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ON THE p' -SECTION SUM IN A FINITE GROUP RING

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Let G be a finite group, and p a prime number. Let $C_1 = \{1\}, C_2, \dots, C_r$ be all the p -regular classes of G . We denote by S_i the p' -section containing C_i , namely $S_i = \{\sigma \in G \mid \sigma' \in C_i\}$, where σ' denotes the p' -component of the element σ of G ($i = 1, 2, \dots, r$). In particular, S_1 is the set consisting of all the p -elements of G . Let k be a field of characteristic p , and J the Jacobson radical of the group ring $k[G]$.

Recently, the author is informed by T. Okuyama that in 1955 R. Brauer stated the following without proof.

Theorem (Brauer [2]). *Let $\hat{S}_j = \sum \sigma$, where the summation is taken over all $\sigma \in S_j$. Then there holds that $\bigcap_{j=1}^r (0 : \hat{S}_j) = J$.*

We recall here the previous paper [6]. There, we showed that $(0 : \hat{S}_1) \supset J$, while the inclusion $\bigcap_{j=1}^r (0 : \hat{S}_j) \subset J$ is an easy deducement of Proposition 1 in [6]. So that, we should like to provide a new proof of the above Theorem along with the arguments used in the proofs of these results.

As a consequence of the Theorem, we have that if e is a primitive idempotent of $k[G]$, then $k[G]\hat{S}_je$ is the socle of $k[G]e$, provided $\hat{S}_je \neq 0$. The condition will be described by the value on C_j of the Brauer character afforded by $k[G]e$. On the other hand, Okuyama's proof of the Theorem is different from ours. There, the condition $\hat{S}_je \neq 0$ is discussed in connection with the coefficients a_i 's which appear in the expression $e = \sum_{i \in G} a_i \tau_i$, $a_i \in k$. We refer to it at the end of this paper.

In the proof of the Theorem, from the beginning, we may assume that k is a splitting field for G . In addition to the notations introduced above, we shall use the following.

Let $\mathfrak{p}$ be a prime divisor of p in an algebraic number field containing the $|G|$ -th roots of unity, and ν the exponential valuation associated with $\mathfrak{p}$ multiplied by a factor to make $\nu(p) = 1$. We assume henceforth k is the residue class field of ν . If α is a $\mathfrak{p}$ -integer, then $\bar{\alpha}$ denotes the residue class of α in k . Let $\{\eta_1, \eta_2, \dots, \eta_r\}$ and $\{\phi_1, \phi_2, \dots, \phi_r\}$ be the set of the principal indecomposable Brauer characters of G and the set of

the irreducible Brauer characters of G respectively, in which we arrange the indices so that $(\eta_j, \phi_j) = \delta_{ij}$ for all i, j ($i, j = 1, 2, \dots, r$). The k -algebra $k[G]/J$ is isomorphic to a direct sum of full matrix algebras over k ; $k[G]/J \simeq \sum_{i=1}^r M(n_i, k)$. We assume that under the isomorphism the simple component corresponding to the irreducible k -character $\bar{\phi}_i$ is mapped onto $M(n_i, k)$ and so $n_i = \phi_i(1)$. If I is a subset of $k[G]$, then $(0: I)$ denotes the set of the right annihilators of I in $k[G]$. Finally, we put $\lambda(\sum_{\sigma \in G} a_\sigma \sigma) = a_1$, where $a_\sigma \in k$ and 1 denotes the identity of G .

Now we enter into the proof of the Theorem. Let S_j be a fixed p' -section and $\sigma \in C_j$. There holds that $\nu(\eta_i(\sigma)) \geq \nu(|C_\sigma(\sigma)|)$ for all η_i ([3], (84.14)). After a suitable change of indices if necessary, we may assume that the first $\eta_1, \eta_2, \dots, \eta_t$ are all that enjoy the equality sign in the above. We put $\eta'_i(\sigma) = \eta_i(\sigma)/p^h$, where $|C_\sigma(\sigma)| = p^h h$, $(p, h) = 1$.

