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ON SOME COHEN-MACAULAY SUBSETS OF A PARTIALLY ORDERED ABELIAN GROUP

To the memory of Professor Gishiro Maruyama

ANDRZEJ NOWICKI, KAZUO KISHIMOTO and TAKASI NAGAHARA

This paper is about some Cohen-Macaulay subsets of a partially ordered abelian group which are useful in the study of Galois extensions of higher derivation type (cf. Remark and [4]).

Let $N = \{1, \dots, n\}$, and $Z = \{\dots, -2, -1, 0, 1, 2, \dots\}$. Now, if $G = (f_1) \times \dots \times (f_n)$ is an abelian group which is the direct product of infinite cyclic groups (f_i) generated by f_i then G becomes a partially ordered group by

$$(\#) : \prod_{i=1}^n f_i^{s_i} \geq \prod_{i=1}^n f_i^{t_i} \Leftrightarrow \sum_{j=k}^n s_j \geq \sum_{j=k}^n t_j \text{ for all } k \in N.$$

This partially ordered group G will be denoted by $(G, \#)$. Clearly $(G, \#)$ can be regarded as the partially ordered additive group $(Z^n, \#) = Z_1 \times \dots \times Z_n$, where $Z_i = Z$ for all $i \in N$. As it is seen later on, $(Z^n, \#)$ is a modular lattice.

For $u_i, v_i \in Z$ with $u_i \leq v_i$ ($i \in N$), we set

$$\Delta = \prod_{i=1}^n [u_i, v_i] = \{(a_1, \dots, a_n) ; u_i \leq a_i \leq v_i, a_i \in Z\}$$

which is a subposet of $(Z^n, \#)$.

Our purpose of this note is to prove that Δ is a modular lattice under the ordering in $(Z^n, \#)$ (Theorem 7), and if, in particular, $u_i < v_i$ for all $i \in N$ then Δ is a modular sublattice of $(Z^n, \#)$ (Theorem 6).

In what follows, we shall use the following conventions :

Let A be a poset with order $\geq$ and Δ a subposet of A . Then, for $a, b \in A$ and $c, d \in \Delta$,

$a > b$ if and only if $a \geq b$ and $a \neq b$.

$a \gg b$ (resp. $c \gg_\Delta d$) if and only if $a > b$ (resp. $c > d$) and there are not elements e in A (resp. e' in Δ) such that $a > e > b$ (resp. $c > e' > d$).

For $a = (a_1, \dots, a_n) \in (Z^n, \#)$, this is sometimes abbreviated to $a = (a_i)$, and for any $k \in N$, $\sum_{j=k}^n a_j$ is denoted by $t_k(a)$.

Let $(Z^n, *)$ be a vector group with order $\geq$ defined by

$$(*) : (a_i) \geq (b_i) \Leftrightarrow a_i \geq b_i \text{ for all } i \in N \text{ (cf. [3])}.$$

Then, one will easily see that $(Z^n, *)$ is a modular lattice. We consider

here the mapping

$$\phi: (Z^n, \#) \longrightarrow (Z^n, *)$$

defined by $\phi(a) = (t_i(a))$. Clearly $\phi(a+b) = \phi(a) + \phi(b)$ for $a, b \in (Z^n, \#)$. By our definition, ϕ is injective. Moreover, since, for $(x_i) \in (Z^n, *)$,

$$\phi((x_1 - x_2, x_2 - x_3, \dots, x_{n-1} - x_n, x_n)) = (x_1, \dots, x_n),$$

ϕ is surjective. Hence ϕ is a group isomorphism which preserves orders, and so, $(Z^n, \#)$ is a modular lattice.

Now, let $a = (a_i)$, $b = (b_i) \in (Z^n, \#)$, and $a > b$. Then $\{c \in (Z^n, \#); a \geq c \geq b\}$ is a finite set whose cardinal number is

$$\prod_{i=1}^n (t_i(a) - t_i(b) + 1) = \prod_{i=1}^n (\sum_{j=i}^n (a_j - b_j) + 1).$$

By $f(a)$, we denote $\sum_{i=1}^n t_i(a)$. Then one will easily see that $f(a) = \sum_{i=1}^n ia_i$.

Additionally, let $a \gg b$, and $\phi(a) = (x_1, \dots, x_n)$. Then $\phi(a) \gg \phi(b)$, and whence

$$\phi(b) = (x_1, \dots, x_{i-1}, x_i - 1, x_{i+1}, \dots, x_n)$$

for some $1 \leq i \leq n$. Hence, it follows that there holds either

$$\begin{aligned} b &= (a_1, \dots, a_{i-2}, a_{i-1} + 1, a_i - 1, a_{i+1}, \dots, a_n) \quad (2 \leq i \leq n) \text{ or} \\ b &= (a_1 - 1, a_2, \dots, a_n). \end{aligned}$$

Moreover, we see that $f(b) = \sum_{i=1}^n t_i(b) = \sum_{i=1}^n x_i - 1 = f(a) - 1$.

