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THE GROUP OF GALOIS H -DIMODULE ALGEBRAS

ANTONIO A. BLANCO-FERRO and MIGUEL A. LOPEZ LOPEZ*

Let R be a commutative ring with identity, and let H be a commutative, cocommutative, finite Hopf R -algebra with antipode. Using the notion of H -dimodule algebra, given by Long in [3], Nakajima defines in [4] Galois H -dimodule algebras generalizing graded Galois algebras and H -objects of Galois in the sense of Chase-Sweedler. In the same paper, he shows that the set of isomorphism classes of Galois H -dimodule algebras admits a structure of monoid provided that the Hopf algebra is a free R -module of finite type.

The purpose of the present paper is to show that for any commutative, cocommutative, finite Hopf R -algebra with antipode the above mentioned monoid is a group which contains the Galois group of H (Chase-Sweedler [2], Beattie [1]), as a subgroup.

0. Preliminaries. In what follows, R is assumed to be a commutative ring with identity, and H a commutative, cocommutative, finite Hopf algebra with antipode. The counit, comultiplication and antipode of H will be denoted by $\varepsilon : H \rightarrow R$, $\delta : H \rightarrow H \otimes H$ and $\lambda : H \rightarrow H$, respectively, where $\otimes$ stands for $\otimes_R$.

For A , any R -algebra, we denote the unit and multiplication with $\eta_A : R \rightarrow A$ and $\mu_A : A \otimes A \rightarrow A$, respectively.

$\phi_A : H \otimes A \rightarrow A$ will denote a structure of H -module and $\chi_B : B \rightarrow B \otimes B$ of H -comodule.

An R -algebra A is called H -module (respectively, H -comodule) algebra if η_A and μ_A are H -module (respectively, H -comodule) morphisms.

An H -module algebra A which is also H -comodule algebra and satisfies $\chi_A \cdot \phi_A = (\phi_A \otimes A) \cdot (H \otimes \chi_A) : H \otimes A \rightarrow A \otimes H$, is called H -dimodule algebra.

For any H -module algebra A , and H -comodule algebra B , the smash product $A \# B$, is a R -algebra: the R -module support is $A \otimes B$, and the multiplication is given by

$$(a \# b) \cdot (c \# d) = \sum_{(b)} a \cdot b_{(1)}(c) \# b_{(0)} \cdot d$$

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1. Galois H -dimodule algebras.

Definition 1.1. An H -dimodule algebra A is called a Galois H -dimodule algebra if the following conditions are satisfied:

- i) A is a faithfully flat R -module
- ii) $\gamma_A = (\mu_A \otimes H) \cdot (A \otimes \chi_A) : A \otimes A \longrightarrow A \otimes H$ is an isomorphism.

Proposition 1.2. For A and B any two Galois H -dimodule algebras, the following statements are true:

- a) A' is a Galois H -dimodule algebra, where A' has the same H -module algebra structure of A and has $\chi_{A'} = (A \otimes \lambda) \cdot \chi_A$ as H -comodule structure.
- b) A^{op} is a Galois H -dimodule algebra.
- c) $\bar{A}$ is a Galois H -dimodule algebra, where $\bar{A}$ has the same H -dimodule structure of A , and its algebra structure is given by $\mu_{\bar{A}} = \mu_A \cdot \tau \cdot (A \otimes \phi_A) \cdot (\chi_{A'} \otimes A)$, τ being the twisting isomorphism.
- d) The smash products, $A \# B$, is an Galois $H \otimes H$ -dimodule algebra. The structure of $H \otimes H$ -module is given by

$$(h_1 \otimes h_2)(a \# b) = h_1(a) \# h_2(b)$$

and the one of $H \otimes H$ -comodule by

$$\chi_{A \# B}(a \# b) = \sum_{(a \# b)} a_{(0)} \# b_{(0)} \otimes a_{(1)} \otimes b_{(1)}.$$

- e) H is a Galois H -dimodule algebra with the trivial structure of H -module, $\phi_H = \varepsilon \otimes H$, and with its comultiplication as H -comodule structure.

