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Subgroup  $SU(8)/Z_2$  of compact simple Lie  
group  $E_7$  and non-compact simple Lie group  
 $E\{7(7)\}$  of type  $E_7$

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## SUBGROUP $SU(8)/Z_2$ OF COMPACT SIMPLE LIE GROUP $E_7$ AND NON-COMPACT SIMPLE LIE GROUP $E_{7(7)}$ OF TYPE $E_7$

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It is known that there exist four simple Lie groups of type  $E_7$  up to local isomorphism, one of them is compact and the others are non-compact. We have shown that in [3], [4] the group

$$E_7 = \{ \alpha \in \text{Iso}_C(\mathfrak{P}^c, \mathfrak{P}^c) \mid \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q, \langle \alpha P, \alpha Q \rangle = \langle P, Q \rangle \}$$

is a simply connected compact simple Lie group of type  $E_7$  and in [5], [7] the groups

$$\begin{aligned} E_{7(-25)} &= \{ \alpha \in \text{Iso}_R(\mathfrak{P}, \mathfrak{P}) \mid \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q \}, \\ E_{7(-5)} &= \{ \alpha \in \text{Iso}_C(\mathfrak{P}^c, \mathfrak{P}^c) \mid \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q, \langle \alpha P, \alpha Q \rangle_\sigma = \langle P, Q \rangle_\sigma \} \end{aligned}$$

are connected non-compact simple Lie groups of type  $E_{7(-25)}$ ,  $E_{7(-5)}$  respectively, and their polar decompositions are given by

$$\begin{aligned} E_{7(-25)} &\simeq (U(1) \times E_6) / Z_3 \times \mathbf{R}^{54}, \\ E_{7(-5)} &\simeq (SU(2) \times Spin(12)) / Z_2 \times \mathbf{R}^{64}. \end{aligned}$$

In this paper, first we find a subgroup in  $E_7$  which is isomorphic to the group  $SU(8)/Z_2$ . Next we show that the group

$$E_{7(7)} = \{ \alpha \in \text{Iso}_R(\mathfrak{P}', \mathfrak{P}') \mid \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q \}$$

is a connected non-compact simple Lie group of type  $E_{7(7)}$  with the center  $z(E_{7(7)}) = \{1, -1\}$  and its polar decomposition is given by

$$E_{7(7)} \simeq SU(8) / Z_2 \times \mathbf{R}^{70}.$$

Our main method used in this paper is to give homomorphisms  $\psi : SU(8) \rightarrow E_7$  and  $\psi : SU(8) \rightarrow E_{7(7)}$  explicitly.

### I. Subgroup $SU(8)/Z_2$ of compact simple Lie group $E_7$

#### 1. Preliminaries

**1.1. Cayley algebras  $\mathfrak{C}$ ,  $\mathfrak{C}^c$  and exceptional Jordan algebras  $\mathfrak{J}$ ,  $\mathfrak{J}^c$ .**  
Let  $\mathfrak{C} = \mathbf{H} \oplus \mathbf{H}e$  ( $\mathbf{H}$  is the quaternionic field) denote the Cayley algebra over the field  $\mathbf{R}$  of real numbers with the multiplication

$$(a + be)(c + de) = (ac - \bar{d}b) + (b\bar{c} + da)e$$

and  $\mathbb{C}^c = \{x_1 + ix_2 \mid x_1, x_2 \in \mathbb{C}\}$  its complexification with respect to the field  $\mathbb{C}$  of complex numbers.

$$\text{Let } \mathfrak{J} = \{X \in M(3, \mathbb{C}) \mid X^* = X\} = \left\{ \begin{bmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{bmatrix} \mid \xi_i \in \mathbf{R}, x_i \in \mathbb{C} \right\}$$

denote the exceptional Jordan algebra with the multiplication

$$X \circ Y = (XY + YX)/2$$

and  $\mathfrak{J}^c = \{X_1 + iX_2 \mid X_1, X_2 \in \mathfrak{J}\}$  its complexification. In  $\mathfrak{J}$  and  $\mathfrak{J}^c$ , inner products  $(X, Y)$ ,  $\langle X, Y \rangle$ , the crossed product  $X \times Y$ , the trilinear form  $(X, Y, Z)$  and the determinant  $\det X$  are defined respectively by

$$\begin{aligned} (X, Y) &= \text{tr}(X \circ Y), & \langle X, Y \rangle &= (\tau X, Y) = (\bar{X}, Y), \\ X \times Y &= (2X \circ Y - \text{tr}(X)Y - \text{tr}(Y)X + (\text{tr}(X)\text{tr}(Y) - \text{tr}(X \circ Y))E), \\ (X, Y, Z) &= (X, Y \times Z), & \det X &= (X, X, X)/3 \end{aligned}$$

where  $\tau : \mathfrak{J}^c \rightarrow \mathfrak{J}^c$  denotes the complex conjugation :  $\tau(X_1 + iX_2) = X_1 - iX_2$ ,  $X_1, X_2 \in \mathfrak{J}$  ( $\tau X$  is also denoted by  $\bar{X}$ ) and  $E$  the  $3 \times 3$  unit matrix. (The other  $n \times n$  unit matrix will be also denoted by  $E$ ).

In  $\mathfrak{J}$  and  $\mathfrak{J}^c$  we adopt the following notations :

$$\begin{aligned} E_1 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, & E_2 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, & E_3 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\ F_1(x) &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & \bar{x} & 0 \end{bmatrix}, & F_2(x) &= \begin{bmatrix} 0 & 0 & \bar{x} \\ 0 & 0 & 0 \\ x & 0 & 0 \end{bmatrix}, & F_3(x) &= \begin{bmatrix} 0 & x & 0 \\ \bar{x} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Then the table of the crossed products among them are given by

$$\begin{cases} E_i \times E_i = 0, & E_i \times E_{i+1} = E_{i+2}/2, \\ E_i \times F_i(x) = -F_i(x)/2, & E_i \times F_j(x) = 0, \quad i \neq j, \\ F_i(x) \times F_i(y) = -(x, y)E_i, & F_i(x) \times F_{i+1}(y) = F_{i+2}(\bar{x}y)/2 \end{cases}$$

where indexes are considered as mod 3.

Finally we define a linear involution  $\gamma : \mathfrak{J} \rightarrow \mathfrak{J}$  (resp.  $\mathfrak{J}^c \rightarrow \mathfrak{J}^c$ ) by

$$\gamma \begin{bmatrix} \xi_1 & a_3 + b_3e & \bar{a}_2 - b_2e \\ \bar{a}_3 - b_3e & \xi_2 & a_1 + b_1e \\ a_2 + b_2e & \bar{a}_1 - b_1e & \xi_3 \end{bmatrix} = \begin{bmatrix} \xi_1 & a_3 - b_3e & \bar{a}_2 + b_2e \\ \bar{a}_3 + b_3e & \xi_2 & a_1 - b_1e \\ a_2 - b_2e & \bar{a}_1 + b_1e & \xi_3 \end{bmatrix}$$

where  $\xi_i \in \mathbf{R}$  (resp.  $\mathbf{C}$ ),  $a_i, b_i \in \mathbf{H}$  (resp.  $\mathbf{H}^c$  (the complexification of  $\mathbf{H}$ )).

**1.2. Compact Lie group  $E_6$  and subgroup  $(E_6)_\rho$ .** We have shown in [8] that the group

$$E_6 = \{\alpha \in \text{Iso}_c(\mathfrak{Z}^c, \mathfrak{Z}^c) \mid \det \alpha X = \det X, \langle \alpha X, \alpha Y \rangle = \langle X, Y \rangle\}$$

$$= \{\alpha \in \text{Iso}_c(\mathfrak{Z}^c, \mathfrak{Z}^c) \mid \tau \alpha \tau(X \times Y) = \alpha X \times \alpha Y, \langle \alpha X, \alpha Y \rangle = \langle X, Y \rangle\}$$

is a simply connected compact simple Lie group of type  $E_6$  and therefore its Lie algebra

$$\mathfrak{e}_6 = \{\phi \in \text{Hom}_c(\mathfrak{Z}^c, \mathfrak{Z}^c) \mid (\phi X, X, X) = 0, \langle \phi X, Y \rangle = -\langle X, \phi Y \rangle\}$$

is a compact simple Lie algebra of type  $E_6$ .

