missing in our discussion is an alternating sign matrix (ASM) [3, 12]. Since inversions and bigrassmannian statistics also make sense for ASMs, we want to generalize some of our results to these matrices (note: we can extend β for ASMs as the rank function of a distributive lattice. For example, what can we say about bdet for ASMs which are not permutations?
- We can also define "λq-determinant" by replacing λ with λq^{n-1} in Robbins-Rumsey condensation (provided all such minors are nonzero). Then we would obtain polynomials of the form $\prod(1 + \lambda q^k)^{n-k}$, say $B_n(\lambda, q)$. Then we can show as Corollary 4.7 that polynomials $\{B_n(\lambda, q)\}$ satisfies

$$B_n(\lambda, q) = \frac{B_{n-1}(\lambda, q)^2}{B_{n-2}(\lambda, q)}(1 + \lambda q^{n-1}).$$

  Recently, there appeared such recursions and polynomials in the literature on Aztec diamonds, perfect matchings and domino tilings; see Brualdi-Kirkland [4], Ciucu [5] and Elkies-Kuperberg-Larsen-Propp [6], for example. It would be nice to give an explicit connection between such work and our results.
- As we mentioned Bruhat order, symmetric groups are Coxeter groups of type A. It makes sense to speak of a signed bigrassmannian statistic even in other situations: let $(W, S, \leq)$ be a finite Coxeter system with Coxeter generators S specified and $\leq$ Bruhat order. Define $\ell(w) = \min\{l \geq 0 \mid w = s_1 \cdots s_l, s_i \in S\}$ and the sign of w to be $(-1)^{\ell(w)}$. Say w is bigrassmannian if there exists a unique pair $(s_1, s_2) \in S \times S$ such that $\ell(s_1 w) < \ell(w)$ and $\ell(w s_2) < \ell(w)$. Define $B(w) = \{u \text{ bigrassmannian} \mid u \leq w\}$ and $\beta(w) = |B(w)|$ in the same way. Find a statistic $\sum_{w \in W} q^{\beta(w)}$.

FIGURE 2. Rothe diagram for 35241



- We can think that each permutation w gives a partition of an integer $\beta(w)$ with $\ell(w)$ parts as $\beta(w) = \sum_{(i,j)\in N(w)}(j-i)$; see Andrews-Eriksson [1] for the theory of integer partitions. Then, it is natural to come to the following idea: *Rothe diagram* for w is the set $\{(i,j) \in [n] \times [n] \mid i < w^{-1}(j) \text{ and } j < w(i)\}$. As is well-known, the cardinality of this set is $\ell(w)$. Figure 2 shows an example; seven circles which does not cross any lines are elements of Rothe diagram for $w = 35241$ (with $\beta(w) = 15$). Is there any formula to compute β from Rothe diagrams?

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