Proof. b) After [3, Prop 1.7 and Prop 2.10], enough to show that $\gamma_{A^{op}}$ is an isomorphism. But this follows from

$$\begin{aligned} & (\gamma_A \otimes H) \cdot (A \otimes \gamma_{A^{op}}) \\ & \text{(definitions of } \gamma_A, \gamma_{A^{op}}, \text{ and } A \text{ is } H\text{-comodule algebra)} \\ & = (\mu_A \otimes H \otimes H) \cdot (A \otimes \mu_A \otimes \mu_H \otimes H) \cdot (A \otimes A \otimes \chi_A \otimes \delta) \cdot \\ & \quad (A \otimes \tau \otimes H) \cdot (A \otimes A \otimes \chi_A) \\ & = (\mu_A \otimes H \otimes H) \cdot (A \otimes A \otimes \mu_H \otimes H) \cdot (\mu_A \otimes \chi_A \otimes \delta) \cdot \\ & \quad (A \otimes A \otimes \tau) \cdot (A \otimes \chi_A \otimes A) \cdot (A \otimes \tau) \\ & \text{(definitions of } \gamma_A \text{ and } \gamma_H) \\ & = (A \otimes \gamma_H) \cdot (\gamma_A \otimes H) \cdot (A \otimes \tau) \cdot (\gamma_A \otimes A) \cdot (A \otimes \tau). \end{aligned}$$

$$c) \quad \gamma_{\bar{A}} = (\mu_{\bar{A}} \otimes H) \cdot (A \otimes \chi_A) = \gamma_{A^{op}} \cdot (A \otimes \phi_A) \cdot (\chi_{A'} \otimes A).$$

From, $\gamma_{A^{op}}$ and $(A \otimes \phi_A) \cdot (\chi_{A'} \otimes A)$ isomorphisms it follows $\gamma_{\bar{A}}$ isomor-

phism.

The rest of the proof is straightforward though tedious.

2. The group $Gal_*(R, H)$. Two Galois H -dimodule algebras are said to be isomorphic if they are so as R -algebras, as H -modules and as H -comodules. The set of isomorphism classes of Galois H -dimodule algebras is denoted by $Gal_*(R, H)$.

From now on, A and B denote two arbitrary Galois H -dimodule algebras.

Definition 2.1. The product $A \cdot B$ is defined by

$$A \cdot B = \{ \sum_{(i)} a_i \# b_i \in A \# B \mid \sum_{(i)} a_{i_{(0)}} \# b_i \otimes a_{i_{(1)}} = \sum_{(i)} a_i \# b_{i_{(0)}} \otimes b_{i_{(1)}} \}$$

and therefore we have the equalizer

$$A \cdot B \xrightarrow{i_{AB}} A \# B \xrightarrow[(A \otimes \tau)(\chi_A \otimes B)]{A \otimes \chi_B} A \# B \otimes H$$

Proposition 2.2. $A \cdot B$ is a Galois H -dimodule algebra.

Proof. The algebra structure is induced by the one of $A \# B$.

The H -module structure $\phi_{A \cdot B}$ is given by the factorization, through the equalizer i_{AB} , of the morphism

$$\begin{array}{ccc} H \otimes A \cdot B & \xrightarrow{\delta \otimes i_{AB}} & H \otimes H \otimes A \otimes B \\ \xrightarrow{H \otimes \tau \otimes B} & H \otimes A \otimes H \otimes B & \xrightarrow{\phi_A \otimes \phi_B} A \otimes B \end{array}$$

i. e.,

$$h(\sum_{(i)} a_i \# b_i) = \sum_{(h, i)} h_{(0)}(a_i) \# h_{(1)}(b_i).$$

The H -comodule structure is given by the map

$$\chi_{A \cdot B} : A \cdot B \longrightarrow A \cdot B \otimes H, \text{ defined by}$$