We have also found in [8] a subgroup of type  $C_4$  in the group  $E_6$ . For later use we review this subgroup. Let  $\mathfrak{Z}(4, \mathbf{H}) = \{X \in M(4, \mathbf{H}) \mid X^* = X\}$  denote the vector space of all  $4 \times 4$  quaternionic Hermitian matrices and  $\mathfrak{Z}(4, \mathbf{H})^c = \{X_1 + iX_2 \mid X_1, X_2 \in \mathfrak{Z}(4, \mathbf{H})\}$  its complexification. In  $\mathfrak{Z}(4, \mathbf{H})$  and  $\mathfrak{Z}(4, \mathbf{H})^c$ , Jordan multiplications  $X \circ Y$  are defined by  $X \circ Y = (XY + YX)/2$ . Put  $\mathfrak{Z}(4, \mathbf{H})_0 = \{X \in \mathfrak{Z}(4, \mathbf{H}) \mid \text{tr}(X) = 0\}$  and let  $\mathfrak{Z}(4, \mathbf{H})_0^c = \{X \in \mathfrak{Z}(4, \mathbf{H})^c \mid \text{tr}(X) = 0\} = \{X_1 + iX_2 \mid X_1, X_2 \in \mathfrak{Z}(4, \mathbf{H})_0\}$  be the complexification of  $\mathfrak{Z}(4, \mathbf{H})_0$ . Now, we define a mapping  $g : \mathfrak{Z}^c \rightarrow \mathfrak{Z}(4, \mathbf{H})_0^c$  by

$$g \left( \begin{bmatrix} \xi_1 & a_3 + b_3 e & \bar{a}_2 - b_2 e \\ \bar{a}_3 - b_3 e & \xi_2 & a_1 + b_1 e \\ a_2 + b_2 e & \bar{a}_1 - b_1 e & \xi_3 \end{bmatrix} + i \begin{bmatrix} \eta_1 & c_3 + d_3 e & \bar{c}_2 - d_2 e \\ \bar{c}_3 - d_3 e & \eta_2 & c_1 + d_1 e \\ c_2 + d_2 e & \bar{c} - d_1 e & \eta_3 \end{bmatrix} \right)$$

$$= \begin{bmatrix} \lambda_1 & -d_1 & -d_2 & -d_3 \\ -\bar{d}_1 & \lambda_2 & a_3 & \bar{a}_2 \\ -\bar{d}_2 & \bar{a}_3 & \lambda_3 & a_1 \\ -\bar{d}_3 & a_2 & \bar{a}_2 & \lambda_4 \end{bmatrix} + i \begin{bmatrix} \mu_1 & b_1 & b_2 & b_2 \\ \bar{b}_1 & \mu_2 & c_3 & \bar{c}_2 \\ \bar{b}_2 & \bar{c}_3 & \mu_3 & c_1 \\ \bar{b}_3 & c_2 & \bar{c}_1 & \mu_4 \end{bmatrix}$$

where  $\xi_i, \eta_i \in \mathbf{R}$ ,  $a_i, b_i, c_i, d_i \in \mathbf{H}$  and  $\lambda_1 = (\xi_1 + \xi_2 + \xi_3)/2$ ,  $\lambda_2 = (\xi_1 - \xi_2 - \xi_3)/2$ ,  $\lambda_3 = (\xi_2 - \xi_1 - \xi_3)/2$ ,  $\lambda_4 = (\xi_3 - \xi_1 - \xi_2)/2$ ,  $\mu_1 = (\eta_1 + \eta_2 + \eta_3)/2$ ,  $\mu_2 = (\eta_1 - \eta_2 - \eta_3)/2$ ,  $\mu_3 = (\eta_2 - \eta_1 - \eta_3)/2$ ,  $\mu_4 = (\eta_3 - \eta_1 - \eta_2)/2$ . And we define a conjugate linear involutive transformation of  $\mathfrak{Z}^c$  by

$$\rho = \tau \gamma = \gamma \tau.$$

**Lemma 1** (1) ([8, Lemma 17]). *The mapping  $g : \mathfrak{Z}^c \rightarrow \mathfrak{Z}(4, \mathbf{H})_0^c$  is a  $\mathbf{C}$ -isomorphism satisfying*

$$g(X \times Y) = g(\gamma X) \circ g(\gamma Y) - ((\gamma X, Y)/4)E$$

where  $E$  is the  $4 \times 4$  unit matrix.

(2) Put  $(\mathfrak{Z}^c)_\rho = \{X \in \mathfrak{Z}^c \mid \rho X = X\}$ . Then  $g$  induces an  $\mathbf{R}$ -isomorphism

$$g : (\mathfrak{I}^c)_\rho \rightarrow \mathfrak{I}(4, \mathbf{H})_0.$$

We have shown in [8, Theorem 18] that a subgroup  $(E_6)_\rho$  of  $E_6$

$$(E_6)_\rho = \{\alpha \in E_6 \mid \rho\alpha = \alpha\rho\}$$

is isomorphic to the group  $Sp(4)/\mathbf{Z}_2$  (where  $Sp(4) = \{A \in M(4, \mathbf{H}) \mid A^*A = E\}$  is the symplectic group and  $\mathbf{Z}_2 = \{E, -E\}$ ) by the correspondence

$$\varphi : Sp(4) \rightarrow (E_6)_\rho, \quad \varphi(C)X = g^{-1}(C(gX)C^*), \quad X \in \mathfrak{I}^c$$

with  $\text{Ker}\varphi = \mathbf{Z}_2$ . Therefore its Lie algebra

$$(\mathfrak{e}_6)_\rho = \{\phi \in \mathfrak{e}_6 \mid \rho\phi = \phi\rho\}$$

is isomorphic to the symplectic Lie algebra  $\mathfrak{sp}(4) = \{C \in M(4, \mathbf{H}) \mid C^* = -C\}$  by the correspondence

$$d\varphi : \mathfrak{sp}(4) \rightarrow (\mathfrak{e}_6)_\rho, \quad d\varphi(C)X = g^{-1}(C(gX) - (gX)C), \quad X \in \mathfrak{I}^c.$$

Finally, we note that the complexification Lie algebra  $\mathfrak{e}_6^C$  of  $\mathfrak{e}_6$  :

$$\mathfrak{e}_6^C = \{\phi \in \text{Hom}_C(\mathfrak{I}^c, \mathfrak{I}^c) \mid (\phi X, X, X) = 0\}$$

is a simple Lie algebra over  $C$  of type  $E_6$ . And, for  $A, B \in \mathfrak{I}^c, A \vee B \in \mathfrak{e}_6^C$  is defined by

$$(A \vee B)X = ((B, X)/2)A + ((A, B)/6)X - 2B \times (A \times X), \quad X \in \mathfrak{I}^c.$$

**1.3. Compact Lie group  $E_7$  and its Lie algebra  $\mathfrak{e}_7$ .** Let  $\mathfrak{F}^C$  be a 56 dimensional vector space over  $C$  defined by

$$\mathfrak{F}^C = \mathfrak{I}^c \oplus \mathfrak{I}^c \oplus C \oplus C.$$

In  $\mathfrak{F}^C$ , the positive definite inner product  $\langle P, Q \rangle$  and the skew-symmetric inner product  $\{P, Q\}$  are defined respectively by

$$\begin{aligned} \langle P, Q \rangle &= \langle X, Z \rangle + \langle Y, W \rangle + \bar{\xi}\zeta + \bar{\eta}\omega, \\ \{P, Q\} &= (X, W) - (Y, Z) + \xi\omega - \eta\zeta \end{aligned}$$

for  $P = (X, Y, \xi, \eta), Q = (Z, W, \zeta, \omega) \in \mathfrak{F}^C$ .

For  $\phi \in \mathfrak{e}_6^C, A, B \in \mathfrak{I}^c$  and  $\nu \in C$ , we define a linear transformation  $\Phi(\phi, A, B, \nu)$  of  $\mathfrak{F}^C$  by

$$\begin{aligned} \Phi(\phi, A, B, \nu) \begin{bmatrix} X \\ Y \\ \xi \\ \eta \end{bmatrix} &= \begin{bmatrix} \phi - (\nu/3)1 & 2B & 0 & A \\ 2A & \phi' + (\nu/3)1 & B & 0 \\ 0 & A & \nu & 0 \\ B & 0 & 0 & -\nu \end{bmatrix} \begin{bmatrix} X \\ Y \\ \xi \\ \eta \end{bmatrix} \\ &= \begin{bmatrix} \phi X - (\nu/3)X + 2B \times Y + \eta A \\ 2A \times X + \phi' Y + (\nu/3)Y + \xi B \\ (A, Y) + \nu \xi \\ (B, X) - \nu \eta \end{bmatrix} \end{aligned}$$

where  $\phi' \in \mathfrak{e}_6^c$  denotes the skew-transpose of  $\phi$  with respect to the inner product  $(X, Y) : (\phi X, Y) + (X, \phi' Y) = 0$ . For  $P = (X, Y, \xi, \eta)$ ,  $Q = (Z, W, \zeta, \omega) \in \mathfrak{P}^c$ , we define a linear transformation  $P \times Q$  of  $\mathfrak{P}^c$  by

$$P \times Q = \Phi(\phi, A, B, \nu), \begin{cases} \phi = -(X \vee W + Z \vee Y)/2, \\ A = -(2Y \times W - \xi Z - \zeta X)/4, \\ B = (2X \times Z - \eta W - \omega Y)/4, \\ \nu = ((X, W) + (Z, Y) - 3(\xi\omega + \zeta\eta))/8. \end{cases}$$

And we define a submanifold  $\mathfrak{M}^c$  of  $\mathfrak{P}^c$ , called Freudenthal manifold, by

$$\begin{aligned} \mathfrak{M}^c &= \{P \in \mathfrak{P}^c \mid P \times P = 0, P \neq 0\} \\ &= \{P = (X, Y, \xi, \eta) \in \mathfrak{P}^c \mid \begin{cases} X \vee Y = 0, X \times X = \eta Y \\ Y \times Y = \xi X, (X, Y) = 3\xi\eta, \end{cases} P \neq 0\}. \end{aligned}$$

Now, as stated in the introduction, a simply connected compact simple Lie group of type  $E_7$  is given by

$$\begin{aligned} E_7 &= \{\alpha \in \text{Isoc}(\mathfrak{P}^c, \mathfrak{P}^c) \mid \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q, \langle \alpha P, \alpha Q \rangle = \langle P, Q \rangle\} \\ &= \{\alpha \in \text{Isoc}(\mathfrak{P}^c, \mathfrak{P}^c) \mid \alpha \mathfrak{M}^c = \mathfrak{M}^c, \{\alpha P, \alpha Q\} = \{P, Q\}, \langle \alpha P, \alpha Q \rangle = \langle P, Q \rangle\} \end{aligned}$$

and its Lie algebra is

$$\mathfrak{e}_7 = \{\Phi(\phi, A, -\bar{A}, \nu) \in \text{Hom}_{\mathbb{C}}(\mathfrak{P}^c, \mathfrak{P}^c) \mid \phi \in \mathfrak{e}_6, A \in \mathfrak{F}^c, \nu \in \mathbb{C}, \bar{\nu} = -\nu\}.$$

The group  $E_7$  contains a subgroup

$$\tilde{E}_6 = \{\alpha \in E_7 \mid \alpha(0, 0, 1, 0) = (0, 0, 1, 0)\}$$

which is isomorphic to the group  $E_6$  by the coorespondence

$$E_6 \ni \alpha \leftrightarrow \alpha = \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & \tau\alpha\tau & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \tilde{E}_6 \subset E_7$$

([3] Proposition 2). From now on, we identify the group  $E_6$  with the group  $\tilde{E}_6$ .

