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# THE CANONICAL LINE BUNDLES OVER EQUIVARIANT REAL PROJECTIVE SPACES

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Abstract. A generator of the reduced $KO$-group of the real projective space of dimension $n$ is related to the canonical line bundle $\gamma$. In the present paper, we will prove that for a finite group $G$ of odd order and a real $G$-representation $U$ of dimension $2n$, in the reduced $G$-equivariant $KO$-group of the real projective space associated with the $G$-representation $\mathbb{R} \oplus U$, the element $2^{n+2}[\gamma]$ is equal to zero.

## 1. Introduction

Let $\mathbb{R}$ (resp. $\mathbb{C}$) be the real (resp. complex) number field. We denote by $\mathbb{R}^n$ (resp. $\mathbb{C}^n$) the $n$-fold cartesian product of $\mathbb{R}$ (resp. $\mathbb{C}$) which will be regarded as a real (resp. complex) vector space. Let $\gamma_n$ be the canonical line bundle of the $n$-dimension real projective space $\mathbb{R}P^n$. By definition, the total space $E(\gamma_n)$ of $\gamma_n$ is

$$\{(\{\pm x\}, v) \in \mathbb{R}P^n \times \mathbb{R}^{n+1} \mid x \in S^n, v \in \mathbb{R} \cdot x\},$$

where $S^n$ is the unit sphere of $\mathbb{R}^{n+1}$. We can regard $E(\gamma_0) \subset E(\gamma_1) \subset E(\gamma_2) \subset \cdots$, in a canonical way. Since the space $E(\gamma_1)$ is homeomorphic to the open Möbius strip and is not homeomorphic to the space $\mathbb{R}P^1 \times \mathbb{R}$, $\gamma_n$ is not a trivial (i.e. product) bundle for each $n \geq 1$. Moreover, by [3, Proposition 3.12], the reduced $K$-group $\widetilde{KO}(\mathbb{R}P^n)$ is isomorphic to the cyclic group $\mathbb{Z}_{2^h}$ of order $2^h$, where

$$h = \sharp\{s \in \mathbb{Z} \mid 0 < s \leq n,\ s \equiv 0,\ 1,\ 2,\ 4 \mod 8\}.$$

The group $\widetilde{KO}(\mathbb{R}P^n)$ has the generator $\xi$ such that $\xi = [\gamma_n] - [\varepsilon^1]$, where $\varepsilon^1$ is the trivial line bundle over $\mathbb{R}P^n$. This generator $\xi$ satisfies the relations $\xi^2 = -2\xi$ and $\xi^{h+1} = 0$.

Let $G$ be a finite group. We would like to consider real projective spaces equipped with smooth action of $G$. Let $U$ be a real $G$-module of finite dimension. We denote by $S(U)$ the unit sphere of $U$ with respect to some $G$-invariant inner product. Then the *real projective space* $P(U)$ associated with $U$ is defined to be the quotient space $S(U)/\{\pm 1\}$. The canonical line bundle $\gamma_M$ of $M = P(U)$ is defined so that the fiber $F_{\{\pm x\}}$ over $\{\pm x\} \in P(U)$

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is the 1-dimensional real vector space spanned by $x$, where $x$ is an element of $S(U)$.

For a $G$-space $N$ and a real (resp. complex) $G$-module $V$, let $\varepsilon_N(V)$ stand for the product bundle over $N$ with fiber $V$. Unless otherwise stated, the real (resp. complex) vector space $\mathbb{R}^n$ (resp. $\mathbb{C}^n$) has the trivial $G$-action. We have seen the next theorem.

**Theorem** ([6]). *Let $G$ be a cyclic group and $V$ a nontrivial real $G$-module of even dimension such that $G$ acts freely on $V$ except the origin. Let $\gamma_M$ be the canonical line bundle of $M = P(\mathbb{R} \oplus V)$. Then the following* (1) *and* (2) *hold.*

(1) *If the order of $G$ is even then for any natural number $m$ and any real $G$-modules $U$, $W$, $\gamma_M^{\oplus m} \oplus \varepsilon_M(U)$ is not isomorphic to $\varepsilon_M(W)$ as real $G$-vector bundles. Hence*

$$m[\gamma] \neq 0 \quad \text{in } \widetilde{KO}_G(M).$$

(2) *If the order of $G$ is odd and* $\dim V = 2$ *then $\gamma_M^{\oplus 4}$ is isomorphic to $\varepsilon_M(\mathbb{R}^4)$ as real $G$-vector bundles. Thus*

$$4[\gamma] = 0 \quad \text{in } \widetilde{KO}_G(M).$$

In this paper, we obtain the next generalization of (2) above. For a complex $G$-module $V$, let $V_{\mathbb{R}}$ denote the realification of $V$.

