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IRREDUCIBILITIES OF THE INDUCED CHARACTERS OF CYCLIC p-GROUPS

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ABSTRACT. We denote by C_n the cyclic group of order p^n, where p is an odd prime. Let ϕ be a faithful irreducible character of C_n. In this paper, we study the p-group G containing C_n such that the induced character ϕ^G is also irreducible. The purpose of this paper is to determine such groups G in the case when G has a subgroup H containing C_n such that $C_n \lhd H$ and $[G : H] = p$.

1. INTRODUCTION

Let G be a finite group. We denote by $\mathrm{Irr}(G)$ the set of complex irreducible characters of G and by $\mathrm{FIrr}(G)$ ($\subset \mathrm{Irr}(G)$) the set of faithful irreducible characters of G.

For a prime p we denote by C_n the cyclic group of order p^n. A finite group G is called an M-group, if every $\phi \in \mathrm{Irr}(G)$ is induced from linear character of a subgroup of G.

It is well-known that every nilpotent group is an M-group. So, for any $\chi \in \mathrm{Irr}(G)$, where G is a p-group, there exists a subgroup H of G and the linear character ϕ of H such that $\phi^G = \chi$. If we set $N = Ker\phi$, then $N \lhd H$ and ϕ is a faithful irreducible character of $H/N \cong C_n$, for some non-negative integer n. In this paper, we will consider the case when $N = 1$, that is, ϕ is a faithful linear character of $H \cong C_n$.

We consider the following :

Problem. *Let p be an odd prime, and ϕ be a faithful irreducible character of C_n. Determine the p-group G such that $C_n \subset G$ and the induced character ϕ^G is also irreducible.*

Since all faithful irreducible character of C_n are algebraically conjugate to each other, the irreducibility of ϕ^G ($\phi \in \mathrm{FIrr}(C_n)$) is independent of the choice of ϕ, but depends only on n.

Recently, Iida [2] solved this problem in the case when C_n is a normal subgroup of G.

The purpose of this paper is to solve this problem in the case when G has a subgroup H containing C_n such that $C_n \lhd H$ and $[G : H] = p$.

The problem of this type was considerd by Yamada and Iida [3]. They studied the 2-groups G such that $H \subset G$ and the similar properties of our problem hold, where $H = Q_n$ or D_n or SD_n. Here, we denote by Q_n and D_n the generalized quaternion group and the dihedral group of order $2^{n+1}(n \geq 2)$, respectively, and by SD_n the semidihedral group of order $2^{n+1}(n \geq 3)$.

Throughout this paper, $\mathbf{Z}$, and $\mathbf{N}$ denote the rational integers and the natural numbers, respectively.

2. Statements of the results

For the rest of this paper, we assume that p is an odd prime.
First, we introduce the following groups :

(i): $G(n,m) = \langle a,\ u_m \rangle$ with
$$a^{p^n} = u_m^{p^m} = 1,\ u_m a u_m^{-1} = a^{1+p^{n-m}}, \quad (m \leq n-1).$$
(ii): $G(n,m,1) = \langle a,\ u_m\ ,\ v \rangle$ ($\rhd G(n,m) = \langle a,\ u_m \rangle$) with
$$a^{p^n} = u_m^{p^m} = 1,\ u_m a u_m^{-1} = a^{1+p^{n-m}},\ vav^{-1} = a^{1+p^{n-m-1}} u_m^{p^{m-1}},$$
$$v^p = u_m,\ v u_m v^{-1} = u_m \quad (2m \leq n-1).$$

We can see that $G(n,m,1)$ is an extension group of $G(m,n)$ by using Proposition 1 below:

Proposition 1. *Let N be a finite group such that $G \rhd N$ and $G/N = \langle uN \rangle$ is a cyclic group of order m. Then $u^m = c \in N$. If we put $\sigma(x) = uxu^{-1}$, $x \in N$, then $\sigma \in Aut(N)$ and (i) $\sigma^m(x) = cxc^{-1}$, $(x \in N)$ (ii) $\sigma(c) = c$.*

Conversely, if $\sigma \in Aut(N)$ and $c \in N$ satisfy (i) and (ii), then there exists one and only one extension group G of N such that $G/N = \langle uN \rangle$ is a cyclic group of order m and $\sigma(x) = vxv^{-1}$ $(x \in N)$ and $v^m = c$.

Proof. For instance, see [4, III, §7] □

We state the theorem of Iida ([2]).