### 2. Subgroup $(E_7)_\rho$ of $E_7$ and its Lie algebra $(\mathfrak{e}_7)_\rho$

We define a conjugate linear involution  $\rho$  of  $\mathfrak{P}^c$  (used the same notation  $\rho$  in  $\mathfrak{F}^c$ ) by

$$\rho(X, Y, \xi, \eta) = (\rho X, \rho Y, \bar{\xi}, \bar{\eta})$$

and we shall investigate a subgroup  $(E_7)_\rho$  of  $E_7$

$$(E_7)_\rho = \{\alpha \in E_7 \mid \rho\alpha = \alpha\rho\}.$$

For this purpose, we give some preliminaries [8].

The quaternionic field  $\mathbf{H} = \mathbf{C} \oplus j\mathbf{C}$  is isomorphic to the space  $\mathfrak{H} = \{\mathbf{x} \in M(2, \mathbf{C}) \mid \mathbf{x}\mathbf{j} = \mathbf{j}\mathbf{x}\}$ , where  $\mathbf{j} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ , as an algebra by the correspondence  $k : \mathbf{H} \rightarrow \mathfrak{H}$ ,

$$k(a+jb) = \begin{bmatrix} a & -\bar{b} \\ b & \bar{a} \end{bmatrix} \quad a, b \in \mathbf{C}.$$

This mapping  $k$  is naturally extended to the mappings

$$k : M(4, \mathbf{H}) \rightarrow M(8, \mathbf{C}), \quad k : M(4, \mathbf{H})^c \rightarrow M(8, \mathbf{C})$$

$$k \left( \begin{matrix} x_{st} \\ y_{st} \end{matrix} \right) = \begin{pmatrix} k(x_{st}) \\ k(y_{st}) \end{pmatrix}, \quad k \left( \begin{matrix} x_{st} + iy_{st} \\ y_{st} \end{matrix} \right) = \begin{pmatrix} k(x_{st}) + ik(y_{st}) \\ k(y_{st}) \end{pmatrix}$$

$x_{st}, y_{st} \in \mathbf{H}$ , respectively. (In the latter equation,  $i$  in the left side is the complexification unit of  $\mathfrak{H}(4, \mathbf{H})^c$  and  $i$  in the right side is the imaginary unit of the field  $\mathbf{C}$ ).

Let  $SU(8) = \{A \in M(8, \mathbf{C}) \mid A^*A = E, \det A = 1\}$  be the special unitary group. The symplectic group  $Sp(4)$  is often regarded as a subgroup of  $SU(8)$  by

$$Sp(4) = k(Sp(4)) = \{A \in SU(8) \mid AJ^tA = J\}, \quad J = \begin{bmatrix} \mathbf{j} & 0 & 0 & 0 \\ 0 & \mathbf{j} & 0 & 0 \\ 0 & 0 & \mathbf{j} & 0 \\ 0 & 0 & 0 & \mathbf{j} \end{bmatrix}$$

**Lemma 2.** Any element  $D$  of the special unitary Lie algebra  $\mathfrak{su}(8) = \{D \in M(8, \mathbf{C}) \mid D^* = -D, \text{tr}(D) = 0\}$  is represented by the form

$$D = k(C) + ik(B) \quad C \in \mathfrak{sp}(4), B \in \mathfrak{H}(4, \mathbf{H})_0.$$

*Proof.* For  $D \in \mathfrak{su}(8)$ , put  $C_1 = (D - J\bar{D}J)/2$  and  $B_1 = -i(D + J\bar{D}J)/2$ , then

$$D = C_1 + iB_1 \quad C_1^* = -C_1, C_1J = J\bar{C}_1, B_1^* = B_1, B_1J = J\bar{B}_1, \text{tr}(B_1) = 0.$$

So,  $C = k^{-1}(C_1)$  and  $B = k^{-1}(B_1)$  are required ones.

**Proposition 3.** The Lie algebra  $(\mathfrak{e}_7)_\rho$  of the group  $(E_7)_\rho$  is

$$\begin{aligned} (\mathfrak{e}_7)_\rho &= \{\Phi \in \mathfrak{e}_7 \mid \rho\Phi = \Phi\rho\} \\ &= \{\Phi_\rho(\phi, A) \in \mathfrak{e}_7 \mid \phi \in \mathfrak{e}_6, \rho\phi = \phi\rho, A \in \mathfrak{H}^c, \rho A = A\} \\ &= \{\Phi_\rho(d\varphi(C), g^{-1}B) \in \mathfrak{e}_7 \mid C \in \mathfrak{sp}(4), B \in \mathfrak{H}(4, \mathbf{H})_0\}. \end{aligned}$$

where  $\Phi_\rho(\phi, A) = \Phi(\phi, A, -\gamma A, 0)$ . The Lie bracket  $[\Phi_1, \Phi_2]$  in  $(\mathfrak{e}_7)_\rho$  is given by

$$[\Phi_\rho(\phi_1, A_1), \Phi_\rho(\phi_2, A_2)] = \Phi_\rho(\phi, A)$$

where

$$\phi = [\phi_1, \phi_2] - 2A_1 \vee \bar{A}_2 + 2A_2 \vee \bar{A}_1, \quad A = \phi_1 A_2 - \phi_2 A_1.$$

And  $(e_7)_\rho$  is isomorphic to the special unitary Lie algebra  $\mathfrak{su}(8)$  by the correspondence

$$k(C) + ik(B) \in \mathfrak{su}(8) \xrightarrow{d\psi} \Phi_\rho(d\varphi(C), g^{-1}B) \in (e_7)_\rho$$

where  $C \in \mathfrak{sp}(4)$ ,  $B \in \mathfrak{I}(4, \mathbf{H})_0$ .

*Proof.* The first statements are easily shown. We shall show that  $d\psi : \mathfrak{su}(8) \rightarrow (e_7)_\rho$  is an isomorphism. (This is the direct consequence of the following section 4, however, here, we will give the direct proof).

(1) For  $C_1, C_2 \in \mathfrak{sp}(4)$ ,  $[k(C_1), k(C_2)] = k[C_1, C_2] \rightarrow \Phi_\rho(d\varphi[C_1, C_2], 0) = [\Phi_\rho(d\varphi(C_1), 0), \Phi_\rho(d\varphi(C_2), 0)]$ .

(2) For  $C \in \mathfrak{sp}(4)$ ,  $B \in \mathfrak{I}(4, \mathbf{H})_0$ ,  $[k(C), ik(B)] = ik[C, B] \rightarrow \Phi_\rho(0, g^{-1}[C, B]) = \Phi_\rho(0, d\varphi(C)(g^{-1}B)) = [\Phi_\rho(d\varphi(C), 0), \Phi_\rho(0, g^{-1}B)]$ .

(3) For  $B_1, B_2 \in \mathfrak{I}(4, \mathbf{H})_0$ ,  $[ik(B_1), ik(B_2)] = -k[B_1, B_2] \rightarrow \Phi_\rho(-d\varphi[B_1, B_2], 0)$ . On the other hand, (put  $A_1 = g^{-1}B_1$ ,  $A_2 = g^{-1}B_2$ )

$$\begin{aligned} [\Phi_\rho(0, g^{-1}B_1), \Phi_\rho(0, g^{-1}B_2)] &= [\Phi_\rho(0, A_1), \Phi_\rho(0, A_2)] \\ &= \Phi_\rho(-2A_1 \vee \bar{A}_2 + 2A_2 \vee \bar{A}_1, 0) \end{aligned}$$

$$\begin{aligned} \text{where } g((2A_1 \vee \bar{A}_2 - 2A_2 \vee \bar{A}_1)X) & \quad X \in \mathfrak{I}^c \\ &= g((2A_1 \vee \gamma A_2 - 2A_2 \vee \gamma A_1)X) \\ &= g((\gamma A_2, X)A_1 + ((A_1, \gamma A_2)/3)X - 4\gamma A_2 \times (A_1 \times X) \\ & \quad - ((\gamma A_1, X)A_2 + ((A_2, \gamma A_1)/3)X - 4\gamma A_1 \times (A_2 \times X))) \\ &= (\gamma A_2, X)gA_1 + ((A_1, \gamma A_2)/3)gX - 4gA_2 \circ g(\gamma A_1 \times \gamma X) + (A_2, A_1 \times X)E \\ & \quad - ((\gamma A_1, X)gA_2 + ((A_2, \gamma A_1)/3)gX - 4gA_1 \circ g(\gamma A_2 \times \gamma X) + (A_1, A_2 \times X)E) \\ &= (\gamma A_2, X)B_1 - 4B_2 \circ (B_1 \circ gX - ((\gamma A_1, X)/4)E) \\ & \quad - ((\gamma A_1, X)B_2 - 4B_1 \circ (B_2 \circ gX - ((\gamma A_2, X)/4)E)) \\ &= 4B_1 \circ (B_2 \circ gX) - 4B_2 \circ (B_1 \circ gX) \\ &= B_1 B_2 (gX) + B_1 (gX) B_2 + B_2 (gX) B_1 + (gX) B_2 B_1 \\ & \quad - (B_2 B_1 (gX) + B_2 (gX) B_1 + B_1 (gX) B_2 + (gX) B_1 B_2) \\ &= [B_1, B_2](gX) - (gX)[B_1, B_2] = g((d\varphi[B_1, B_2])X). \end{aligned}$$

Therefore we have  $\Phi_\rho(-d\varphi[B_1, B_2], 0) = [\Phi_\rho(0, g^{-1}B_1), \Phi_\rho(0, g^{-1}B_2)]$ . Thus Proposition 3 has been proved.