**Theorem 1.** *Let $G$ be a finite group of odd order and let $V$ be a direct sum of 1-dimensional complex $G$-modules $V(i)$, $i = 1, \ldots, n$. Then for $M = P(\mathbb{R} \oplus V_{\mathbb{R}})$, the complexification ${\gamma_M}_{\mathbb{C}}^{\oplus 2^{n+1}}$ ($= {\gamma_M}^{\oplus 2^{n+1}} \otimes_{\mathbb{R}} \mathbb{C}$) of ${\gamma_M}^{\oplus 2^{n+1}}$ is isomorphic to $\varepsilon_M(\mathbb{C}^{2^{n+1}})$ as complex $G$-vector bundles, and hence ${\gamma_M}^{\oplus 2^{n+2}}$ is isomorphic to $\varepsilon_M(\mathbb{R}^{2^{n+2}})$ as real $G$-vector bundles.*

We have the sequence

$$M_1 \subset M_2 \subset \cdots \subset M_k \subset \cdots \subset M_n = M$$

consisting of $G$-submanifolds of $M$ with

$$M_k = P(\mathbb{R} \oplus (V(1) \oplus \cdots \oplus V(k))_{\mathbb{R}}).$$

We will prove the theorem above by induction on $k$; in other words, we will show that $(\gamma_{M_k}^{\oplus 2^{k+1}})_{\mathbb{C}}$ $(= ((\gamma_M^{\oplus 2^{k+1}})|_{M_k})_{\mathbb{C}})$ is isomorphic to $\varepsilon_M(\mathbb{C}^{2^{k+1}})$ as complex $G$-vector bundles. Note that $M_1$ is obtained by gluing a disk with the Möbius strip along the boundary. Similarly $M_{k+1}$ is obtained by gluing a $G$-submanifold $Y_{k+1}(\supset M_k)$ with another $Z_{k+1}(\supset P(V(k+1)_{\mathbb{R}}))$ along the boundary such that $M_k$ and $P(V(k+1)_{\mathbb{R}})$ are strong $G$-deformation retracts of $Y_{k+1}$ and $Z_{k+1}$, respectively. In the inductive step to show the triviality of

$(\gamma_{M_{k+1}}^{\oplus 2^{k+2}})_{\mathbb{C}}$ from the triviality of $(\gamma_{M_k}^{\oplus 2^{k+1}})_{\mathbb{C}}$, we will employ Eilenberg's extension theorem. Recall that the step-wise obstructions in the extension theorem lie in the cohomology groups $H^m(Z_{k+1}/G, \partial Z_{k+1}/G; \pi_{m-1}(\mathbf{U}(2^{k+2})))$, where $1 \le m \le 2k+2$ and $\mathbf{U}(\ell)$ is the unitary group of degree $\ell$. What we need to handle the step-wise obstructions in this paper is the next result.

**Theorem 2.** *Let $G$ and $V(i)$ be as in Theorem 1 and $k$ an integer with $0 \le k \le n-1$. Set $U = V(1) \oplus \cdots \oplus V(k)$, $W = V(k+1)$, $N = S(W) \times D(\mathbb{R} \oplus U)$, and $\widehat{G} = G \times \{1, -1\}$. Then the cohomology group $H^*(N/\widehat{G}, \partial N/\widehat{G}; \mathbb{Z})$ is as follows.*

$$H^m(N/\widehat{G}, \partial N/\widehat{G}; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}_2 & (m = 2k+2) \\ O & (m \neq 2k+2). \end{cases}$$

In Section 2, we prove Theorem 1 using Theorem 2. By the universal coefficient theorem, Theorem 2 immediately follows from Theorem 3, a result on the homology group $H_*(N/\widehat{G}, \partial N/\widehat{G}; \mathbb{Z})$, which is given in Section 3. In the computation of this homology group, Serre's spectral sequence is helpful. In Section 3, we recall results on Serre's spectral sequence which we need in this paper. Sections 4–5 are devoted to the computation of relevant homology groups and the proof of Theorem 3.

## 2. Proof of Theorem 1

Let $G$ and $V(i)$ be as in Theorem 1, and let $k$ be an integer with $0 \le k \le n-1$. We set

$$U = \bigoplus_{i=1}^{k} V(i) \quad \text{and} \quad W = V(k+1).$$

Let $I$ denote the closed interval $[-1, 1]$ with the trivial $G$-action. Let $T$ denote the group $\{-1, 1\}$ of order 2. In this paper, $T$ acts on each $S(W_{\mathbb{R}})$, $I$ and $D(U_{\mathbb{R}})$ by the scalar multiplication. Let $\widehat{G}$ denote $G \times T$ and let $K$ denote the kernel of the $G$-representation $W$, i.e. of the homomorphism $G \to \mathrm{Aut}_{\mathbb{C}}(W)$.

