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TWO COMMUTATIVITY THEOREMS FOR RINGS

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Throughout R will represent a ring (with or without 1), N_0 the set of all nilpotent elements of R , N the prime radical of R , and J the Jacobson radical of R . Given subsets S, T of R , we set $V_s(T) = \{s \in S \mid st = ts \text{ for all } t \in T\}$ and $V_{\bar{s}}(T) = \{s \in S \mid st = -ts \text{ for all } t \in T\}$. We denote by C the center of R , and by C' the set of all $x \in R$ such that for each $y \in R$ there holds $[x, y - yy'] = 0$ with some y' in the subring $[y]$ generated by y . In [1], C' is called the *cohypercenter* of R .

In this paper, we consider the following conditions:

A) For each $x \in R$ there exists a positive integer n such that $x - x^{n+1} \in N_0$.

A') For each $x \in R$ there exist positive integers m and n such that $x^m - x^{m+n} \in N_0$.

A'') For each $x \in R$ there exist a positive integer n and an element $x' \in [x]$ such that $x^n = x^n x'$.

A''') For each $x \in R$ there exists an element $x' \in [x]$ such that $x - xx' \in N_0$.

B) $x - y \in N_0$ and $y - z \in N_0$ ($x, y, z \in R$) imply $x^2 = z^2$ or $xy = yx$.

B') $x - y \in N_0$ ($x, y \in R$) implies that $x^2 = y^2$ or both x and y are contained in $V_R(N_0)$.

B'') Either R is commutative or $R = V_{\bar{R}}(N_0)$ and $u^2 = 0$ for all $u \in N_0$.

C) For each $x, y \in R$ there exist $x' \in [x]$ and $y' \in [y]$ such that $[x - xx', y - yy'] = 0$.

C') For each $x, y \in R$ there exists some $x' \in [x]$ such that $x - xx'$ is in $V_{N_0}(y)$.

Recently, in [1, Theorem 3], M. Chacron proved that if R satisfies the condition C) then both R/N and N are commutative. The proof depends heavily on another (perhaps more difficult) result in [1] that the cohypercenter of a semi-prime ring coincides with its center. In §1 of this paper, we shall give a somewhat direct and economical proof to the above theorem. And in §2, we shall deal with the commutativity theorem of S. Ligh and J. Luh (cf. [3]) and D. L. Outcalt and A. Yaqub [4] without assuming the existence of 1.

1. The next has been shown in [1, Remarks 12 and 14]. However, for the sake of completeness, we shall give the proof.

Lemma 1. *If C) is satisfied then C' is a commutative subring of R containing N_0 .*

Proof. Let $x \in N_0$, and $y \in R$. There exist $x'_1 \in [x]$ and $y'_1 \in [y]$ such that $[x - xx'_1, y - yy'_1] = 0$. To be easily seen, there exist $x'_2 \in [xx'_1]$ and $y'_2 \in [y]$ such that $[x - xx'_1x'_2, y - yy'_2] = 0$. Repeating the same procedure, we obtain eventually $0 = [x - xx'_1x'_2 \cdots x'_k, y - yy'_k] = [x, y - yy'_k]$ for some k , which proves $N_0 \subseteq C'$. Next, for any $a, b \in C'$ we consider the subring S generated by $\{a, b\}$. Given $x \in S$, one will easily see that there exist some $x' \in [x]$ such that $[a, x - xx'] = 0 = [b, x - xx']$, namely, $x - xx' \in V_S(S)$. Then, by [2, Theorem 19], S is a commutative ring contained in C' . This proves that C' is a commutative subring of R .

Corollary 1. *Assume that R satisfies the condition C). If R is either a division ring or a radical ring without non-zero zero-divisors, then R is commutative.*

Proof. If z is a quasi-regular element of R with the quasi-inverse z^* then $y - yz$ and $y - yz^*$ will be written formally as $y(1 - z)$ and $y(1 - z)^{-1}$ respectively. Obviously, the map defined by $y \mapsto (1 - z)y(1 - z)^{-1}$ ($y \in R$) is an automorphism of R . Now, let a and x be elements of R such that $a \notin C[x]$ and $x \notin C[a]$. First we shall show that there exist $a', a'_0 \in [a]$ such that

$$(1) \quad \begin{aligned} &[(1 - x)(a - aa')(1 - x)^{-1}, a - aa'_0] = 0, \\ &[(1 - ax)(a - aa')(1 - ax)^{-1}, a - aa'_0] = 0. \end{aligned}$$