### 3. $C$ -isomorphism $\chi$ between $\mathfrak{P}^c$ and $\mathfrak{E}(8, C) \oplus \mathfrak{E}(8, C)$

Let  $\mathfrak{E}(8, C)$  denote the 28 dimensional vector space over  $C$  of all  $8 \times 8$  complex skew-symmetric matrices :

$$\mathfrak{E}(8, C) = \{S \in M(8, C) \mid {}^t S = -S\}.$$

We give a  $C$ -isomorphism  $\chi : \mathfrak{B}^c \rightarrow \mathfrak{S}(8, C) \oplus \mathfrak{S}(8, C)$  by

$$\chi = h\epsilon\tilde{g}\gamma_2$$

where  $\gamma_2, \tilde{g}, \epsilon, h$  are  $C$ -isomorphisms defined by

$$\begin{aligned} \gamma_2 : \mathfrak{B}^c &\rightarrow \mathfrak{B}^c, \gamma_2(X, Y, \xi, \eta) = (X, \gamma Y, \xi, \eta), \\ \tilde{g} : \mathfrak{B}^c &\rightarrow \mathfrak{S}(4, \mathbf{H})^c \oplus \mathfrak{S}(4, \mathbf{H})^c, \\ &\tilde{g}(X, Y, \xi, \eta) = (gX - (\xi/2)E, gY - (\eta/2)E), \\ \epsilon : \mathfrak{S}(4, \mathbf{H})^c \oplus \mathfrak{S}(4, \mathbf{H})^c &\rightarrow \mathfrak{S}(4, \mathbf{H})^c \oplus \mathfrak{S}(4, \mathbf{H})^c, \epsilon(M + iN, M' + iN') \\ &= (M + iM', N + iN'), \text{ where } M, N, M', N' \in \mathfrak{S}(4, \mathbf{H}), \\ h : \mathfrak{S}(4, \mathbf{H})^c \oplus \mathfrak{S}(4, \mathbf{H})^c &\rightarrow \mathfrak{S}(8, C) \oplus \mathfrak{S}(8, C), h(K, L) = (k(K)J, k(L)J). \end{aligned}$$

**Remark.** put  $(\mathfrak{B}^c)_\rho = \{P \in \mathfrak{B}^c \mid \rho P = P\}$ . Then the complexification  $((\mathfrak{B}^c)_\rho)^c = \{P_1 + iP_2 \mid P_1, P_2 \in (\mathfrak{B}^c)_\rho\}$  is  $\mathfrak{B}^c$ . Now, the restriction  $\chi' = \chi \mid (\mathfrak{B}^c)_\rho : (\mathfrak{B}^c)_\rho \rightarrow \mathfrak{S}(8, C)$  of  $\chi$  is

$$\chi'(P) = \chi'(X, Y, \xi, \eta) = k(gX - (\xi/2)E)J + ik(g(\gamma Y) - (\eta/2)E)J, P \in (\mathfrak{B}^c)_\rho$$

and the original  $\chi$  is the complexification of this  $\chi'$ .

#### 4. Homomorphism $\psi : SU(8) \rightarrow (E_7)_\rho$

We define a homomorphism  $\psi : SU(8) \rightarrow (E_7)_\rho$  by

$$\psi(A)P = \chi^{-1}(A(\chi(P))'A), \quad P \in \mathfrak{B}^c.$$

First of all, we must show  $\psi(A) \in (E_7)_\rho$  for  $A \in SU(8)$ . To show this, it suffices to prove for their Lie algebras (because  $SU(8)$  is connected), that is, the differential homomorphism  $d\psi : \mathfrak{su}(8) \rightarrow \text{Hom}_C(\mathfrak{B}^c, \mathfrak{B}^c)$  of  $\psi$  defined by

$$d\psi(D)P = \chi^{-1}(D(\chi(P)) + (\chi(P))'D), \quad P \in \mathfrak{B}^c$$

coincides with the mapping  $d\psi : \mathfrak{su}(8) \rightarrow (\mathfrak{e}_7)_\rho$  defined in Proposition 3.

(1) For  $D = k(C), C \in \mathfrak{sp}(4), (X, Y, \xi, \eta) \in \mathfrak{B}^c, d\psi(k(C))(X, Y, \xi, \eta)$  is

$$\begin{aligned} \begin{bmatrix} X \\ Y \\ \xi \\ \eta \end{bmatrix} &\xrightarrow{\gamma_2} \begin{bmatrix} X \\ \gamma Y \\ \xi \\ \eta \end{bmatrix} \xrightarrow{\tilde{g}} \begin{bmatrix} gX - (\xi/2)E \\ g(\gamma Y) - (\eta/2)E \end{bmatrix} \stackrel{\text{put}}{=} \begin{bmatrix} M + iN \\ M' + iN' \end{bmatrix}, \quad M, N, M', N' \in \mathfrak{S}(4, \mathbf{H}), \\ \xrightarrow{\epsilon} \begin{bmatrix} M + iM' \\ N + iN' \end{bmatrix} &\xrightarrow{h} \begin{bmatrix} k(M + iM')J \\ k(N + iN')J \end{bmatrix} \rightarrow \begin{bmatrix} k(C)k(M + iM')J + k(M + iM')J'k(C) \\ k(C)k(N + iN')J + k(N + iN')J'k(C) \end{bmatrix} \\ &= \begin{bmatrix} k(C(M + iM'))J + k((M + iM')C^*)J \\ k(C(N + iN'))J + k((N + iN')C^*)J \end{bmatrix} \xrightarrow{h^{-1}} \begin{bmatrix} C(M + iM') + (M + iM')C^* \\ C(N + iN') + (N + iN')C^* \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \xrightarrow{\varepsilon^{-1}} & \begin{bmatrix} C(M+iN)-(M+iN)C \\ C(M'+iN')-(M'+iN')C \end{bmatrix} = \begin{bmatrix} C(gX-(\xi/2)E)-(gX-(\xi/2)E)C \\ C(g(\gamma Y)-(\eta/2)E)-(g(\gamma Y)-(\eta/2)E)C \end{bmatrix} \\ = & \begin{bmatrix} C(gX)-(gX)C \\ C(g(\gamma Y)-(\eta/2)E)-(g(\gamma Y)-(\eta/2)E)C \end{bmatrix} \xrightarrow{\tilde{g}^{-1}} \begin{bmatrix} g^{-1}(C(gX)-(gX)C) \\ g^{-1}(C(g(\gamma Y)-(\eta/2)E)-(g(\gamma Y)-(\eta/2)E)C) \\ 0 \\ 0 \end{bmatrix} \xrightarrow{\gamma_2^{-1}} \begin{bmatrix} d\varphi(C)X \\ \gamma d\varphi(C)\gamma Y \\ 0 \\ 0 \end{bmatrix} \\ = & \begin{bmatrix} d\varphi(C)X \\ \tau d\varphi(C)\tau Y \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} d\varphi(C) & 0 & 0 & 0 \\ 0 & \tau d\varphi(C)\tau & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ \xi \\ \eta \end{bmatrix} = \Phi_\rho(d\varphi(C), 0) \begin{bmatrix} X \\ Y \\ \xi \\ \eta \end{bmatrix}. \end{aligned}$$

Thus we have  $d\psi(k(C)) = \Phi_\rho(d\varphi(C), 0)$ .

$$(2) \text{ For } D = ik(B), B = gA = \begin{bmatrix} 0 & -p & 0 & 0 \\ -\bar{p} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, A = iF_1(pe), p \in \mathbf{H},$$

$d\psi(ik(B))(0, 0, 1, 0)$  is

$$\begin{aligned} & \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \xrightarrow{\tilde{g}\gamma_2} \begin{bmatrix} -E/2 \\ 0 \end{bmatrix} \xrightarrow{\varepsilon} \begin{bmatrix} -E/2 \\ 0 \end{bmatrix} \xrightarrow{h} \begin{bmatrix} -J/2 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} -(ik(B)J + Ji^t k(B))/2 \\ 0 \end{bmatrix} \\ & = \begin{bmatrix} -ik(B)J \\ 0 \end{bmatrix} \xrightarrow{h^{-1}} \begin{bmatrix} -iB \\ 0 \end{bmatrix} \xrightarrow{\varepsilon^{-1}} \begin{bmatrix} 0 \\ -B \end{bmatrix} \xrightarrow{\tilde{g}^{-1}} \begin{bmatrix} 0 \\ -A \\ 0 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 \\ A \\ 0 \\ 0 \end{bmatrix} \\ & = \begin{bmatrix} 0 & 2A & 0 & A \\ 2A & 0 & A & 0 \\ 0 & A & 0 & 0 \\ A & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \Phi_\rho(0, A) \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}. \end{aligned}$$