Since $M_0$ is a point, it follows immediately that $(\gamma_{M_0}^{\oplus 2})_{\mathbb{C}} \cong_G \varepsilon_{M_0}(\mathbb{C}^2)$. For fixed $k$, suppose $(\gamma_{M_k}^{\oplus 2^{k+1}})_{\mathbb{C}} \cong_G \varepsilon_{M_k}(\mathbb{C}^{2^{k+1}})$. By the definition in Section 1,

$$M_{k+1} = P(\mathbb{R} \oplus (U \oplus W)_{\mathbb{R}}) = S(\mathbb{R} \oplus U \oplus W)/\{\pm 1\}.$$

Set

$$Y_{k+1} = (S(\mathbb{R} \oplus U) \times D(W))/\{\pm 1\},$$

and

$$Z_{k+1} = (D(\mathbb{R} \oplus U) \times S(W))/\{\pm 1\}.$$

It is easily verified that $M_{k+1} = Y_{k+1} \cup Z_{k+1}$. Furthermore $M_k$ is a strong $G$-deformation retract of $Y_{k+1}$ and

$$(\gamma_{M_{k+1}}^{\oplus 2^{k+1}})_{\mathbb{C}}|_{Y_{k+1}} \cong_G \varepsilon_{Y_{k+1}}(\mathbb{C}^{2^{k+1}}).$$

Evidently $P(W_{\mathbb{R}}) = S(W)/\{\pm 1\}$ is a strong $G$-deformation retract of $Z_{k+1}$. Since $W$ is a 1-dimensional complex $G$-representation space and $G/K$ is a cyclic group of odd order, the realification $W_{\mathbb{R}}$ of $W$ is a 2-dimensional real $G/K$-module and $G/K$ acts on $W_{\mathbb{R}}$ freely except the origin. Applying the proof of Theorem 2 in [6], we obtain $(\gamma_{M_{k+1}})|_{Z_{k+1}}^{\oplus 2} \cong_{G/K} \varepsilon_{Z_{k+1}}(\mathbb{R}^2)$. Therefore we get

$$(\gamma_{M_{k+1}})|_{Z_{k+1}}^{\oplus 2^{k+1}} \cong_{G/K} \varepsilon_{Z_{k+1}}(\mathbb{R}^{2^{k+1}}),$$

and hence

$$(\gamma_{M_{k+1}}^{\oplus 2^{k+1}})_{\mathbb{C}}|_{Z_{k+1}} \cong_G \varepsilon_{Z_{k+1}}(\mathbb{C}^{2^{k+1}}).$$

Hereafter we identify $(\gamma_{M_{k+1}\mathbb{C}}|_{Y_{k+1}})^{\oplus 2^{k+1}}$ with $\varepsilon_{Y_{k+1}}(\mathbb{C}^{2^{k+1}})$ and $(\gamma_{M_{k+1}\mathbb{C}}|_{Z_{k+1}})^{\oplus 2^{k+1}}$ with $\varepsilon_{Z_{k+1}}(\mathbb{C}^{2^{k+1}})$, respectively. With these identifications, we take the canonical unitary framings $(e_1, \ldots, e_{2^{k+1}})$ and $(f_1, \ldots, f_{2^{k+1}})$ over $(\gamma_{M_{k+1}\mathbb{C}}|_{Y_{k+1}})^{\oplus 2^{k+1}}$ and $(\gamma_{M_{k+1}\mathbb{C}}|_{Z_{k+1}})^{\oplus 2^{k+1}}$, respectively. Let $A = [a_{ij}] : Y_{k+1} \cap Z_{k+1} \to \mathbf{U}(2^{k+1})$ be the matrix function defined by

$$f_i(x) = \sum_{j=1}^{2^{k+1}} a_{ji}(x) e_j(x) \quad (x \in Y_{k+1} \cap Z_{k+1},\ i = 1, \ldots, 2^{k+1}).$$

We regard the unitary group $\mathbf{U}(2^{k+1})$ as a space with the trivial $G$-action. The next (1) and (2) can be verified without difficulties

(1) $Y_{k+1} \cap Z_{k+1}$ is equal to the boundary of $Z_{k+1}$.
(2) The map $A$ is $G$-invariant.

We remark that the $G$-maps from $Z_{k+1}$ to $\mathbf{U}(2^{k+1})$ correspond in a one-to-one way to the maps from $Z_{k+1}/G$ to $\mathbf{U}(2^{k+1})$. Thus if the map $A' : \partial Z_{k+1}/G \to \mathbf{U}(2^{k+1})$ defined by $A'([x]) = A(x)$ $(x \in \partial Z_{k+1})$ extends to $Z_{k+1}/G$, then $(\gamma_{M_{k+1}}^{\oplus 2^{k+1}})_{\mathbb{C}}$ is $G$-isomorphic to $\varepsilon_{M_{k+1}}(\mathbb{C}^{2^{k+1}})$. By Eilenberg's theorem (see [1, Chapter 4]), the step-wise obstruction classes