$$\chi_{A \cdot B}(\sum_{(i)} a_i \# b_i) = \sum_{(i)} a_{i_{(0)}} \# b_i \otimes a_{i_{(1)}} = \sum_{(i)} a_i \# b_{i_{(0)}} \otimes b_{i_{(1)}},$$

which is the only one satisfying

$$(i_{AB} \otimes H) \cdot \chi_{A \cdot B} = (A \otimes \tau) \cdot (\chi_A \otimes B) \cdot i_{AB} = (A \otimes \chi_B) \cdot i_{AB}.$$

It readily follows that $A \cdot B$ is an H -dimodule algebra.

Now, $A \# B$ is a faithfully flat R -module; therefore

$$\begin{array}{ccc}
 A \# B \otimes A \cdot B & \xrightarrow{A \# B \otimes i_{AB}} & A \# B \otimes A \# B \\
 \xrightarrow{A \# B \otimes A \otimes \chi_B} & & \xrightarrow{A \# B \otimes A \# B \otimes H} \\
 A \# B \otimes ((A \otimes \tau)(\chi_A \otimes B)) & & \\
 \\
 A \# B \otimes B & \xrightarrow{A \# B \otimes \delta^\epsilon} & A \# B \otimes H \otimes H \\
 \xrightarrow{A \# B \otimes H \otimes \delta} & & \xrightarrow{A \# B \otimes H \otimes H \otimes H} \\
 A \# B \otimes ((H \otimes \tau)(\delta \otimes H)) & &
 \end{array}$$

are diagrams of equalizer.

The isomorphism $\gamma_{A \# B} : (A \# B) \otimes (A \# B) \longrightarrow A \# B \otimes H \otimes H$ factors through $A \# B \otimes \delta$ yielding, this way, morphisms h and h' , inverse to each other, making the diagram

$$\begin{array}{ccc}
 (A \# B) \otimes A \cdot B & \xrightarrow{A \# B \otimes i_{AB}} & (A \# B) \otimes (A \# B) \\
 \downarrow h \quad \uparrow h' & & \downarrow \gamma_{A \# B} \quad \uparrow \gamma_{A \# B}^{-1} \\
 (A \# B) \otimes H & \xrightarrow{A \# B \otimes \delta} & (A \# B) \otimes H \otimes H
 \end{array}$$

to commute.

It follows, then, that

$$\begin{aligned}
 & ((A \# B) \otimes \delta \otimes H) \cdot (h \otimes H) \cdot ((A \# B \otimes \gamma_{A \cdot B}) \\
 & = ((A \# B) \otimes \delta \otimes H) \cdot ((A \# B) \otimes (\gamma_H \cdot \tau)) \cdot \\
 & (h \otimes H) \cdot ((A \# B) \otimes \tau) \cdot (h \otimes A \cdot B)
 \end{aligned}$$

and therefore $\gamma_{A \cdot B}$ is an isomorphism, because $(A \# B) \otimes \delta \otimes H$ is an equalizer and the involved morphisms are isomorphisms.

Corollary 2.3. *The product of Galois H -dimodule algebras induces an operation on $\text{Gal}_*(R, H)$ whose identity element is the class of H .*

Proposition 2.4. *$\text{Gal}_*(R, H)$ is a group.*

Proof. We will show that $|A|^{-1} = |\bar{A}|$.
Indeed, since A is a Galois H -object [2],

$$H \xrightarrow{\eta_A \otimes H} A \otimes H \text{ is the equalizer of } A \otimes H \xrightarrow[A \otimes \eta_H \otimes H]{\chi_A \otimes H} A \otimes H \otimes H$$