And  $d\psi(ik(B))(X, 0, 0, 0)$  is

$$\begin{bmatrix} X \\ 0 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{\text{put}} \begin{bmatrix} \begin{bmatrix} \xi_1 & a_3 + b_3e & \bar{a}_2 - b_2e \\ \bar{a}_3 - b_3e & \xi_2 & a_1 + b_1e \\ a_2 + b_2e & \bar{a}_1 - b_1e & \xi_3 \end{bmatrix} + i \begin{bmatrix} \eta_1 & c_3 + d_3e & \bar{c}_2 - d_2e \\ \bar{c}_3 - d_3e & \eta_2 & c_1 + d_1e \\ c_2 + d_2e & \bar{c}_2 - d_1e & \eta_3 \end{bmatrix} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} & \xrightarrow{\underline{g}_1} \left[ \begin{array}{cccc} \lambda_1 & -d_1 & -d_2 & -d_3 \\ -\bar{d}_1 & \lambda_2 & a_3 & \bar{a}_2 \\ -\bar{d}_2 & \bar{a}_3 & \lambda_3 & a_1 \\ -\bar{d}_3 & a_2 & \bar{a}_1 & \lambda_4 \end{array} \right] + i \left[ \begin{array}{cccc} \mu_1 & b_1 & b_2 & b_3 \\ \bar{b}_1 & \mu_2 & c_3 & \bar{c}_2 \\ \bar{b}_2 & \bar{c}_3 & \mu_3 & c_1 \\ \bar{b}_3 & c_2 & \bar{c}_1 & \mu_4 \end{array} \right] \xrightarrow{\text{put}} \begin{bmatrix} M+iN \\ 0 \end{bmatrix} \\ & \xrightarrow{\underline{\varepsilon}} \begin{bmatrix} M \\ N \end{bmatrix} \xrightarrow{\underline{h}} \begin{bmatrix} k(M)J \\ k(N)J \end{bmatrix} \rightarrow \begin{bmatrix} ik(B)k(M)J+k(M)Ji'k(B) \\ ik(B)k(N)J+k(N)Ji'k(B) \end{bmatrix} = \begin{bmatrix} ik(BM+MB)J \\ ik(BN+NB)J \end{bmatrix} \\ & \xrightarrow{\underline{h}^{-1}} \begin{bmatrix} i(BM+MB) \\ i(BN+NB) \end{bmatrix} \xrightarrow{\underline{\varepsilon}^{-1}} \begin{bmatrix} 0 \\ (BM+MB)+i(BN+NB) \end{bmatrix} \\ & = \left[ \begin{array}{cccc} & & & 0 \\ 2(p, d_1) & -\xi_1 p & -pa_3 & -p\bar{a}_2 \\ -\xi_1 \bar{p} & 2(p, d_1) & \bar{p}d_2 & \bar{p}d_3 \\ -\bar{a}_3 \bar{p} & \bar{d}_2 p & 0 & 0 \\ -a_2 \bar{p} & \bar{d}_3 p & 0 & 0 \end{array} \right] + i \left[ \begin{array}{cccc} & & & 0 \\ -2(p, b_1) & -\eta_1 p & -pc_3 & -p\bar{c}_2 \\ -\eta_1 \bar{p} & -2(p, b_1) & -\bar{p}b_2 & -\bar{p}b_3 \\ -\bar{c}_3 \bar{p} & -\bar{b}_2 p & 0 & 0 \\ -c_2 \bar{p} & -b_3 p & 0 & 0 \end{array} \right] \\ & \xrightarrow{\underline{g}_2^{-1}} \left[ \begin{array}{cccc} & & & 0 \\ 2(p, d_1) & \bar{p}d_2-(p\bar{c}_2)e & \bar{p}d_3+(pc_3)e & \\ \bar{d}_2 p+(p\bar{c}_2)e & 0 & -\eta_1 pe & \\ \bar{d}_3 p-(pc_3)e & \eta_1 pe & 0 & \\ & & & 0 \\ & & & -2(p, d_1)+2i(p, b_1) \end{array} \right] + i \left[ \begin{array}{cccc} & & & 0 \\ -2(p, b_1) & -\bar{p}b_2+(p\bar{a}_2)e & -\bar{p}b_3-(pa_2)e & \\ -\bar{b}_2 p-(p\bar{a}_2)e & 0 & \xi_1 pe & \\ -\bar{b}_3 p+(pa_3)e & -\xi_1 pe & 0 & \\ & & & 0 \end{array} \right] \\ & \xrightarrow{\underline{\gamma}_2^{-1}} \left[ \begin{array}{cccc} & & & 0 \\ 2(p, d_1) & \bar{p}d_2+(p\bar{c}_2)e & \bar{p}d_3-(pc_3)e & \\ \bar{d}_2 p-(p\bar{c}_2)e & 0 & \eta_1 pe & \\ \bar{d}_3 p+(pc_3)e & -\eta_1 pe & 0 & \\ & & & 0 \\ & & & -2(p, d_1)+2i(p, b_1) \end{array} \right] + i \left[ \begin{array}{cccc} & & & 0 \\ -2(p, b_1) & -\bar{p}b_2-(p\bar{a}_2)e & -\bar{p}b_3+(pa_3)e & \\ -b_2 p+(p\bar{a}_2)e & 0 & -\xi_1 pe & \\ -\bar{b}_3 p-(pa_3)e & \xi_1 pe & 0 & \\ & & & 0 \end{array} \right] \\ & \stackrel{**}{=} \begin{bmatrix} 0 \\ 2A \times X \\ 0 \\ (A, X) \end{bmatrix} = \begin{bmatrix} 0 & 2A & 0 & A \\ 2A & 0 & A & 0 \\ 0 & A & 0 & 0 \\ A & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} X \\ 0 \\ 0 \\ 0 \end{bmatrix} = \Phi_\rho(\phi, A) \begin{bmatrix} X \\ 0 \\ 0 \\ 0 \end{bmatrix}. \quad (**, \text{ in fact,} \end{aligned}$$

$$\begin{aligned} 2A \times X &= 2iF_1(pe) \times (\sum_{i=1}^3 (\xi_i E_i + F_i(a_i + b_i e)) + i \sum_{i=1}^3 (\eta_i E_i + F_i(c_i + d_i e))) \\ &= i(-\xi_1 F_1(pe) - 2(p, b_1) E_1 + F_3((pe)(a_2 + b_2 e)) + F_2((a_3 + b_3 e)(pe))) \\ &\quad - (-\eta_1 F_1(pe) - 2(p, d_1) E_1 + F_3((pe)(c_2 + d_2 e)) + F_2((c_3 + d_3 e)(pe))) \\ &= (2(p, d_1) E_1 + \eta_1 F_1(pe) + F_2(\bar{d}_3 p + (pc_3)e) + F_3(\bar{p}d_2 + (p\bar{c}_2)e)) \\ &\quad + i(-2(p, b_1) E_1 - \xi_1 F_1(pe) - F_2(\bar{b}_3 p + (pa_3)e) - F_3(\bar{p}b_2 + (p\bar{a}_2)e)), \\ (A, X) &= (2iF_1(pe), (\sum_{i=1}^3 (\xi_i E_i + F_i(a_i + b_i e)) + i \sum_{i=1}^3 (\eta_i E_i + F_i(c_i + d_i e)))) \\ &= -2(p, d_1) + 2i(p, b_1). \end{aligned}$$

Similarly, we have  $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \rightarrow \Phi_\rho(0, A) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ Y \\ 0 \\ 0 \end{bmatrix} \rightarrow \Phi_\rho(0, A) \begin{bmatrix} 0 \\ Y \\ 0 \\ 0 \end{bmatrix}$ . Thus

we have  $d\psi(ik(B)) = \Phi_\rho(0, A) = \Phi_\rho(0, g^{-1}B)$ .

(3) For other generators of  $\mathfrak{su}(8)$ , that is, for  $D = ik(B)$ ,  $B = gA$ , where  $A = iF_j(p_e)$ ,  $p \in \mathbf{H}$ ,  $j = 2, 3$ ,  $A = E_1 - E_j$ ,  $j = 2, 3$  and  $A = F_j(p)$ ,  $P \in \mathbf{H}$ ,  $j = 1, 2, 3$ , we have also

$$d\psi(ik(B)) = \Phi_\rho(0, g^{-1}B)$$

by the straightforward calculations as the above (1), (2).

All together (1),(2),(3). we see that the homomorphism  $\psi : SU(8) \rightarrow (E_7)_\rho$  is well-defined.

### 5. Isomorphism $(E_7)_\rho \cong SU(8)/Z_2$

Our aim of this section is to prove that  $\psi : SU(8) \rightarrow (E_7)_\rho$  is onto.

**Lemma 4.** For  $a \in \mathbf{R}$ , the linear transformation of  $\mathfrak{B}^C$  defined by

$$\alpha_i(a) = \begin{bmatrix} 1 + (\cos|a| - 1)p_i & (2a/|a|)\sin|a|E_i & 0 & -(a/|a|)\sin|a|E_i \\ -(2a/|a|)\sin|a|E_i & 1 + (\cos|a| - 1)p_i & (a/|a|)\sin|a|E_i & 0 \\ 0 & -(a/|a|)\sin|a|E_i & \cos|a| & 0 \\ (a/|a|)\sin|a|E_i & 0 & 0 & \cos|a| \end{bmatrix}$$

(if  $a = 0$ , then  $(a/|a|)\sin|a|$  means 0) belongs to the group  $\psi(SU(8))$ ,  $i = 1, 2, 3$ , where the mapping  $p_i : \mathfrak{F}^C \rightarrow \mathfrak{F}^C$  is defined by

$$p_i \begin{bmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{bmatrix} = \begin{bmatrix} \xi_1 & \delta_{i3}x_3 & \delta_{i2}\bar{x}_2 \\ \delta_{i3}\bar{x}_3 & \xi_2 & \delta_{i1}x_1 \\ \delta_{i2}x_2 & \delta_{i1}\bar{x}_1 & \xi_3 \end{bmatrix}$$

(where  $\delta_{ij}$  is the Kronecker's delta) and the action of  $\alpha_i(a)$  on  $\mathfrak{B}^C$  is defined as similar to that of  $\Phi(\phi, A, B, \nu)$  in §1.3.