$$\sigma_{m+1}(A') \in H^{m+1}(Z_{k+1}/G, \partial Z_{k+1}/G; \pi_m(\mathbf{U}(2^{k+1})))$$

of the map $A'$, where $0 \leq m \leq \dim M_{k+1} - 1$, are well-defined and equal to zero, then the map $A'$ has an extension to $Z_{k+1}/G$. So we look into the group

$$H^{m+1}(Z_{k+1}/G, \partial Z_{k+1}/G; \pi_m(\mathbf{U}(2^{k+1}))).$$

As for the coefficient group of the cohomology group, the result

$$\pi_i(\mathbf{U}(2^{k+1})) = \pi_i(\mathbf{U}) \cong \begin{cases} 0 & (i\text{: even}) \\ \mathbb{Z} & (i\text{: odd}) \end{cases}$$

is known for integers $i$ with $0 \leq i < 2^{k+2}$ $(= 2(2^{k+1}+1)-2)$(see [5, p. 216]). Our integer $m$ satisfies $m+1 \leq \dim M_{k+1} = 2(k+1)$, and hence $m \leq 2k+1 < 2^{k+2}$. Since $Z_{k+1} = (D(\mathbb{R} \oplus U) \times S(W))/\{\pm 1\}$ by definition, $Z_{k+1}/G$ is homeomorphic to $(S(W) \times I \times D(U))/\widehat{G}$, where $\widehat{G} = G \times T$. By Theorem 2, we get

$$H^{m+1}(Z_{k+1}/G, \partial Z_{k+1}/G; \pi_m(\mathbf{U}(2^{k+1}))) = \begin{cases} \mathbb{Z}_2 & (m+1 = 2k+2) \\ O & (m+1 \neq 2k+2). \end{cases}$$

Thus $\sigma_{m+1}(A')$ are well-defined and trivial for all $m$ with $0 \leq m \leq \dim M_{k+1} - 2$, and hence $\sigma_{m+1}(A')$ is well-defined for $m = \dim M_{k+1} - 1$. Now define the map

$$A'^2 : \partial Z_{k+1}/G \to U(2^{(k+1)+1})$$

by

$$A'^2([x]) = \begin{pmatrix} A'([x]) & 0 \\ O & A'([x]) \end{pmatrix} \quad ([x] \in \partial Z_{k+1}/G).$$

Then all $\sigma_{m+1}(A'^2) \in H^{m+1}(Z_{k+1}/G, \partial Z_{k+1}/G; \pi_m(\mathbf{U}(2^{(k+1)+1}))$ are well-defined and trivial for $m$ with $0 \leq m \leq \dim M_{k+1} - 1$. Hence $A'^2$ extends to $Z_{k+1}/G$, which implies

$$\gamma_{M_{k+1}\mathbb{C}}^{\oplus 2^{(k+1)+1}} \cong_G \varepsilon_{M_{k+1}}(\mathbb{C}^{2^{(k+1)+1}}),$$

i.e.

$$((\gamma_M^{\oplus 2^{(k+1)+1}})|_{M_{k+1}})_{\mathbb{C}} \cong_G \varepsilon_{M_{k+1}}(\mathbb{C}^{2^{(k+1)+1}}).$$

## 3. Preparation to use Serre's spectral sequences

Let $G$ and $V(i)$ be as in Theorem 1, and set

$$U = \bigoplus_{i=1}^{k} V(i) \quad \text{and} \quad W = V(k+1).$$

Let $T$ denote the group $\{-1, 1\}$ of order 2 and let $\widehat{G}$ denote the group $G \times T$. We regard each $V(i)$ as a complex $\widehat{G}$-module such that $T$-acts on $V(i)$ by the scalar multiplication. The closed interval $I = [-1, 1]$ has the $\widehat{G}$-action such that $T$ acts on $I$ by the scalar multiplication and $G$ acts trivially on $I$. We set $N = S(W) \times I \times D(U)$. Let $K$ denote the kernel of the $G$-representation $W$, i.e. of the homomorphism $G \to \mathrm{Aut}_{\mathbb{C}}(W)$.

In this paper, the orbit space of a $\widehat{G}$-space $X$ will be denoted by $\overline{X}$, i.e. $\overline{X} = X/\widehat{G}$. We can compute the cohomology group $H^*(\overline{N}, \overline{\partial N}; \mathbb{Z})$ from the homology group $H_*(\overline{N}, \overline{\partial N}; \mathbb{Z})$ by the universal coefficient theorem. Thus the proof of Theorem 2 deduces to the proof of

**Theorem 3.** *The homology group $H_*(\overline{N}, \overline{\partial N}; \mathbb{Z})$ is as follows.*

$$H_m(\overline{N}, \overline{\partial N}; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}_2 & (m = 2k+1) \\ O & (m \neq 2k+1) \end{cases}$$

The remainder of this paper is devoted to proving this theorem.