Our study starts with the following

Lemma 1. Let $a = (a_i)$, $b = (b_i) \in (Z^n, \#)$, and $a > b$. If

$$a = a^{(0)} \gg a^{(1)} \gg \dots \gg a^{(p)} = b \quad (a^{(s)} \in (Z^n, \#))$$

then $p = \sum_{i=1}^n i(a_i - b_i)$, whence the length p is uniquely determined by $a > b$.

Proof.

$$\begin{aligned} \sum_{i=1}^n i(a_i - b_i) &= \sum_{i=1}^n ia_i - \sum_{i=1}^n ib_i = f(a) - f(b) \\ &= \sum_{i=0}^{p-1} (f(a^{(i)}) - f(a^{(i+1)})) = p. \end{aligned}$$

The above p will be denoted by $|a > b|$.

Lemma 2. Let $\Delta = \prod_{i=1}^n [u_i, v_i]$ where $u_i, v_i \in Z$ and $u_i < v_i$ ($i \in N$). If $a, b \in \Delta$ and $a \gg_\Delta b$ then $a \gg b$.

Proof. Let $a = (a_i)$ and $b = (b_i)$. One will easily see that our assertion is true for $(Z, \#)$. Hence we assume that our lemma holds for $(Z^{n-1}, \#)$. For any $r = (r_1, \dots, r_n) \in (Z^n, \#)$, we set $C_2(r) = (r_2, \dots, r_n)$, and $C_2(\Delta) = \{C_2(r) ; r \in \Delta\}$. Then $C_2(\Delta)$ can be regarded as a subset of $(Z^{n-1}, \#)$, and $C_2(a), C_2(b) \in C_2(\Delta)$. Clearly $C_2(a) \geq C_2(b)$. In case $C_2(a) = C_2(b)$, one will easily see that $b = (a_1 - 1, a_2, \dots, a_n)$ where $a_1 > u_1$, and so, $a \gg b$. Hence, let $C_2(a) > C_2(b)$. Then, there exists an element e' in $C_2(\Delta)$ such that

$$C_2(a) \gg_{C_2(\Delta)} e' \geq C_2(b)$$

By the induction assumption, we have $C_2(a) \gg e'$. Hence e' coincides with either

$$f' = (a_2, \dots, a_{k-2}, a_{k-1} + 1, a_k - 1, a_{k+1}, \dots, a_n)$$

where $3 \leq k \leq n$, $a_{k-1} < v_{k-1}$ and $a_k > u_k$ (in case $k = 3$, f' is taken as $(a_2 + 1, a_3 - 1, a_4, \dots, a_n)$), or

$$g' = (a_2 - 1, a_3, \dots, a_n)$$

where $a_2 > u_2$. In case $e' = f'$, we set

$$f = (a_1, a_2, \dots, a_{k-2}, a_{k-1} + 1, a_k - 1, a_{k+1}, \dots, a_n).$$

Then $f \in \Delta$, $t_1(f) = t_1(a) \geq t_1(b)$, $C_2(f) = e' \geq C_2(b)$, and whence $a > f \geq b$ in Δ . Hence $f = b$, and so, $a \gg b$. In case $e' = g'$, we set

$$g = (a_1 + 1, a_2 - 1, a_3, \dots, a_n).$$

If $a_1 < v_1$ then there exists a chain $a > g \geq b$ in Δ , whence $g = b$, and so, $a \gg b$. If $a_1 = v_1$ then there is a chain

$$a > (a_1 - 1, a_2, \dots, a_n) > (a_1, a_2 - 1, a_3, \dots, a_n) \geq b \text{ in } \Delta$$

which is a contradiction. This completes the proof.

By virtue of the results of Lemma 1 and Lemma 2, we obtain the following

Theorem 3. Let $\Delta = \prod_{i=1}^n [u_i, v_i]$ where $u_i, v_i \in Z$ and $u_i < v_i$ ($i \in N$). Then, if $a = (a_i)$, $b = (b_i) \in \Delta$, $a > b$, and

$$a = a^{(0)} \gg_{\Delta} a^{(1)} \gg_{\Delta} \dots \gg_{\Delta} a^{(q)} = b \quad (a^{(t)} \in \Delta)$$

then $q = |a > b| = \sum_{i=1}^n i(a_i - b_i)$, and $a^{(t)} \gg a^{(t+1)}$ for $t = 0, 1, \dots, q-1$.