*Proof.* For  $\Phi_\rho(0, -aE_i) \in d\psi(\mathfrak{su}(8))$ , we have  $\alpha_i(a) = \exp \Phi_\rho(0, -aE_i)$ , hence  $\alpha_i(a) \in \psi(SU(8))$ ,  $i = 1, 2, 3$ .

We define a subspace  $(\mathfrak{M}^C)_\rho$  of  $\mathfrak{B}^C$  by

$$(\mathfrak{M}^C)_\rho = \{P \in \mathfrak{M}^C \mid \rho P = P\}.$$

**Lemma 5.** Any element  $P \in (\mathfrak{M}^C)_\rho$  can be transformed in a real diagonal form by a certain element  $\alpha \in \psi(SU(8))$ :

$\alpha P = (X, Y, \xi, \eta)$ ,  $X, Y$  are real diagonal forms and  $\xi, \eta \in \mathbf{R}$ .

Moreover we can choose  $\alpha \in \psi(SU(8))$  so that  $\xi > 0$ .

*Proof.* Let  $P = (X, Y, \xi, \eta) \in (\mathfrak{M}^c)_\rho$ . First assume that  $\xi \neq 0$ . Then  ${}_\rho Y = Y$ ,  $\bar{\xi} = \xi$ ,  $\bar{\eta} = \eta$  and  $X = (Y \times Y)/\xi$ . Since  $g(\gamma Y) \in \mathfrak{I}(4, \mathbf{H})_0$  (Lemma 1 (2)), we can choose  $C \in Sp(4)$  so that  $C(g(\gamma Y))C^*$  is real diagonal, so  $\gamma\phi(C)\gamma Y = g^{-1}(C(g(\gamma Y))C^*)$  has a real diagonal form. In this case,  $\phi(C)X = \phi(C)((Y \times Y)/\xi) = (\gamma\phi(C)\gamma Y \times \gamma\phi(C)\gamma Y)/\xi$  is also real diagonal, hence  $\psi(C)P$  is a diagonal form. In the case of  $\eta \neq 0$ , the statement is also valid. Next we consider the case  $P = (X, Y, 0, 0)$ ,

$Y \neq 0$ . Choose  $C \in Sp(4)$  such that  $\gamma\phi(C)\gamma Y = \begin{bmatrix} \eta_1 & 0 & 0 \\ 0 & \eta_2 & 0 \\ 0 & 0 & \eta_3 \end{bmatrix}$ ,  $\eta_i \in \mathbf{R}$ .

Since  $\gamma\phi(C)\gamma Y \neq 0$ , we may assume  $\eta_1 \neq 0$ . Operate  $\alpha_1(-\pi/2) \in \psi(SU(8))$  of Lemma 4 on  $\psi(C)P$ . Then

$$\alpha_1(-\pi/2)\psi(C)P = (*, *, \eta_1, *).$$

So, we can reduce to the first case  $\xi \neq 0$ . In the case of  $P = (X, Y, 0, 0)$ ,  $X \neq 0$ , the statement is also valid. If  $\xi < 0$ , then operate  $\alpha_1(\pi)$  on  $\alpha P$ . Then  $\xi$  becomes a positive number. Noting  $\alpha_3(-\pi/2)\alpha_2(\pi/2)\alpha_1(\pi/2)(0, 0, 0, 1) = (0, 0, 1, 0)$ , then we can always reduce to the case  $\xi \neq 0$ . Thus Lemma 5 is proved.

Now, we shall prove that  $\psi : SU(8) \rightarrow (E_7)_\rho$  is onto. For a given  $\alpha \in (E_7)_\rho$ , consider an element  $P = \alpha(0, 0, 1, 0) \in (\mathfrak{M})_\rho$ . From Lemma 5, there exists  $\beta \in \psi(SU(8))$  such that

$$\beta P = \left( (1/\xi) \begin{bmatrix} \eta_2\eta_3 & 0 & 0 \\ 0 & \eta_3\eta_1 & 0 \\ 0 & 0 & \eta_1\eta_2 \end{bmatrix}, \begin{bmatrix} \eta_1 & 0 & 0 \\ 0 & \eta_2 & 0 \\ 0 & 0 & \eta_3 \end{bmatrix}, \xi, (\eta_1\eta_2\eta_3)/\xi^2 \right), \xi > 0, \eta_i \in \mathbf{R}.$$

Then the condition  $\langle P, P \rangle = 1$  is

$$(1 + (|\eta_1|/\xi)^2)(1 + (|\eta_2|/\xi)^2)(1 + (|\eta_3|/\xi)^2) = 1/\xi^2.$$

Choose  $r_i \in \mathbf{R}$ ,  $\pi/2 > r_i \geq 0$ , such that  $\tan r_i = |\eta_i|/\xi$ ,  $i = 1, 2, 3$ . Then we have

$$\xi = \cos r_1 \cos r_2 \cos r_3.$$

Put  $a_i = (\eta_i/|\eta_i|)r_i$  (if  $\eta_i = 0$ , then  $(\eta_i/|\eta_i|)r_i$  means 0),  $i = 1, 2, 3$ . Then we have

$$\beta P = \alpha_3(a_3)\alpha_2(a_2)\alpha_1(a_1)(0, 0, 1, 0)$$

(cf. [3, Theorem 9]), that is,

$$\alpha_1(a_1)^{-1}\alpha_2(a_2)^{-1}\alpha_3(a_3)^{-1}\beta\alpha(0, 0, 1, 0) = (0, 0, 1, 0).$$

Hence  $\tilde{\alpha} = \alpha_1(a_1)^{-1}\alpha_2(a_2)^{-1}\alpha_3(a_3)^{-1}\beta\alpha \in E_6$ , moreover  $\rho\tilde{\alpha} = \tilde{\alpha}\rho$ , therefore

$$\tilde{\alpha} = \alpha_1(a_1)^{-1}\alpha_2(a_2)^{-1}\alpha_3(a_3)^{-1}\beta\alpha \in (E_6)_\rho = \varphi(Sp(4)) \subset \psi(SU(8)).$$

Since  $\alpha_i(a_i)$  and  $\beta \in \psi(SU(8))$ ,  $\alpha$  is also  $\alpha \in \psi(SU(8))$ , that is,  $\psi$  is onto.

Finally,  $\text{Ker } \psi = \{E, -E\}$  is easily obtained. Thus we have the following theorem which was our first aim.

**Theorem 6.** *The subgroup  $(E_7)_\rho = \{\alpha \in E_7 \mid \rho\alpha = \alpha\rho\}$  of the group  $E_7$  is isomorphic to the group  $SU(8)/Z_2$ , where  $Z_2 = \{E, -E\}$ .*

## II. Lie group $E_{7(G)}$

### 6. Preliminaries

#### 6.1. Split Cayley algebra $\mathfrak{C}'$ and split exceptional Jordan algebra $\mathfrak{J}'$ .

Let  $\mathfrak{C}' = \mathbf{H} \oplus \mathbf{H}e$  denote the split Cayley algebra with the multiplication

$$(a + be)(c + de) = (ac + \bar{d}b) + (b\bar{c} + da)e.$$

And let  $\mathfrak{J}' = \{X \in M(3, \mathfrak{C}') \mid X^* = X\}$  denote the split exceptional Jordan algebra with the multiplication  $X \circ Y = (XY + YX)/2$ . In  $\mathfrak{J}'$  also, the inner product  $(X, Y)' = \text{tr}(X \circ Y)$ , the crossed product  $X \times Y$ , the trilinear form  $(X, Y, Z)' = (X, Y \times Z)'$ , and the determinant  $\det X = (X, X, X)'/3$  are defined as same as in  $\mathfrak{J}$  and  $\mathfrak{J}^c$ . Moreover we define a positive definite inner product  $(X, Y)$  in  $\mathfrak{J}'$  by

$$(X, Y) = (X, \gamma Y)' = (\gamma X, Y)'$$

where  $\gamma : \mathfrak{J}' \rightarrow \mathfrak{J}'$  is the involution defined as  $\gamma : \mathfrak{J} \rightarrow \mathfrak{J}$  in §1.1.