Now we set

$$\begin{aligned} &U^+ = D(U)/S(U), \quad \{\infty\} = S(U)/S(U), \\ &X = S(W) \times I \times U^+, \quad \partial X = S(W) \times \partial I \times U^+ \\ &A = S(W) \times I \times \{\infty\}, \quad \partial A = S(W) \times \partial I \times \{\infty\}. \end{aligned} \tag{3.1}$$

Then $N/\partial N$ is $\widehat{G}$-homeomorphic to $(X/A)/(\partial X/\partial A)$. The homology group $H_*(\overline{N}, \partial\overline{N}; \mathbb{Z})$ will be computed from the homology groups $H_*(\overline{X}, \overline{A}; \mathbb{Z})$ and $H_*(\overline{\partial X}, \overline{\partial A}; \mathbb{Z})$.

We remark

$$\begin{aligned} X/\widehat{G} &= \left(S(W) \times I \times U^+\right)/\widehat{G} \\ &= (S(W) \times I \times F)/(G/K \times T), \end{aligned}$$

where

$$F = U^+/K.$$

The cyclic group $G/K \times T$ acts freely on $S(W) \times I$ and the orbit space

$$M = (S(W) \times I)/\widehat{G}$$

is homeomorphic to the Möbius strip. The canonical projection $\pi_X : \overline{X} \to M$ can be regarded as a fiber bundle with fiber $F$. Similarly the canonical projection $\pi_{\partial X} : \overline{\partial X} \to \partial M$ can be regarded as a fiber bundle with fiber $F$. For the computation of $H_*(\overline{X}; \mathbb{Z})$, $H_*(\overline{\partial X}; \mathbb{Z})$ and $H_*(\overline{N}, \partial\overline{N}; \mathbb{Z})$, Serre's spectral sequence is helpful.

Let $\pi : E \to B$ be a fiber bundle such that the base $B$ is a connected finite CW-complex of dimension $d$ with base point $b_0$ and the fiber $F_{b_0} = \pi^{-1}(b_0)$ is a connected CW-complex. The space $B$ has the filtration $\mathfrak{F}_B$:

$$B^{(0)} \subset B^{(1)} \subset \cdots \subset B^{(i)} \subset \cdots \subset B^{(d)} = B$$

consisting of the $i$-skeletons $B^{(i)}$ of $B$. This induces the filtration $\mathfrak{F}_E$ of $E$:

$$E_0 \subset E_1 \subset \cdots \subset E_i \subset \cdots \subset E_d = E$$

with $E_i = \pi^{-1}(B^{(i)})$. The theory of Serre's spectral sequence [7, p.480, Theorem 16] says

**Lemma 4.** *Suppose the fiber bundle $\pi : E \to B$ is orientable. Then there is a convergent $E^2$ spectral sequence $E(\pi)$ with $E^2_{s,t} \cong H_s(B; H_t(F; \mathbb{Z}))$ and $E^\infty$ the bigraded module associated to the filtration of $H_*(E; \mathbb{Z})$ defined by*

$$F_s H_*(E; \mathbb{Z}) = \mathrm{Im}[H_*(E_s; \mathbb{Z}) \to H_*(E; \mathbb{Z})],$$

*where $F = F_{b_0}$. In addition, this spectral sequence is functorial on the category of orientable fiber bundles.*

*Remark* 5. In the lemma above, we have

$$\begin{aligned} O \subset F_0 H_*(E;\mathbb{Z}) \subset F_1 H_*(E;\mathbb{Z}) \subset \cdots \\ \subset F_{s-1} H_*(E;\mathbb{Z}) \subset F_s H_*(E;\mathbb{Z}) \subset \cdots \\ \subset F_d H_*(E;\mathbb{Z}) = H_*(E;\mathbb{Z}) \end{aligned}$$

and

$$E^\infty_{s,t} = F_s H_{s+t}(E;\mathbb{Z}) / F_{s-1} H_{s+t}(E;\mathbb{Z}).$$

**Lemma 6.** *Suppose the fiber bundle $\pi : E \to B$ is orientable and $H_s(B; R) = O$ for any finitely generated abelian group $R$ and all $s \neq 0,\ 1$. Then the spectral sequence $E(\pi)$ converges in the strong sense, i.e.*

$$E^2_{s,t} = E^3_{s,t} = E^4_{s,t} = \cdots = E^\infty_{s,t}. \tag{1}$$

*Furthermore there is a natural exact sequence*

$$O \to H_0(B; H_m(F;\mathbb{Z})) \to H_m(E;\mathbb{Z}) \to H_1(B; H_{m-1}(F;\mathbb{Z})) \to O. \tag{2}$$

*Proof.* The differential $d^r_{s,t}$ of $E(\pi)$ is a homomorphism $E^r_{s,t} \to E^r_{s-r,t+r-1}$. Since $E^2_{s,t} = O$ for $s \neq 0,\ 1$, if $r \geq 2$ then $d^r_{s,t}$ is the trivial homomorphism for all $s$ and $t$. As $E^{r+1}_{s,t} = \mathrm{Ker}\, d^r_{s,t} / \mathrm{Image}\, d^r_{s+r,t-r+1}$, we obtain the conclusion (1) of the lemma.