Lemma 4. Let $\Delta = \prod_{i=1}^n [u_i, v_i]$ where $u_i, v_i \in Z$ and $u_i < v_i$ ($i \in N$). Let $b = (b_i)$, $c, d \in \Delta$, $c \neq d$, $c \gg b$, $d \gg b$, and $e = c \cup d$ in the lattice $(Z^n, \#)$. Then

$$e \in \Delta, e \gg c \gg b, e \gg d \gg b,$$

and for $r \in \Delta$ with $r \geq c$ and $r \geq d$,

$$r \geq e \text{ and } |r \geq e| = |r > b| - 2.$$

Proof. We have the two cases (1) and (2) :

(1) $c = (b_1+1, b_2, \dots, b_n)$ and

$$d = (b_1, \dots, b_{j-2}, b_{j-1}-1, b_j+1, b_{j+1}, \dots, b_n) \quad (2 \leq j \leq n);$$

(2) $c = (b_1, \dots, b_{i-2}, b_{i-1}-1, b_i+1, b_{i+1}, \dots, b_n)$ ($2 \leq i \leq n$) and

$$d = (b_1, \dots, b_{j-2}, b_{j-1}-1, b_j+1, b_{j+1}, \dots, b_n) \quad (i \leq j-1 < n).$$

In case (1) and $j-1 > 1$, we set

$$e' = (b_1+1, b_2, \dots, b_{j-2}, b_{j-1}-1, b_{j+1}+1, b_{j+2}, \dots, b_n).$$

In case (1) and $j-1 = 1$, we set

$$e' = (b_1, b_2+1, b_3, \dots, b_n).$$

In case (2) and $j-1 > i$, we set

$$e' = (b_1, \dots, b_{i-2}, b_{i-1}-1, b_i+1, b_{i+1}, \dots, \\ \dots, b_{j-2}, b_{j-1}-1, b_j+1, b_{j+1}, \dots, b_n).$$

In case (2) and $j-1 = i$, we set

$$e' = (b_1, \dots, b_{i-2}, b_{i-1}-1, b_i, b_{i+1}+1, b_{i+2}, \dots, b_n).$$

Then we have

$$e' \in \Delta, \quad e' \gg c \gg b, \text{ and } e' \gg d \gg b.$$

Since $(Z^n, \#)$ is a lattice, e' coincides with $c \cup d$ in $(Z^n, \#)$, that is, $e' = e$. Moreover, for $r \in \Delta$ with $r \geq c$ and $r \geq d$, we see that

$$r \geq e' \text{ and } |r \geq e'| = |r > b| - 2.$$

Lemma 5. Let $\Delta = \prod_{i=1}^n [u_i, v_i]$ where $u_i, v_i \in Z$ and $u_i < v_i$ ($i \in N$). Let $c, d \in \Delta$, $e = c \cup d$, and $f = c \cap d$ in the lattice $(Z^n, \#)$. Then $e, f \in \Delta$.

Proof. Clearly $u = (u_i) \in \Delta$, $c \geq u$, and $d \geq u$. Hence, we have $e \geq u$ and a finite length $|e \geq u|$. Let $m(c, d)$ be the smallest integer in $\{|e \geq w|; c \geq w, d \geq w, w \in \Delta\}$. If $m(c, d) = 0$ then $c = d = e$. By the induction with respect to $m(c, d)$, we shall prove that $e \in \Delta$. Hence, let $m(c, d) = t \geq 1$, and assume that $c' \cup d' \in \Delta$ for $c', d' \in \Delta$ with $m(c', d') < t$. If either $c \geq d$ or $d \geq c$ then our assertion holds trivially. Hence, let $c \not\geq d$ and $d \not\geq c$. Let w_0 be an element of Δ such that $|e \geq w_0| = m(c, d)$, and consider the following chains (note Theorem 3):

$$\begin{aligned} e \geq c &= c^{(0)} \gg c^{(1)} \gg \dots \gg c^{(p)} = w_0 \quad (c^{(r)} \in \Delta), \\ e \geq d &= d^{(0)} \gg d^{(1)} \gg \dots \gg d^{(q)} = w_0 \quad (d^{(s)} \in \Delta). \end{aligned}$$

Clearly $c^{(p-1)} \neq d^{(q-1)}$. To prove $e \in \Delta$, we shall distinguish the following cases :

- (1) $c \gg w_0, d \gg w_0$;
- (2) $c \gg w_0, d \geq d^{(q-2)} (q \geq 2)$;
- (3) $c \geq c^{(p-2)}, d \geq d^{(q-2)} (p \geq 2, q \geq 2)$.