In any rate, $[(1 - x)(a - aa'')(1 - x)^{-1}, a - aa'_0] = 0$ for some $a'', a'_0 \in [a]$. Then $[(1 - ax)\{(a - aa'')(1 - b')\}(1 - ax)^{-1}, (a - aa'_0)(1 - b'_0)] = 0$ for some $b' \in [a - aa'']$ and $b'_0 \in [a - aa'_0]$. Evidently, setting $a' = a'' + b' - a''b'$ and $a'_0 = a'_0 + b'_0 - a'_0b'_0$, we obtain (1). For the brevity, we set $\alpha = a - aa'$, $\alpha_0 = a - aa'_0$, $\beta = (1 - x)\alpha(1 - x)^{-1}$, and $\beta' = (1 - ax)\alpha(1 - ax)^{-1}$. Then (1) becomes $[\beta, \alpha_0] = 0 = [\beta', \alpha_0]$, and we have

$$(2) \quad (1 - x)\alpha = \beta(1 - x),$$

$$(3) \quad (1 - ax)\alpha = \beta'(1 - ax).$$

Now, subtracting (3) from (2) multiplied by a , we get

$$(4) \quad (a - 1)\alpha = (\beta'a - a\beta)x + a\beta - \beta'.$$

This deduces $(\beta'a - a\beta)x = (a - 1)\alpha - a\beta + \beta' \in V_R(\alpha_0)$, so that $(\beta'a - a\beta)$

$[x, \alpha_0] = 0$. If $[x, \alpha_0] \neq 0$, then $a\beta = \beta'a$, and by (4) it follows $\alpha(a-1) = a\beta - \beta' = \beta'(a-1)$, that is, $\alpha = \beta'$. Going back to (3), we get $(1-ax)\alpha = \alpha(1-ax)$, namely, $a[x, \alpha] = 0$. Hence, $[x, \alpha] = 0$. We have therefore seen that every element of R is in C' , whence it follows the commutativity of R by Lemma 1.

Lemma 2. *If R is a semi-prime ring satisfying C) then R is commutative.*

Proof. Without loss of generality, we may assume R is prime.

Case I: R is semi-primitive. We may assume further R is primitive. Every homomorphic image of a subring of R inherits the condition C). Since matrix rings over division rings of degree > 1 contains non-commutative nilpotent elements, by Lemma 1 a routine argument enables us to see that R is a division ring, so R is commutative by Corollary 1.

Case II: R is not semi-primitive. If $J \neq 0$ is shown to be commutative, then one will easily see that $V_R(J) \subseteq C$, so that $R = C$. Therefore, we assume henceforth $R = J$. Suppose R contains a non-zero element a with $a^2 = 0$. Then for each $y \in aR$ there exists some $y' \in [y]$ such that $0 = [a, y - yy'] = yy'a - ya$. Hence, we have $y^2 = y^2y'$. This implies evidently $y^2 = 0$. Combining this with $0 = yy'a - ya$, we readily obtain $ya = 0$, namely, $aRa = 0$. This contradiction shows that R is a reduced ring. Thus, R is a ring without non-zero zero-divisors, and so commutative by Corollary 1.

Now, as a combination of Lemmas 1 and 2, we readily obtain

Theorem 1 ([1, Theorem 3]). *If R satisfies the condition C) then both R/N and N are commutative.*

2. Evidently, in A'' , $x^n = x^n x'$ may be replaced by $x^n = x^{n+1} x'$. Similarly, in A''' (resp. C'), $x - xx' \in N_0$ (resp. $x - xx' \in V_{N_0}(y)$) may be replaced by $x - x^2 x' \in N_0$ (resp. $x - x^2 x' \in V_{N_0}(y)$).

Lemma 3. (1) *If B) is satisfied then N_0 is an ideal and $x^2 \in V_R(N_0)$ for all $x \in R$, especially, every idempotent of R is central.*

(2) *If N_0 is an ideal then the conditions A)–A''') are equivalent.*

(3) *B'') implies B'), and B') implies B).*

(4) *If for each $x, y \in R$ there exists some $z \in R$ such that $[x - x^2 z, y] = 0$ then $N_0 \subseteq C$.*

Proof. (1) is contained in [3, Lemma 1], and (3) is easy.

(2) Obviously, $A) \Rightarrow A') \Rightarrow A'')$. For any $x' \in [x]$ we have $(x - xx')^n = (x^n - x^n x') - (x^n - x^n x')x''$ with some $x'' \in [x]$, which proves $A'') \Rightarrow A''')$. Finally, $A''') \Rightarrow A)$ by [5, Corollary 3.5].

(4) Let $x^n = 0$ ($n > 1$). We proceed by the induction with respect to n . Given $y \in R$, there exists z with $[x - x^2 z, y] = 0$. Since x^2 is central by $(x^2)^{n-1} = 0$, it follows $(x^2 z)^{n-1} = 0$, so that $x^2 z$ is central. Hence, $[x, y] = [x^2 z, y] = 0$.

Lemma 4. (1) If B) is satisfied then N_0 is either commutative or anti-commutative with $u^2 = 0$ for all $u \in N_0$.

(2) If A) and B) are satisfied and R is left (or right) s-unital then N_0 is commutative.

(3) If A) and B) are satisfied and N_0 is commutative then R is commutative.

Proof. (1) By Lemma 3 (1), N_0 is an ideal of R and $z^2 \in V_R(N_0)$ for all $z \in R$. Suppose there exist $x, y \in N_0$ such that $xy \neq yx$. Since $x + y \equiv x \equiv 0 \pmod{N_0}$ and $(x + y)x \neq x(x + y)$, B) implies $0 = (x + y)^2 = x^2$, and similarly $0 = (x + y)^2 = y^2$. From these it follows $xy = -yx$. Now, by making use of Brauer's trick, one will easily see that N_0 is anti-commutative. If v is an arbitrary element of the center of N_0 , then $xv = vx = -xv$. Hence, we obtain $v^2 = (x + v)^2 - 2xv - x^2 = 0$.