The exact sequence (2) is an interpretation of the exact sequence

$$O \to E^2_{0,m} \to H_m(E;\mathbb{Z}) \to E^2_{1,m-1} \to O.$$

□

In the next section, we apply Lemmas 4 and 6 to the fiber bundles $\pi_X : \overline{X} \to M$ and $\pi_{\partial X} : \overline{\partial X} \to \partial M$. For this, we need to show the next fact.

**Proposition 7.** *The fiber bundle $\pi_X : \overline{X} \to M$ is orientable, i.e. $\pi_1(M, b_0)$ acts trivially on $H_*(F;\mathbb{Z})$, where $b_0 \in M$ and $F = \pi_X^{-1}(b_0)$. Hence the fiber bundle $\pi_{\partial X} : \overline{\partial X} \to \partial M$ is also orientable.*

*Proof.* It suffices to prove that the fiber bundle $\pi_Y : \overline{Y} \to \overline{S(W)}$, where $Y = S(W) \times U^+$, is orientable.

Recall that $G$ is a group of odd order and each $V(i)$ is a 1-dimensional complex $\widehat{G}$-module. We can regard $V(i)$ as a complex $S^1$-module and the original $\widehat{G}$-action on $V(i)$ is obtained through a group homomorphism $\psi_i : \widehat{G} \to S^1$. Let $g$ be an element of $\widehat{G}$. The class $[g]$ in the cyclic group $\widehat{G}/K = C \times T$ determines a map $F \to F$, where $F = U^+/K$. We have to show that the homomorphism $[g]_* : H_*(F;\mathbb{Z}) \to H_*(F;\mathbb{Z})$ is the identity map. Let $\rho : [0,1] \to S^1 \times \cdots \times S^1$ ($k$-fold) be a continuous map such that $\rho(0) = (e, \cdots, e)$ and $\rho(1) = (\psi_1(g), \cdots \psi_k(g))$. Then for each $t \in [0,1]$, $\rho(t)$ determines a map $U^+ \to U^+$, and furthermore a map $F \to F$. Hence we obtain the homomorphism $\rho(t)_* : H_*(F;\mathbb{Z}) \to H_*(F;\mathbb{Z})$. Since $\rho(0)_*$ is the identity map, $\rho(1)_*$ $(= [g]_*)$ is the identity map. □

## 4. Computation of $H_*(\overline{X}, \overline{A}; \mathbb{Z})$ and $H_*(\overline{\partial X}, \overline{\partial A}; \mathbb{Z})$

First recall that

$$M = (S(W) \times I)/\widehat{G} = (S(W) \times I)/(C \times T),$$

where $C = G/K$, is homeomorphic to the Möbius strip. Clearly $H_s(M;R) = O$ for any abelian group $R$ and all $s \neq 0,\ 1$. Thus the Serre spectral sequence $E(\pi_X)$ converges in the strong sense of Lemma 6.

**Proposition 8.** *The homology group $H_*(F;\mathbb{Z})$ of the fiber $F = U^+/K$ is as follows:*

$$H_t(F;\mathbb{Z}) = \begin{cases} \mathbb{Z} & (t = 0,\ 2k) \\ A_t & (0 < t < 2k) \\ O & (t > 2k), \end{cases}$$

*where $A_t$ is a finite abelian group of odd order for each integer $t$ with $0 < t < 2k$.*

*Proof.* First note

$$H_t(U^+) = \begin{cases} \mathbb{Z} & (t = 0,\ 2k) \\ O & (t \neq 0,\ 2k). \end{cases}$$

We have a transfer homomorphism $tr : H_*(F;\mathbb{Z}) \to H_*(U^+;\mathbb{Z})$ such that $\pi_* \circ tr = |K|$, where $\pi : U^+ \to F$ is the canonical projection (see [2, Chapter 5.3] . Thus $|K| H_t(F;\mathbb{Z}) = O$ for $t$ with $0 < t < 2k$.