Theorem 0 (Iida [2].). *Let G be a p-group which contains C_n as a normal subgroup of index p^m. Let $\phi \in$ FIrr(C_n). Suppose that $\phi^G \in$ Irr(G). Then $G \cong G(n,m)$.*

In particular, when $C_n \subset G$ and $[G : C_n] = p$, C_n is always a normal subgroup of G, so we have

Corollary 0. *Let $\phi \in$ FIrr(C_n). Suppose that $C_n \subset G$ such that $[G : C_n] = p$ and $\phi^G \in$ Irr(G). Then $G \cong G(n,1)$.*

Our main theorem is the following:

Theorem. *Let G be a p-group which contains C_n with $[G : C_n] = p^{m+1}$, where p is an odd prime. Let $\phi \in \mathrm{FIrr}(C_n)$. Suppose that $\phi^G \in \mathrm{Irr}(G)$, and $n-3 \geq 2m$. Further, suppose that there exists a subgroup H of G such that $H \rhd C_n$ and $[G : H] = p$. Then $G \cong G(n, m+1)$ or $G(n, m, 1)$.*

Corollary. *Let G be a p-group which contains C_n with $[G : C_n] = p^2$. Let $\phi \in \mathrm{FIrr}(C_n)$. Suppose that $\phi^G \in \mathrm{Irr}(G)$ and $n \geq 5$. Then $G \cong G(n, 2)$ or $G(n, 1, 1)$.*

3. Some preleminary results

In this section, we state some results concerning the criterion of the irreducibilities of induced characters and others, which we need in section 4.

We denote by $\zeta = \zeta_{p^n}$ a primitive p^nth root of unity. It is known that, for $C_n = \langle a \rangle$, there are p^m irreducible characters ϕ_ν $(1 \leq \nu \leq p^n)$ of C_n:

$$\phi_\nu(a^i) = \zeta^{\nu i}, \qquad (1 \leq i \leq p^n).$$

The irreducible character ϕ_ν is faithful if and only if $(\nu, p) = 1$.
First, we state the following result of Shoda (cf [1, p.329]):

Proposition 2. *Let G be a group and H be a subgroup of G. Let ϕ be a linear character of H. Then the induced character ϕ^G of G is irreducible if and only if , for each $x \in G - H = \{ g \in G | \ g \notin H \}$, there exists $h \in xHx^{-1} \cap H$ such that $\phi(h) \neq \phi(x^{-1}hx)$. In particular, when ϕ is faithful, the condition $\phi(h) \neq \phi(x^{-1}hx)$ is equivalent to that of $h \neq x^{-1}hx$.*

Using this result, we have the following:

Proposition 3. *Let $\langle a \rangle = C_n \subset G$, and ϕ be a faithful irreducible character of C_n. Then the following conditions are equivalent*

1. *ϕ^G is irreducible.*
2. *For each $x \in G - C_n$, there exists $y \in \langle a \rangle \cap x\langle a \rangle x^{-1}$ such that $xyx^{-1} \neq y$.*

Definition. *When the condition (2) of Proposition 3 holds, we say that G satisfies (EX, C).*

Finally, we state the following :

Lemma 1. *Let p be an odd prime and n, m, k, j be integers satisfying $0 \leq m \leq n$. Then , if we put $s = 1 + kp^{n-m}$, we have the following equality :*

$$\frac{s^{jp^m}-1}{s^j-1} \equiv p^m \pmod{p^n}.$$

4. Proof of Theorem

Let $G \supset H$ be a p-group as is stated in Theorem, and let $\phi \in \mathrm{FIrr}(C_n)$. Since $\phi^G = (\phi^H)^G \in \mathrm{Irr}(G)$, we must have $\phi^H \in \mathrm{Irr}(H)$. Therefore, by Theorem 0, we can take an element u_m in H such that $H = \langle\, a, u_m \rangle \cong G(n,m)$. For the sake of simplicity, we write u instead of u_m. Since $[G:H]=p$, we may write as

$$G = \langle\, H\ , y\ \rangle\ (\rhd H),$$

where $y \in G - H = \{\ g \in G|\ g \notin H\ \}$ and $y^p \in H$.

Note that any element in $H = \langle a, u\rangle$ is represented as $a^i u^j$ for some $i, j \in \mathbf{Z}$, $0 \le i \le p^n - 1$, $0 \le j \le p^m - 1$.
Further, if we put $s = 1 + p^{n-m}$, we have

$$(a^i u^j)^{p^m} = a^{i(\frac{s^{p^m j}-1}{s^j-1})} u^{p^m j} = a^{p^m i}$$

by Lemma 1.
First, we consider the elements yay^{-1} and yuy^{-1}.