(2) Suppose there exist $x, y \in N_0$ such that $xy \neq yx$. Then, by (1), N_0 is an anti-commutative ideal and $u^2 = 0$ for all $u \in N_0$. By [6, Theorem 1], there exists an element c such that $cx = x$ and $cy = y$. Choose an element $d \in [c]$ such that $c^n = c^{n+1}d$ for some positive integer n (Lemma 3 (2)). Then $e = c^n d^n$ is a central idempotent with $ec^n = c^n$ (Lemma 3 (1)). Hence, $ex = ec^n x = x$ and $ey = y$. Noting that $e + x + y \equiv e + x \pmod{N_0}$, B) implies then $(e + x + y)^2 = (e + x)^2$, whence it follows $2(x + y) = 2x$, namely, $2y = 0$. This forces a contradiction $xy = -yx = yx$.

(3) Let x, y be arbitrary elements of R , and $S = [x, y, N_0]$. Then, by [5, Theorem 3.4], $\bar{S} = S/N_0 = S_1/N_0 \oplus \cdots \oplus S_m/N_0$, where each S_i/N_0 is a finite field. As is well-known, the identity of $\bar{S}$ can be lifted to an idempotent e of S , which is central by Lemma 3 (1). Suppose there exist $s \in S$ and $t \in N_0$ such that $st \neq ts$. Then there exists some $s_j \in S$ such that $s_j t \neq ts_j$. We set $es_j = s_j + u$ with some $u \in N_0$. By Lemma 3 (1) and the assumption, one will easily see that $2s_j = (e + s_j)^2 - e - s_j^2 - 2u \in V_R(N_0)$. Hence, there holds $2(s_j t - ts_j) = 0$. If the characteristic p of S_j/N_0 is different from 2 then the last together with $p(s_j t - ts_j) = (ps_j)t -$

$t(ps_j)=0$ deduces a contradiction $s_jt-ts_j=0$. While, if $S_j/N_0=\text{GF}(2^k)$ then $0=(s_j^{2^k}-s_j)t-t(s_j^{2^k}-s_j)=ts_j-s_jt$ by $s_j^2t=ts_j^2$, which is a contradiction. Thus, we have seen that N_0 is contained in the center of S . Consequently, S is commutative by [2, Theorem 19], which means that R is commutative.

Now, we are ready to prove the principal theorem of this section.

Theorem 2. *The following statements are equivalent :*

- 1) A) and B) are satisfied.
- 1') A') and B) are satisfied.
- 1'') A'') and B) are satisfied.
- 1''') A''') and B) are satisfied.
- 2) A) and B') are satisfied.
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- 2''') A''') and B') are satisfied.
- 3) A) and B'') are satisfied.
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Proof. By Lemma 1 (3), $B'' \Rightarrow B' \Rightarrow B$. Hence, by Lemma 3 (1) and (2), $1)-1''')$, $2)-2''')$ and $3)-3''')$ are respectively equivalent and $3) \Rightarrow 2) \Rightarrow 1)$. It remains therefore to prove $1) \Rightarrow 3)$. Suppose R is not commutative. By Lemma 4 (1) and (3), N_0 is anti-commutative and $u^2=0$ for all $u \in N_0$. If $xv \neq vx$ for some $x \in R$ and $v \in N_0$, then B) implies $(x+v)^2=x^2$, whence it follows $xv=-vx$. Now, by making use of Brauer's trick, one can easily see that $R=V_R(N_0)$ or $V_R^-(N_0)$. Since $R=V_R(N_0)$ yields the commutativity of R by [2, Theorem 19], R must be $V_R^-(N_0)$.

The next includes [3, Theorem 2] as well as [4, Theorem 2].

Corollary 2. *If R is left (or right) s-unital, then the following statements are equivalent :*

- 1) A) and B) are satisfied.
- 1') A') and B) are satisfied.
- 1'') A'') and B) are satisfied.
- 1''') A''') and B) are satisfied.
- 2) A) and B') are satisfied.

- 2') $A')$ and $B')$ are satisfied.
- 2'') $A'')$ and $B')$ are satisfied.
- 2''') $A''')$ and $B')$ are satisfied.
- 3) R is a commutative ring satisfying $A)$.
- 3') R is a commutative ring satisfying $A')$.
- 3'') R is a commutative ring satisfying $A'')$.
- 3''') R is a commutative ring satisfying $A''')$.
- 4) $C')$ is satisfied.

Proof. 4) implies 3''') by Lemma 3 (4) and [2, Theorem 19], and 1) implies 3) by Lemma 4 (2). Hence, the corollary is evident by Theorem 2.

Remark. Let R be the module $Z \oplus Z \oplus Z$. If we define the multiplication by $(a_1, a_2, a_3)(b_1, b_2, b_3) = (0, 0, a_1b_2 - a_2b_1)$, then R is an anti-commutative, non-commutative ring and the square of each element is 0.

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