For the computation of $H_{2k}(F;\mathbb{Z})$, we may assume that $K$ acts effectively on $S(U)$. Under this assumption, $\dim S(U)^g \leq 2k - 3$ for all $g \in G \setminus \{e\}$. Let $\Sigma$ be the singular set of $S(U)$, i.e.

$$\Sigma = \bigcup_{g \in G \setminus \{e\}} S(U)^g.$$

Let $N(\Sigma)$ be the $K$-equivariant closed regular neighborhood of $\Sigma$ in $S(U)$. Since the action of $K$ on $S(U)$ is orientation preserving, $B = (S(U) \setminus N(\Sigma)^\circ)/K$ is a compact connected orientable manifold with boundary, where

$N(\Sigma)^\circ$ is the interior of $N(\Sigma)$. Hence we get $H_{2k-1}(B, \partial B; \mathbb{Z}) \cong \mathbb{Z}$. It follows that

$$H_{2k-1}(S(U)/K; \mathbb{Z}) = H_{2k-1}(S(U)/K, N(\Sigma)/K; \mathbb{Z}) = H_{2k-1}(B, \partial B; \mathbb{Z}) \cong \mathbb{Z}.$$

Since the space $F$ is the suspension of $S(U)/K$, we obtain $H_{2k}(F; \mathbb{Z}) \cong \mathbb{Z}$. □

By Lemma 4 and Proposition 7, we have

$$E^2_{s,t}(\pi_X) = H_s\left(M; H_t(F; \mathbb{Z})\right), \text{ where } M = \overline{S(W) \times I}.$$

Thus Proposition 8 implies

$$E^2_{s,t}(\pi_X) \cong \begin{cases} \mathbb{Z} & (s = 0,\ 1, t = 0,\ 2k) \\ A_t & (s = 0,\ 1,\ 0 < t < 2k) \\ O & (s \neq 0,\ 1). \end{cases} \tag{4.1}$$

Furthermore by Lemma 6, we get the following result.

**Proposition 9.** *The homology group $H_*(\overline{X}; \mathbb{Z})$ is as follows:*

$$H_0(\overline{X}; \mathbb{Z}) = H_0(M; H_0(F; \mathbb{Z})) \cong \mathbb{Z},$$
$$H_{2k+1}(\overline{X}; \mathbb{Z}) = H_1(M; H_{2k}(F; \mathbb{Z})) \cong \mathbb{Z}.$$

*In addition, the sequence*

$$O \to H_0(M; H_m(F; \mathbb{Z})) \to H_m(\overline{X}; \mathbb{Z}) \to H_1(M; H_{m-1}(F; \mathbb{Z})) \to O$$

*is exact for each $m$ with $1 \leq m \leq 2k$.*

Regard $\pi_A : \overline{S(W) \times I \times \{\infty\}} \to M$, where $M = \overline{S(W) \times I}$, as a fiber bundle with trivial fiber $\{\infty\}$.

It is easy to see

$$E^2_{s,t}(\pi_A) \cong \begin{cases} \mathbb{Z} & (s = 0,\ 1,\ t = 0) \\ O & (s = 0,\ 1,\ t > 0) \\ O & (s \neq 0,\ 1). \end{cases} \tag{4.2}$$

**Proposition 10.** *The homology group $H_*(\overline{X}, \overline{A}; \mathbb{Z})$ is as follows:*

$$H_0(\overline{X}, \overline{A}; \mathbb{Z}) = O,$$
$$H_1(\overline{X}, \overline{A}; \mathbb{Z}) = H_0(M; H_1(F; \mathbb{Z})),$$
$$H_{2k+1}(\overline{X}, \overline{A}; \mathbb{Z}) = H_1(M; H_{2k}(F; \mathbb{Z})) \cong \mathbb{Z}.$$

*Moreover the sequence*

$$O \to H_0(M; H_m(F; \mathbb{Z})) \to H_m(\overline{X}, \overline{A}; \mathbb{Z}) \to H_1(M; H_{m-1}(F; \mathbb{Z})) \to O$$

*is exact for each $m$ with $2 \leq m \leq 2k$.*

*Proof.* Using the result in Lemma 6, we can get the natural exact sequence of $\pi_A$, where $B$ is $M$, $E$ is $\overline{S(W) \times I \times \{\infty\}}$ and $F$ is $\{\infty\}$. On the other hand, we have known the homology group of $\overline{X}$ with coefficients in $\mathbb{Z}$, using the formula (4.2), and a straightforward argument completes the proof. □

By Lemma 4 and Proposition 7, we have $\partial M = \overline{S(W) \times \partial I}$ is homeomorphic to $S^1$. Thus it holds that

$$E^2_{s,t}(\pi_{\partial X}) \cong \begin{cases} \mathbb{Z} & (s = 0,\ 1,\ t = 0,\ 2k) \\ A_t & (s = 0,\ 1,\ 0 < t < 2k) \\ O & (s \neq 0,\ 1). \end{cases} \tag{4.3}$$

This affords the following two propositions.