We will show the following

Claim I. *We can write as*

$$yay^{-1} = a^{1+kp^{n-m-1}} u^{p^{m-1}j},$$
$$yuy^{-1} = a^{p^{n-m}d} u,$$

for some $k,\ j,\ d\ \in \mathbf{Z}$.

Proof of Claim I.. Write $yay^{-1} = a^{i_0}u^{j_0}$ and $yuy^{-1} = a^{d_0}u^{t_0}$. Since

$$ya^{p^m}y^{-1} = a^{p^m i_0},$$

we must have $(p, i_0) = 1$.
On the other hand, since

$$1 = yu^{p^m}y^{-1} = a^{d_0 p^m},$$

we have

$$d_0 \equiv 0 \pmod{p^{n-m}}.$$

Therefore, we may write $d_0 = p^{n-m}d$ and

$$yuy^{-1} = a^{p^{n-m}d} u^{t_0},$$

for some $d \in \mathbf{Z}$. Since $n - m \ge m$, by our assumption, we have

(1) $$ya^{p^{n-m}}y^{-1} = a^{p^{n-m}i_0}.$$

Taking the conjugate of both sides of the equality, $uau^{-1} = a^{1+p^{n-m}}$ by y, we get

$$(a^{p^{n-m}d}u^{t_0})(a^{i_0}u^{j_0})(a^{p^{n-m}d}u^{t_0})^{-1} = a^{i_0}u^{j_0}a^{p^{n-m}i_0}.$$

Hence, we have

$$a^{i_0(1+p^{n-m})^{t_0}}u^{j_0} = a^{i_0(1+p^{n-m})}u^{j_0}.$$

Therefore,

$$i_0(1+t_0\cdot p^{n-m}) \equiv i_0(1+p^{n-m}) \pmod{p^n}.$$

But $(i_0, p) = 1$, so we get $t_0 \equiv 1 \pmod{p^m}$, and hence

$$yuy^{-1} = a^{p^{n-m}d}u.$$

For a normal subgroup N of G, and any g, $h \in G$, we write

$$g \equiv h \pmod{N}$$

when $gh^{-1} \in N$.

Note that $\langle a^{p^{n-m}}\rangle$ is a normal subgroup of G, by (1).
It is easy to see that

$$yuy^{-1} \equiv u \pmod{\langle a^{p^{n-m}}\rangle}.$$
$$ua \equiv au \pmod{\langle a^{p^{n-m}}\rangle}.$$

Further, we have

$$ya^ly^{-1} = (a^{i_0}u^{j_0})^l \equiv a^{i_0l}u^{j_0l} \pmod{\langle a^{p^{n-m}}\rangle}.$$

for any $l \in \mathbf{N}$.
Using these relations repeatedly, we get

$$y^say^{-s} \equiv a^{i_0^s}u^{j_0(i_0^{s-1}+\cdot+i_0+1)} \pmod{\langle a^{p^{n-m}}\rangle},$$

for any $s \in \mathbf{N}$.
In particular,

$$y^pay^{-p} \equiv a^{i_0^p}u^{j_0(i_0^{p-1}+\cdot+i_0+1)} \pmod{\langle a^{p^{n-m}}\rangle}.$$

Hence we may write as

$$y^pay^{-p} = a^{i_0^p+rp^{n-m}}u^{j_0(i_0^{p-1}+\cdot+i_0+1)},$$

for some integer r.
Since $y^p \in H = \langle a, u\rangle$, we must have

(2) $$i_0^p \equiv 1 \pmod{p^{n-m}},$$

and

(3) $$j_0(i_0^{p-1}+\cdot+i_0+1) \equiv 0 \pmod{p^m}.$$

By (2), we can write as $i_0 = 1 + kp^{n-m-1}$, for some integer k. So,

$$j_0(i_0^{p-1} + \cdot + i_0 + 1) = j_0(\frac{i_0^p - 1}{i_0 - 1}) \equiv j_0 p \pmod{p^{n-m}}.$$

Since $n - m \geq m$, by our assumption, we have

$$j_0 p \equiv 0 \pmod{p^m},$$

by (3). Therefore we can write $j_0 = p^{m-1}j$, for some $j \in \mathbf{Z}$. Thus the proof of Claim I is completed. □

Hence, in order to prove the theorem, we have only to consider the following two cases:

Case I.: $yay^{-1} = a^{1+kp^{n-m-1}}$, and $yuy^{-1} = a^{p^{n-m}d}u$,
Case II.: $yay^{-1} = a^{1+kp^{n-m-1}}u^{p^{m-1}j}$, $(j,p) = 1$, and $yuy^{-1} = a^{p^{n-m}d}u$,

First, we consider Case I. But in this case we can see that $G \rhd C_n$. Hence, by Iida's result, we have

$$G \cong G(n, m+1).$$

Next, we consider Case II.
In this case, we have

$$y\langle a\rangle y^{-1} \cap \langle a\rangle = \langle a^p\rangle,$$

because

$$ya^py^{-1} = (a^{1+kp^{n-m-1}}u^{p^{m-1}j})^p = a^{(1+kp^{n-m-1})p} \in \langle a^p\rangle.$$

Suppose that $k \equiv 0 \pmod p$, then there exists $s_0 \in \mathbf{Z}$, $0 \leq s_0 \leq p^m - 1$, such that

$$(u^{s_0}y)a^p(u^{s_0}y)^{-1} = a^p.$$

This contradicts the hypothesis that the condition (EX,C) holds. So, we must have

$$(k,p) = 1.$$

Next, we consider the element y^p ($\in H = \langle a, u\rangle$). Write $y^p = a^{l_0}u^{k_0}$. Since

$$ya^{p^{m+1}}y^{-1} = (a^{1+kp^{n-m-1}}u^{p^{m-1}j})^{p^{m+1}} = a^{p^{m+1}},$$

we have

$$ya^{p^t}y^{-1} = a^{p^t},$$

for any $t \geq m+1$. In particular, since $n - m - 1 \geq m + 1$, by our assumption, we have

$$ya^{p^{n-m-1}}y^{-1} = a^{p^{n-m-1}}.$$

By a direct calculation, we have

$$y^p a y^{-p} = a^{1+kp^{n-m}+p^{n-1}dj(1+2+\cdot+(p-1))} u^{p^m j} = a^{1+kp^{n-m}}.$$

On the other hand, we have

$$y^p a y^{-p} = (a^{l_0}u^{k_0})a(a^{l_0}u^{k_0})^{-1} = a^{1+k_0p^{n-m}}.$$

Hence, we have

$$k \equiv k_0 \pmod{p^m},$$

so, we may write

$$y^p = a^{l_0}u^k.$$

We show the following

Claim II. *There exists an integer e, such that* $(a^e y)^p = u^k$.

Proof of Claim II. Since

$$y a^{l_0} y^{-1} \equiv a^{l_0} u^{p^{m-1}jl_0} \pmod{\langle a^{p^{n-m-1}}\rangle},$$

we have

$$\begin{aligned} a^{l_0}u^k =& y^p = yy^py^{-1} = y(a^{l_0}u^k)y^{-1} \\ \equiv& a^{l_0}u^{p^{m-1}jl_0+k} \pmod{\langle a^{p^{n-m-1}}\rangle}. \end{aligned}$$

Therefore we have

$$jl_0 \equiv 0 \pmod{p}.$$

But, $(j,p)=1$, by our assumption, so

$$l_0 \equiv 0 \pmod{p}.$$

Hence we may write as $l_0 = pl$ and

$$y^p = a^{pl}u^k$$

for some $l \in Z$.
By a direct calculation, we get

$$\begin{aligned} (a^s y)^p \equiv& a^{ps} u^{p^{m-1}js(1+2+\cdot+(p-1))} y^p \\ =& a^{ps} u^{p^{m-1}js\frac{p(p-1)}{2}} y^p \\ =& a^{p(s+l)} u^k \pmod{\langle a^{p^{n-m-1}}\rangle}, \end{aligned}$$

for any $s \in \mathbf{N}$.
Therefore we may write as

$$(a^s y)^p = a^{p(s+l+p^{n-m-2}\beta_{p,s})} u^k$$

for some integer $\beta_{p,s}$. Note that $\beta_{p,s}$ is not independent of the choice of s. If we set $y_1 = a^{-l}y$, we can write as

$$y_1^p = a^{\beta p^{n-m-1}} u^k,$$

for some integer β. Further, set $e = -\beta p^{n-m-2} - l$, and

$$y_2 = a^e y = a^{-\beta p^{n-m-2} - l} y = a^{-\beta p^{n-m-2}} y_1.$$

Since $n - m - 2 \geq m + 1$, by our assumption, we have

$$y_1 a^{p^{n-m-2}} y_1^{-1} = a^{p^{n-m-2}}.$$

So,

$$(a^e y)^p = y_2^p = (a^{-\beta p^{n-m-2}} y_1)^p = a^{-\beta p^{n-m-1}} y_1^p = u^k.$$