On the other hand, the morphism

$$A \cdot \bar{A}' \xrightarrow{i_{A\bar{A}'}} A \# \bar{A}' \xrightarrow{\gamma_{A'}} A \otimes H$$

equalizes this pair because

$$\begin{aligned} (\chi_A \otimes H) \cdot \gamma_{A'} \cdot i_{A\bar{A}'} &= (\chi_A \otimes H) \cdot (\mu_A \otimes H) \cdot (A \otimes \chi_{A'}) \cdot i_{A\bar{A}'} \\ &\text{(\chi_A is a morphism of algebras)} \\ &= (\mu_A \otimes \mu_H \otimes H) \cdot (A \otimes \tau \otimes H \otimes H) \cdot (\chi_A \otimes \chi_A \otimes H) \cdot (A \otimes \chi_{A'}) \cdot i_{A\bar{A}'} \\ &\text{(A is an H-comodule)} \\ &= (\mu_A \otimes \mu_H \otimes H) \cdot (A \otimes \tau \otimes H \otimes \lambda) \cdot (A \otimes H \otimes A \otimes \delta) \cdot (\chi_A \otimes \chi_A) \cdot i_{A\bar{A}'} \\ &\text{(definition 2.1 and } \tau^2 = 1) \\ &= (\mu_A \otimes \mu_H \otimes H) \cdot (A \otimes \tau \otimes H \otimes \lambda) \cdot (A \otimes H \otimes A \otimes \delta) \cdot \\ &\quad (A \otimes H \otimes \chi_A) \cdot (A \otimes \tau) \cdot (A \otimes \chi_{A'}) \cdot i_{A\bar{A}'} \\ &\text{(A is an H-comodule)} \\ &= (\mu_A \otimes \mu_H \otimes H) \cdot (A \otimes A \otimes \tau \otimes \lambda) \cdot (A \otimes A \otimes H \otimes \tau) \cdot \\ &\quad (A \otimes A \otimes H \otimes H \otimes \lambda) \cdot (A \otimes A \otimes H \otimes \delta) \cdot (A \otimes A \otimes \delta) \cdot (A \otimes \chi_A) \cdot i_{A\bar{A}'} \\ &\text{(cocommutativity of H)} \\ &= (A \otimes \eta_H \otimes H) \cdot (\mu_A \otimes \lambda) \cdot (A \otimes \chi_A) \cdot i_{A\bar{A}'} = (A \otimes \eta_H \otimes H) \cdot \gamma_{A'} \cdot i_{A\bar{A}'} \end{aligned}$$

Therefore, there exists a morphism $f: A \cdot \bar{A}' \longrightarrow H$ so that $(\eta_A \otimes H) \cdot f = \gamma_{A'} \cdot i_{A\bar{A}'}$ (i.e. calling q to $f(a \# \bar{a})$, $1_A \# q = \sum_{(\bar{a})} a\bar{a}_{(0)} \# \lambda(\bar{a}_{(1)})$)

This f is an H -module morphism, i.e., the diagram

$$\begin{array}{ccc} H \otimes A \cdot \bar{A}' & \xrightarrow{H \otimes f} & H \otimes H \\ \downarrow \phi_{A \cdot \bar{A}'} & & \downarrow \varepsilon \otimes H \\ A \cdot \bar{A}' & \xrightarrow{f} & H \end{array}$$

commutes. Indeed.

Since $\eta_A \otimes H$ is an equalizer,

$$\begin{aligned} (\eta_A \otimes H) \cdot f \cdot \phi_{A \cdot \bar{A}'} &= \gamma_{A'} \cdot i_{A\bar{A}'} \cdot \phi_{A \cdot \bar{A}'} \\ &\text{(definition of } \phi_{A \cdot \bar{A}'} \text{)} \\ &= (\mu_A \otimes H) \cdot (A \otimes \chi_{A'}) \cdot (\phi_A \otimes \phi_{\bar{A}'}) \cdot (H \otimes \tau \otimes A) \cdot (\delta \otimes A \otimes A) \cdot (H \otimes i_{A\bar{A}'}) \\ &\text{(A' is an H-dimodule)} \\ &= (\mu_A \otimes H) \cdot (\phi_A \otimes \phi_{\bar{A}'} \otimes H) \cdot (H \otimes \tau \otimes \chi_{A'}) \cdot (\delta \otimes A \otimes A) \cdot (H \otimes i_{A\bar{A}'}) \\ &\text{(A is an H-module algebra)} \end{aligned}$$