**Proposition 11.** *The homology group $H_*(\overline{\partial X}; \mathbb{Z})$ is as follows:*

$$H_0(\overline{\partial X}; \mathbb{Z}) = H_0(\partial M; H_0(F; \mathbb{Z})) \cong \mathbb{Z},$$
$$H_{2k+1}(\overline{\partial X}; \mathbb{Z}) = H_1(\partial M; H_{2k}(F; \mathbb{Z})) \cong \mathbb{Z}.$$

*Furthermore the sequence*

$$O \to H_0(\partial M; H_m(F; \mathbb{Z})) \to H_m(\overline{\partial X}; \mathbb{Z}) \to H_1(\partial M; H_{m-1}(F; \mathbb{Z})) \to O$$

*is exact for each $m$ with $1 \leq m \leq 2k$.*

**Proposition 12.** *The homology group $H_*(\overline{\partial X}, \overline{\partial A}; \mathbb{Z})$ is as follows:*

$$H_0(\overline{\partial X}, \overline{\partial A}; \mathbb{Z}) = O,$$
$$H_1(\overline{\partial X}, \overline{\partial A}; \mathbb{Z}) = H_0(\partial M; H_1(F; \mathbb{Z})),$$
$$H_{2k+1}(\overline{\partial X}, \overline{\partial A}; \mathbb{Z}) = H_1(\partial M; H_{2k}(F; \mathbb{Z})) \cong \mathbb{Z}.$$

*Moreover the sequence*

$$O \to H_0(\partial M; H_m(F; \mathbb{Z})) \to H_m(\overline{\partial X}, \overline{\partial A}; \mathbb{Z}) \to H_1(\partial M; H_{m-1}(F; \mathbb{Z})) \to O$$

*is exact for each $m$ with $2 \leq m \leq 2k$.*

*Proof.* The proof is essentially same as that of Proposition 10. □

## 5. Computation of $H_*(\overline{N}, \overline{\partial N}; \mathbb{Z})$

In this section, we use notation in the previous section.

Since we have the commutative diagram

$$\begin{array}{ccc} H_0(\partial M; H_1(F; \mathbb{Z})) & \xrightarrow{\cong} & H_1(\overline{\partial X}, \overline{\partial A}; \mathbb{Z}) \\ {\scriptstyle\cong}\downarrow & & \downarrow \\ H_0(M; H_1(F; \mathbb{Z})) & \xrightarrow[\cong]{} & H_1(\overline{X}, \overline{A}; \mathbb{Z}) \end{array}$$

we get $H_1(\overline{N}, \overline{\partial N}; \mathbb{Z}) = O$.

For $m$ with $2 \le m \le 2k$, we have the commutative diagram

$$\begin{array}{ccccccccc}
O & \longrightarrow & H_0(\partial M; H_m(F;\mathbb{Z})) & \longrightarrow & H_m(\overline{\partial X}, \overline{\partial A}; \mathbb{Z}) & \longrightarrow & H_1(\partial M; H_{m-1}(F;\mathbb{Z})) & \longrightarrow & O \\
 & & \cong\downarrow & & \downarrow & & \downarrow & & \\
O & \longrightarrow & H_0(M; H_m(F;\mathbb{Z})) & \longrightarrow & H_m(\overline{X}, \overline{A}; \mathbb{Z}) & \longrightarrow & H_1(M; H_{m-1}(F;\mathbb{Z})) & \longrightarrow & O,
\end{array}$$

where the horizontal sequences are exact. In addition we have the commutative diagram

$$\begin{array}{ccc}
H_1(\partial M; H_{m-1}(F;\mathbb{Z})) & \xrightarrow{\cong} & H_{m-1}(F;\mathbb{Z}) \\
\downarrow & & \downarrow \times 2 \\
H_1(M; H_{m-1}(F;\mathbb{Z})) & \xrightarrow[\cong]{} & H_{m-1}(F;\mathbb{Z})
\end{array}$$

and $H_{m-1}(F;\mathbb{Z}) = A_{m-1}$ is a finite abelian group of odd order. It follows that $A_{m-1}/2A_{m-1} = O$. Thus we conclude $H_m(\overline{N}, \overline{\partial N}; \mathbb{Z}) = O$.

Next observe the commutative diagram

$$\begin{array}{ccccc}
H_{2k+1}(\overline{\partial X}, \overline{\partial A}; \mathbb{Z}) & \xrightarrow{\cong} & H_1(\partial M; H_{2k}(F;\mathbb{Z})) & \xrightarrow{\cong} & \mathbb{Z} \\
\downarrow & & \downarrow & & \downarrow \times 2 \\
H_{2k+1}(\overline{X}, \overline{A}; \mathbb{Z}) & \xrightarrow[\cong]{} & H_1(M; H_{2k}(F;\mathbb{Z})) & \xrightarrow[\cong]{} & \mathbb{Z}
\end{array}$$

This implies $H_{2k+1}(\overline{N}, \overline{\partial N}; \mathbb{Z}) = \mathbb{Z}_2$.

We have completed the proof of Theorem 3.

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