Thus the proof of Claim II is completed. □

Since $(k, p) = 1$, there exists $k' \in \mathbf{Z}$, such that $kk' \equiv 1 \pmod{p^m}$. Hence

$$y_2^{k'p} = u^{kk'} = u.$$

Therefore

$$y_2 u y_2^{-1} = u.$$

Further, we have

$$\begin{aligned} y_2 a y_2^{-1} &= a^{-\beta p^{n-m-2} - l} y a y^{-1} a^{l + \beta p^{n-m-2}} \\ &= a^{-l}(a^{1 + kp^{n-m-1}} u^{p^{m-1} j}) a^l \\ &= a^{1 + (k + ljp^m) p^{n-m-1}} u^{p^{m-1} j}. \end{aligned}$$

If we set $k_1 = k + ljp^m$, then

$$y_2 a y_2^{-1} = a^{1 + k_1 p^{n-m-1}} u^{p^{m-1} j},$$

and

$$y_2^p = u^k = u^{k_1}.$$

Summarizing the results, we have

$$\begin{aligned} &y_2 a y_2^{-1} = a^{1 + k_1 p^{n-m-1}} u^{jp^{m-1}}, \\ &y_2^p = u^{k_1}, \\ &y_2 u y_2^{-1} = u. \end{aligned}$$

There exists an integer l_1, such that

$$l_1 k_1 \equiv 1 \pmod{p^{m+1}}.$$

Set $y_3 = y_2^{l_1}$. Since $n - m - 1 \geq m + 1$, by our assumption, we have

$$y_2 a^{p^{n-m-1}} y_2^{-1} = a^{p^{n-m-1}}$$

Hence,

$$y_3ay_3^{-1} = y_2^{l_1}ay_2^{-l_1} = a^{1+p^{n-m-1}k_1l_1}u^{p^{m-1}l_1j} = a^{1+p^{n-m-1}}u^{p^{m-1}l_1j}.$$

and

$$y_3^p = y_2^{pl_1} = u^{k_1l_1} = u, \quad y_3uy_3^{-1} = u.$$

Take an integer s_1, such that

$$l_1js_1 \equiv 1 \pmod p.$$

Then

$$\begin{aligned} y_3a^{s_1}y_3^{-1} &= (a^{1+p^{n-m-1}}u^{p^{m-1}l_1j})^{s_1} \\ &= a^{(1+p^{n-m-1})(s_1+l_1jp^{n-1}\frac{s_1(s_1-1)}{2})}u^{l_1js_1p^{m-1}} \\ &= a^{(1+p^{n-m-1})(s_1+l_1jp^{n-1}\frac{s_1(s_1-1)}{2}l_1js_1)}u^{l_1js_1p^{m-1}} \\ &= a^{s_1(1+p^{n-m-1}+k_2p^{n-1})}u^{p^{m-1}}, \end{aligned}$$

where, $k_2 = \frac{s_1(s_1-1)}{2}l_1^2j^2$.
Set $a_1 = a^{s_1}$, then

$$\langle a\rangle = \langle a^{s_1}\rangle$$

and

$$y_3a_1y_3^{-1} = a_1^{1+p^{n-m-1}+k_2p^{n-1}}u^{p^{m-1}}.$$

Further,

$$a_1^{p^n} = 1, \quad ua_1u^{-1} = a_1^{1+p^{n-m}}, \quad u^{p^m} = 1, \tag{4}$$

Finally, we set

$$y_4 = u^{-k_2p^{m-1}}y_3.$$

Then,

$$\begin{aligned} y_4a_1y_4^{-1} &= u^{-k_2p^{m-1}}y_3a_1y_3^{-1}u^{k_2p^{m-1}} \\ &= u^{-k_2p^{m-1}}(a_1^{1+p^{n-m-1}+k_2p^{n-1}}u^{p^{m-1}})u^{k_2p^{m-1}} \\ &= a_1^{1+p^{n-m-1}}u^{p^{m-1}}, \end{aligned} \tag{5}$$

and

$$y_4^p = y_3^p = u, \quad y_4uy_4^{-1} = u. \tag{6}$$

Therefore, G is generated by a_1, u and y_4 with relations (4), (5) and (6). These relations are the same as that of $G(n,m,1)$. Hence

$$G = \langle a_1,\ u,\ y_4\rangle \cong G(n,m,1),$$

as desired. This completes the proof of Theorem.

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