**6.2. Lie group  $E_{6(6)}$  and subgroup  $(E_{6(6)})_K$ .** We have shown in [6] that the group

$$E_{6(6)} = \{\alpha \in \text{Iso}_{\mathbf{R}}(\mathfrak{J}', \mathfrak{J}') \mid \det \alpha X = \det X\}$$

is a connected non-compact simple Lie group of type  $E_{6(6)}$  and found a subgroup of type  $C_4$  in  $E_{6(6)}$ . To find this subgroup, we used a mapping  $f : \mathfrak{J}' \rightarrow \mathfrak{J}(4, \mathbf{H})_0$ ,

$$f \begin{bmatrix} \xi_1 & a_3 + b_3e & \bar{a}_2 - b_2e \\ \bar{a}_3 - b_3e & \xi_2 & a_1 + b_1e \\ a_2 + b_2e & \bar{a}_1 - b_1e & \xi_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 & b_1 & b_2 & b_3 \\ \bar{b}_1 & \lambda_2 & a_3 & \bar{a}_2 \\ \bar{b}_2 & \bar{a}_3 & \lambda_3 & a_1 \\ \bar{b}_3 & a_2 & \bar{a}_1 & \lambda_4 \end{bmatrix}$$

where  $\xi_i \in \mathbf{R}$ ,  $a_i, b_i \in \mathbf{H}$  and  $\lambda_1 = (\xi_1 + \xi_2 + \xi_3)/2$ ,  $\lambda_2 = (\xi_1 - \xi_2 - \xi_3)/2$ ,  $\lambda_3 = (\xi_2 - \xi_1 - \xi_3)/2$ ,  $\lambda_4 = (\xi_3 - \xi_1 - \xi_2)/2$ .

**Lemma 7** ([6, Lemma 1]). *The mapping  $f : \mathfrak{F} \rightarrow \mathfrak{F}(4, \mathbf{H})_0$  is a  $\mathbf{R}$ -isomorphism satisfying*

$$f(X \times Y) = f(\gamma X) \circ f(\gamma Y) - ((X, Y)/4)E.$$

Now, we have shown that a subgroup  $(E_{6(6)})_K$  of  $E_{6(6)}$

$$(E_{6(6)})_K = E_{6(6)} \cap O(\mathfrak{F}) = \{a \in E_{6(6)} \mid (aX, aY) = (X, Y)\}$$

is isomorphic to the group  $Sp(4)/\mathbf{Z}_2$  by the correspondence

$$\varphi : Sp(4) \longrightarrow (E_{6(6)})_{K'} \quad \varphi(C)X = f^{-1}(C(fX)C^*), \quad X \in \mathfrak{F}'$$

with  $\text{Ker } \varphi = \mathbf{Z}_2 = \{E, -E\}$ . Therefore its Lie algebra

$$(\mathfrak{e}_{6(6)})_K = \{\phi \in \mathfrak{e}_{6(6)} \mid (\phi X, Y) = -(X, \phi Y)\}.$$

is isomorphic to  $\mathfrak{sp}(4)$  by the correspondence

$$d\varphi : \mathfrak{sp}(4) \longrightarrow (\mathfrak{e}_{6(6)})_{K'} \quad d\varphi(C)X = f^{-1}(C(fX) - (fX)C), \quad X \in \mathfrak{F}'.$$

Finally, we note that, for  $A, B \in \mathfrak{F}'$ ,  $A \vee B \in \mathfrak{e}_{6(6)}$  is defined by  $(A \vee B)X = ((B, X')/2)A + ((A, B')/6)X - 2B \times (A \times X)$ ,  $X \in \mathfrak{F}'$  analogously as  $A \vee B \in \mathfrak{e}_6^c$  in §1.2.

**7. Lie group  $E_{7(7)}$**

Let  $\mathfrak{F}'$  be a 56 dimensional vector space over  $\mathbf{R}$  defined by

$$\mathfrak{F}' = \mathfrak{F}' \oplus \mathfrak{F}' \oplus \mathbf{R} \oplus \mathbf{R}.$$

In  $\mathfrak{F}'$ , the symmetric inner product  $(P, Q)'$ , the skew-symmetric inner product  $\{P, Q\}'$  and one more positive definite inner product  $(P, Q)$  are defined respectively by

$$\begin{aligned} (P, Q)' &= (X, Z)' + (Y, W)' + \xi\zeta + \eta\omega, \\ \{P, Q\}' &= (X, W)' - (Y, Z)' + \xi\omega - \eta\zeta, \\ (P, Q) &= (X, Z) + (Y, W) + \xi\zeta + \eta\omega \end{aligned}$$

for  $P = (X, Y, \xi, \eta)$ ,  $Q = (Z, W, \zeta, \omega) \in \mathfrak{F}'$ .

For  $\phi \in \mathfrak{e}_{6(6)}$ ,  $A, B \in \mathfrak{F}'$  and  $\nu \in \mathbf{R}$ , a linear transformation  $\Phi(\phi, A, B, \nu)$  of  $\mathfrak{F}'$  is defined by

$$\Phi(\phi, A, B, \nu) \begin{bmatrix} X \\ Y \\ \xi \\ \eta \end{bmatrix} = \begin{bmatrix} \phi X - (\nu/3)X + 2B \times Y + \eta A \\ 2A \times X + \phi' Y + (\nu/3)Y + \xi B \\ (A, Y)' + \nu\xi \\ (B, X)' - \nu\eta \end{bmatrix}$$

where  $\phi' \in \mathfrak{e}_{6(6)}$  denotes the skew-transpose of  $\phi$  with respect to the inner product  $(X, Y) : (\phi X, Y) + (X, \phi' Y) = 0$ . For  $P, Q \in \mathfrak{F}'$ , a linear transformation

$$P \times Q = \mathcal{O}(\phi, A, B, \nu)$$

of  $\mathfrak{F}'$  is analogously defined as  $P \times Q$  in  $\mathfrak{F}$  or  $\mathfrak{F}^C$  (use  $(X, Y)$  instead of  $(X, Y)$ ) and define a submanifold  $\mathfrak{M}'$  of  $\mathfrak{F}'$  by

$$\mathfrak{M}' = \{P \in \mathfrak{F}' \mid P \times P = 0, P \neq 0\}.$$

Now, we define a group  $E_{7(7)}$  by

$$E_{7(7)} = \{\alpha \in \text{Iso}_R(\mathfrak{F}', \mathfrak{F}') \mid \alpha \mathfrak{M}' = \mathfrak{M}', \{\alpha P, \alpha Q\}' = \{P, Q\}'\}.$$

(Later, we see that this group  $E_{7(7)}$  is connected (Theorem 13), therefore it may be also defined by (see [4])

$$E_{7(7)} = \{\alpha \in \text{Iso}_R(\mathfrak{F}', \mathfrak{F}') \mid \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q\}.$$

The Lie algebra  $\mathfrak{e}_{7(7)}$  of the group  $E_{7(7)}$  is

$$\mathfrak{e}_{7(7)} = \{\mathcal{O}(\phi, A, B, \nu) \in \text{Hom}_R(\mathfrak{F}', \mathfrak{F}') \mid \phi \in \mathfrak{e}_{6(6)}, A, B \in \mathfrak{F}', \nu \in \mathbf{R}\}$$

(see [2],[3]). Since the complexification of the Lie algebra  $\mathfrak{e}_{7(7)}$  is  $\mathfrak{e}_{7^C}$ , the group  $E_{7(7)}$  is a simple Lie group of type  $E_7$ .

The group  $E_{7(7)}$  contains a subgroup

$$\tilde{E}_{6(6)} = \left\{ \alpha \in E_{7(7)} \mid \begin{array}{l} \alpha(0, 0, 1, 0) = (0, 0, 1, 0) \\ \alpha(0, 0, 0, 1) = (0, 0, 0, 1) \end{array} \right\}$$

which is isomorphic to the group  $E_{6(6)}$  by the correspondence

$$E_{6(6)} \ni \alpha \longleftrightarrow \alpha = \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & \alpha^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \tilde{E}_{6(6)} \subset E_{7(7)}$$

where  $\alpha'$  is the transpose of  $\alpha \in E_{6(6)}$  with respect to the inner product  $(X, Y) : (\alpha X, Y) = (X, \alpha' Y)$ .

In order to investigate the group  $E_{7(7)}$ , we consider a subgroup  $(E_{7(7)})_K$  of  $E_{7(7)}$ :

$$(E_{7(7)})_K = E_{7(7)} \cap O(\mathfrak{F}') = \{\alpha \in E_{7(7)} \mid (\alpha P, \alpha Q) = (P, Q)\}$$

**Proposition 8.** *The Lie algebra  $(\mathfrak{e}_{7(7)})_K$  of the group  $(E_{7(7)})_K$  is*

$$\begin{aligned} (\mathfrak{e}_{7(7)})_K &= \{\mathcal{O} \in \mathfrak{e}_{7(7)} \mid (\mathcal{O}P, Q) = -(P, \mathcal{O}Q)\} \\ &= \{\mathcal{O}'(\phi, A) \in \mathfrak{e}_{7(7)} \mid \phi \in (\mathfrak{e}_{6(6)})_K, A \in \mathfrak{F}'\} \\ &= \{\mathcal{O}'(d\varphi(C), f^{-1}B) \in \mathfrak{e}_{7(7)} \mid C \in \mathfrak{sp}(4), B \in \mathfrak{F}(4, \mathbf{H})_0\} \end{aligned}$$

where  $\Phi'(\phi, A) = \Phi(\phi, A, -\gamma A, 0)$ . And it is isomorphic to the special unitary Lie algebra  $\mathfrak{su}(8)$  by the correspondence

$$k(C) + ik(B) \in \mathfrak{su}(8) \xrightarrow{d\psi} \Phi'(d\varphi(C), f^{-1}B) \in (\mathfrak{e}_{7(7)})_K$$

where  $C \in \mathfrak{su}(4)$ ,  $B \in \mathfrak{J}(4, \mathbf{H})_0$ .

*Proof.* The proof is similar to that of Proposition 3 by using Lemma 7.