$$\begin{aligned}
 &= (\phi_A \otimes H) \cdot (H \otimes \mu_A \otimes H) \cdot (H \otimes A \otimes \chi_{A'}) \cdot (H \otimes i_{A\bar{A}'}) \\
 &= (\phi_A \otimes H) \cdot (H \otimes \eta_A \otimes f) \\
 &\quad (\eta: R \longrightarrow A \text{ is an } H\text{-module morphism}) \\
 &= (\eta_A \otimes H) \cdot (\varepsilon \otimes H) \cdot (H \otimes f).
 \end{aligned}$$

f is a morphism of H -comodules or, what is the same, the diagram

$$\begin{array}{ccc}
 A \cdot \bar{A}' & \xrightarrow{\chi_{A \cdot \bar{A}'}} & A \cdot \bar{A}' \otimes H \\
 \downarrow f & & \downarrow f \otimes H \\
 H & \xrightarrow{\delta} & H \otimes H
 \end{array}$$

commutes. Indeed,

Since $\eta_A \otimes H \otimes H$ is an equalizer,

$$\begin{aligned}
 &(\eta_A \otimes H \otimes H) \cdot (f \otimes H) \cdot \chi_{A \cdot \bar{A}'} = (\eta_A \otimes f \otimes H) \cdot \chi_{A \cdot \bar{A}'} \\
 &\quad (\text{definition of } f) \\
 &= (\gamma_{A'} \otimes H) \cdot (i_{A\bar{A}'} \otimes H) \cdot \chi_{A \cdot \bar{A}'} \\
 &\quad (\text{definition of } \chi_{A \cdot \bar{A}'}) \\
 &= (\gamma_{A'} \otimes H) \cdot (A \otimes \chi_{A'}) \cdot i_{A\bar{A}'} \\
 &\quad (\text{definition of } \gamma_{A'}) \\
 &= (\mu_A \otimes H \otimes H) \cdot (A \otimes \chi_{A'} \otimes H) \cdot (A \otimes \chi_{A'}) \cdot i_{A\bar{A}'} \\
 &\quad (A' \text{ is an } H\text{-comodule}) \\
 &= (\mu_A \otimes H \otimes H) \cdot (A \otimes A \otimes \delta) \cdot (A \otimes \chi_{A'}) \cdot i_{A\bar{A}'} \\
 &= (A \otimes \delta) \cdot (\mu_A \otimes H) \cdot (A \otimes \chi_{A'}) \cdot i_{A\bar{A}'} \\
 &\quad (\text{definition of } \gamma_{A'}) \\
 &= (A \otimes \delta) \cdot \gamma_{A'} \cdot i_{A\bar{A}'} = (A \otimes \delta) \cdot (\eta_A \otimes f) = (\eta_A \otimes H \otimes H) \cdot \delta \cdot f.
 \end{aligned}$$

Analogously f is a morphism of R -algebras and by ([1], Lemma 1.1., p. 688) is an isomorphism.

Corollary 2.5. *The map*

$$\begin{aligned}
 \text{Gal}(R, H) &\longrightarrow \text{Gal}_\#(R, H) \\
 |A| &\longrightarrow |A|
 \end{aligned}$$

obtained by regarding any Galois H -object as a Galois H -dimodule algebra with the trivial action ([1], [2]) is a monomorphism of groups, in general not surjective ([4], Remark 2.5, p. 174).

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