From the orthogonality relation

$$\sum_{i=1}^t \eta_i(\sigma) \phi_i(\tau^{-1}) = \begin{cases} |C_\sigma(\sigma)| & \text{if } \tau \text{ is conjugate to } \sigma \\ 0 & \text{otherwise} \end{cases}$$

and that $\overline{\phi_i(\tau)} = \overline{\phi_i(\tau')}$ for any element τ of G , we get (reducing mod p)

$$(*) \quad \sum_{i=1}^t \overline{\eta'_i(\sigma)} \overline{\phi_i(\tau^{-1})} = \begin{cases} \bar{h} & \text{if } \tau \in S_j \\ 0 & \text{otherwise.} \end{cases}$$

Let $U_j = \sum_{i=1}^t k[G]e + J$, where e runs over the primitive idempotents such that $k[G]e$ affords a Brauer character η_s with $s > t$. Then U_j is a two sided ideal of $k[G]$. In fact, it is the inverse image of $\sum_{i=t+1}^r M(n_i, k)$ by the composite map $k[G] \longrightarrow k[G]/J \simeq \sum_{i=1}^r M(n_i, k)$. We identify $k[G]/U_j$ with $\sum_{i=1}^t M(n_i, k)$ and denote by ρ_i the projection of $k[G]/U_j$ onto $M(n_i, k)$. If we put $\mu = \sum_{i=1}^t \overline{\eta'_i(\sigma)} \text{tr } \rho_i$, then μ is a (symmetric) non-singular linear function on $k[G]/U_j$. Hence by Theorem 9 of Nakayama [3] (or see [1], (55.11)), there exists an element c of $k[G]$ such that $(0: U_j) = k[G]c$ and $\eta\phi(x) = \lambda(cx)$ for all $x \in k[G]$, where ϕ is the natural map $k[G] \longrightarrow k[G]/U_j$. From this, by making use of (*), we get easily that $c = \bar{h}\hat{S}_j$ and hence $(0: \hat{S}_j) = U_j$, as $k[G]$ is Frobeniusean. It is clear that $\cap_{j=1}^r U_j = J$ (namely, for any i , there exists a p -regular element σ such that $\nu(\eta_i(\sigma)) = \nu(|C_\sigma(\sigma)|)$). This follows easily, for instance, from the relation $(\eta_i, \phi_i) = 1$. Thus we conclude that $\cap_{j=1}^r (0: \hat{S}_j) = J$ and the proof is complete.

From the above argument, we get also

Corollary A. Let e be a primitive idempotent of $k[G]$, and let η

be the Brauer character afforded by $k[G]e$. If S_j is a p' -section of G , then the following are equivalent:

- (1) $\hat{S}_j e \neq 0$.
- (2) $\nu(\gamma(\sigma)) = \nu(|C_G(\sigma)|)$, where σ is a p -regular element in S_j .

We continue our argument to give an alternative proof to the following result of Okuyama.

Corollary B (Okuyama [5]). *Under the same notation as in Corollary A, let $e = \sum_{\tau \in G} a_\tau \tau$. Then the following are equivalent:*

- (1) $\hat{S}_j e \neq 0$.
- (2) $\sum a_\tau \neq 0$, where the summation is taken over all $\tau \in S_j^{-1} = \{\sigma^{-1} | \sigma \in S_j\}$.

Proof. Recall that if ξ is a symmetric linear function on $M(n, k)$, then there exists some $a \in k$ such that $\xi(x) = a \cdot \text{tr}(x)$ for all $x \in M(n, k)$ (since the set $\{xy - yx | x, y \in M(n, k)\}$ spans the subspace consisting of the elements of trace zero). In particular, we have $\xi(e) \neq 0$ for any primitive idempotent e . Keeping the notation used in the proof of the Theorem, we know that μ is a symmetric, non-singular linear function on $k[G]/U_j = \sum_{i=1}^t M(n_i, k)$ and hence $\mu\psi(e) \neq 0$ for any primitive idempotent e of $k[G]$ not contained in U_j . And if $e = \sum a_\tau \tau$, then $\mu\psi(e) = \sum_{\tau \in G} a_\tau \overline{\eta_i(\sigma)} \overline{\phi_i(\tau)} = (\sum a_\tau) \bar{h}$, where the second summation is taken over all $\tau \in S_j^{-1}$. From these observations, we get the above assertion.

Remark. According to a result of Okuyama [5], there holds that the summation $\sum a_\tau$ in the above (2) is equal to the restricted summation $\sum' a_\tau$, where τ runs over all $\tau \in C_j^{-1}$.

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