**8.  $\mathbf{R}$ -isomorphism  $\chi$  between  $\mathfrak{B}'$  and  $\mathfrak{E}(8, \mathbf{C})$**

We give an  $\mathbf{R}$ -isomorphism  $\chi: \mathfrak{B}' \rightarrow \mathfrak{E}(8, \mathbf{C})$  by

$$\chi = h\epsilon\bar{f}\gamma_2$$

(cf. Remark of §3), where  $\gamma_2, \bar{f}, \epsilon, h$  are  $\mathbf{R}$ -isomorphisms defined by

$$\begin{aligned} \gamma_2: \mathfrak{B}' &\rightarrow \mathfrak{B}', \quad \gamma_2(X, Y, \xi, \eta) = (X, \gamma Y, \xi, \eta), \\ \bar{f}: \mathfrak{B}' &\rightarrow \mathfrak{J}(4, \mathbf{H}) \oplus \mathfrak{J}(4, \mathbf{H}), \quad \bar{f}(X, Y, \xi, \eta) = (fX - (\xi/2)E, fY - (\eta/2)E), \\ \epsilon: \mathfrak{J}(4, \mathbf{H}) \oplus \mathfrak{J}(4, \mathbf{H}) &\rightarrow \mathfrak{J}(4, \mathbf{H})^c, \quad \epsilon(M, N) = M + iN, \\ h: \mathfrak{J}(4, \mathbf{H})^c &\rightarrow \mathfrak{E}(8, \mathbf{C}), \quad h(L) = k(L)J. \end{aligned}$$

**9. Homomorphism  $\psi: SU(8) \rightarrow (E_{7(7)})_K$**

We define a homomorphism  $\psi: SU(8) \rightarrow (E_{7(7)})_K$  by

$$\psi(A)P = \chi^{-1}(A(\chi(P))'A) \quad P \in \mathfrak{B}'.$$

To show that  $\psi(A) \in (E_{7(7)})_K$  for  $A \in SU(8)$ , it suffices to prove for their Lie algebras, that is, the differential homomorphism  $d\psi: \mathfrak{su}(8) \rightarrow \text{Hom}_{\mathbf{R}}(\mathfrak{B}', \mathfrak{B}')$  of  $\psi$ :

$$d\psi(D)P = \chi^{-1}(D(\chi(P)) + \chi(P)'D) \quad P \in \mathfrak{B}'$$

coincides with the mapping  $d\psi: \mathfrak{su}(8) \rightarrow (\mathfrak{e}_{7(7)})_K$  defined in Proposition 8. However the proof is the same as the proof of the section 4.

**10. Isomorphism  $(E_{7(7)})_K \cong SU(8)/Z_2$**

We shall show that  $\psi: SU(8) \rightarrow (E_{7(7)})_K$  is onto. For this purpose, if we prepare the following three lemmas, we can attain our aim by using the same way as the section 5.

**Lemma 9.** For  $a \in \mathbf{R}$ , linear transformations  $\alpha_i(a)$  of  $\mathfrak{B}'$  defined analogously as Lemma 4 belongs to  $\psi(SU(8))$ ,  $i = 1, 2, 3$ .

**Lemma 10.** Any element  $P \in (\mathfrak{M}')_K = \{P \in \mathfrak{M}' \mid (P, P) = 1\}$  can be transformed in a diagonal form by a certain element  $a \in \psi(SU(8))$ :

$$\alpha P = (X, Y, \xi, \eta) \quad X, Y \text{ are diagonal forms, } \xi > 0, \eta \in \mathbf{R}.$$

**Lemma 11.** *If  $\alpha \in (E_{7(\gamma)})_K$  satisfies  $\alpha(0, 0, 1, 0) = (0, 0, 1, 0)$ , then  $\alpha(0, 0, 0, 1) = (0, 0, 0, 1)$ .*

Now, since it is easy to see that the kernel of  $\psi: SU(8) \rightarrow (E_{7(\gamma)})_K$  is  $\{E, -E\}$ , thus we have

**Proposition 12.** *The group  $(E_{7(\gamma)})_K = \{\alpha \in E_{7(\gamma)} \mid (\alpha P, \alpha Q) = (P, Q)\}$  is isomorphic to the group  $SU(8)/Z_2$ , where  $Z_2 = \{E, -E\}$ .*

### 11. Polar decomposition of $E_{7(\gamma)}$

We define a linear transformation  $i'$  of  $\mathfrak{P}'$  by

$$i'(X, Y, \xi, \eta) = (Y, -X, \eta, -\xi).$$

Then  $i' \in E_{7(\gamma)}$ ,  $i'^2 = -1$  and  $\{P, Q\}' = -(\iota P, Q)' = (P, \iota Q)'$  for  $P, Q \in \mathfrak{P}'$ . Therefore put

$$v = \gamma i' = i' \gamma.$$

Then  $v \in E_{7(\gamma)}$ ,  $v^2 = -1$  and we have

$$\{P, Q\}' = -(vP, Q) = (P, vQ), \quad P, Q \in \mathfrak{P}'.$$

To give a polar decomposition of the group  $E_{7(\gamma)}$ , we prepare

**Lemma 13.** *The group  $E_{7(\gamma)}$  is an algebraic subgroup of the general linear group  $GL(56, \mathbf{R}) = \text{Isor}(\mathfrak{P}', \mathfrak{P}')$  and satisfies the condition that  $\alpha \in E_{7(\gamma)}$  implies  ${}^t\alpha \in E_{7(\gamma)}$ , where  ${}^t\alpha$  is the transpose of  $\alpha$  with respect to the inner product  $(P, Q): (\alpha P, Q) = (P, {}^t\alpha Q)$ .*

*Proof.* Since  $({}^t\alpha P, Q) = (P, \alpha Q) = (vP, \alpha Q)' = (\alpha^{-1}vP, Q)' = -(v\alpha^{-1}vP, Q)$  for  $\alpha \in E_{7(\gamma)}$ , we have

$${}^t\alpha = -v\alpha^{-1}v \in E_{7(\gamma)}.$$

And it is obvious that the group  $E_{7(\gamma)}$  is algebraic, because it is defined by algebraic relations  $\alpha\mathfrak{M}' = \mathfrak{M}'$ ,  $\{\alpha P, \alpha Q\}' = \{P, Q\}'$ .

Using Chevalley's lemma ([1, Lemma 2, p. 201]), we have a homeomorphism

$$\begin{aligned} E_{7(\gamma)} &\simeq (E_{7(\gamma)} \cap O(\mathfrak{P}')) \times \mathbf{R}^d \\ &\simeq SU(8)/Z_2 \times \mathbf{R}^d \quad (\text{Proposition 12}). \end{aligned}$$

Since  $E_{7(\gamma)}$  is a simple Lie group of type  $E_7$ , the dimension of  $E_{7(\gamma)}$  is 133. Hence the dimension  $d$  of the Euclidean part and the Cartan index  $i$  of  $E_{7(\gamma)}$  are calculated respectively as follows:

$$d = \dim E_{7(7)} - \dim SU(8) = 133 - 63 = 70,$$

$$i = \dim E_{7(7)} - 2\dim SU(8) = 133 - 2 \times 63 = 7.$$

Thus we have the following theorem which was our second aim.

**Theorem 14.** *The group  $E_{7(7)}$  is homeomorphic to the topological product of the group  $SU(8)/\mathbf{Z}_2$  and a Euclidean space  $\mathbf{R}^{70}$ :*

$$E_{7(7)} \simeq SU(8)/\mathbf{Z}_2 \times \mathbf{R}^{70}.$$

*In particular,  $E_{7(7)}$  is a connected non-compact simple Lie group of type  $E_{7(7)}$ .*

### 12. Center $z(E_{7(7)})$ of $E_{7(7)}$

**Theorem 15.** *The center  $z(E_{7(7)})$  of the group  $E_{7(7)}$  is the cyclic group of order 2:*

$$z(E_{7(7)}) = \{1, -1\}.$$

*Proof.* Let  $\alpha \in z(E_{7(7)})$ . From the commutativity with  $\nu \in E_{7(7)}$ , we have  $\nu\alpha = -\nu\alpha^{-1}\nu = -\nu\nu\alpha^{-1} = \alpha^{-1}$ . Hence  $\alpha \in (E_{7(7)})_K$  therefore there exists  $A \in SU(8)$  such that  $\alpha = \psi(A)$ . Furthermore

$\alpha \in z((E_{7(7)})_K) = z(\psi(SU(8)))$  ( $z(G)$  denotes always the center of a group  $G$  and in this case we have)

$$\begin{aligned} &= \psi(z(SU(8))) \\ &= \psi\{\rho^i E, i = 0, 1, 2, \dots, 7\}, \rho = e^{i\pi/4} \\ &= \psi\{E, \rho E, \rho^2 E, \rho^3 E\} \\ &= \{1, -\nu, -1, \nu\} \text{ (because } \psi(\rho E) = -\nu). \end{aligned}$$

However,  $\nu, -\nu$  are not contained in  $z(E_{7(7)})$ . In fact, for  $r \in \mathbf{R}, r \neq 0$ , define a linear transformation  $r$  of  $\mathfrak{K}$  by

$$r(X, Y, \xi, \eta) = (r^{-1}X, rY, r^3\xi, r^{-3}\eta).$$

Then  $r \in E_{7(7)}$  and  $r\nu \neq \nu r, r(-\nu) \neq (-\nu)r$  for  $r \neq 1$ , because  $r\nu(0, 0, 1, 0) = r(0, 0, 0, -1) = (0, 0, 0, -r^{-3}), \nu r(0, 0, 1, 0) = \nu(0, 0, r^3, 0) = (0, 0, 0, -r^3)$ . Thus we have  $z(E_{7(7)}) = \{1, -1\}$ .

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