In case (1), we have $e = c \cup d \in \Delta$ by Lemma 4.

In case (2), we set $w_1 = c \cup d^{(q-1)}$. Then, we have $w_1 \in \Delta$ by Lemma 4, and $w_1 \cup d = e$. Moreover, $w_1 \geq d^{(q-1)}, d \geq d^{(q-1)}$, and so, $m(w_1, d) < t$. Hence, we have $w_1 \cup d \in \Delta$ by the induction assumption, that is, $e \in \Delta$.

In case (3), we set $w_1 = c^{(p-1)} \cup d^{(q-1)}$. Then, we have $w_1 \in \Delta$ by Lemma 4, and $(c \cup w_1) \cup (w_1 \cup d) = e$. Since $m(c, w_1) < t$ and $m(w_1, d) < t$, we have $c \cup w_1 \in \Delta$ and $w_1 \cup d \in \Delta$. Moreover, since $m(c \cup w_1, w_1 \cup d) < t$, it follows that $(c \cup w_1) \cup (w_1 \cup d) \in \Delta$, that is, $e \in \Delta$. By the duality of the lattice $(Z^n, \#)$, we have $f \in \Delta$. This completes the proof.

Now, by virtue of the result of Lemma 5, we obtain the following theorem which is our main result.

Theorem 6. *Let $\Delta = \prod_{i=1}^n [u_i, v_i]$ where $u_i, v_i \in Z$ and $u_i < v_i$ ($i \in N$). Then Δ is a modular sublattice of $(Z^n, \#)$.*

By the general theory of modular lattices, it has been known that for any modular lattice with both chain conditions and for any chain $a \geq b$ in its lattice, the composition chains $a \gg \dots \gg b$ has a unique length (cf. Lemma 1, Theorem 3, Theorem 6, [1], [2], and [5]). Now, we shall prove the following

Theorem 7. *Let $\Delta = \prod_{i=1}^n [u_i, v_i]$ where $u_i, v_i \in Z$ and $u_i \leq v_i$ ($i \in N$). Then Δ is a modular lattice under the ordering in $(Z^n, \#)$, and whence it is a Cohen-Macaulay poset. If $a = (a_i), b = (b_i) \in \Delta, a > b$ and*

$$a = a^{(0)} \gg_{\Delta} a^{(1)} \gg_{\Delta} \dots \gg_{\Delta} a^{(q)} = b \ (a^{(i)} \in \Delta)$$

then $q = |a > b| - \sum_{i=1}^n i(0)(a_i - b_i) = \sum_{i=1}^n (i - i(0))(a_i - b_i)$ where $i(0)$ is the cardinal number of the set $\{j \in N; u_j = v_j, j < i\}$.

Proof. Let $\{\varepsilon(1), \dots, \varepsilon(m)\} = \{i \in N; u_i < v_i\}$ where $\varepsilon(1) < \dots < \varepsilon(m)$. Then, there is an ordered isomorphism

$$\psi: \Delta = \prod_{i=1}^n [u_i, v_i] \longrightarrow \prod_{j=1}^m [u_{\varepsilon(j)}, v_{\varepsilon(j)}] \ (\subset (Z^m, \#))$$

such that $\psi((d_1, \dots, d_n)) = (d_{\varepsilon(1)}, \dots, d_{\varepsilon(m)})$. By Theorem 6, $\psi(\Delta)$ is a mod-

ular sublattice of $(Z^m, \#)$. Hence Δ is a modular lattice under the ordering in $(Z^n, \#)$. Since $u_{\varepsilon(j)} < v_{\varepsilon(j)}$ for $j = 1, \dots, m$ and $\psi(a^{(t)}) \gg_{\psi(\Delta)} \psi(a^{(t+1)})$ in $\psi(\Delta)$ for $t = 0, 1, \dots, q-1$, we have $q = \sum_{j=1}^m j(a_{\varepsilon(j)} - b_{\varepsilon(j)})$ by Theorem 3. Hence, it follows from Lemma 1 that

$$\begin{aligned} |a > b| &= \sum_{i=1}^n i(a_i - b_i) = \sum_{j=1}^m (j + \varepsilon(j)(0))(a_{\varepsilon(j)} - a_{\varepsilon(j)}) \\ &= \sum_{j=1}^m j(a_{\varepsilon(j)} - b_{\varepsilon(j)}) + \sum_{j=1}^m \varepsilon(j)(0)(a_{\varepsilon(j)} - b_{\varepsilon(j)}) \\ &= q + \sum_{i=1}^n i(0)(a_i - b_i). \end{aligned}$$

Remark. Let B be a ring with an identity 1 and A a subring of B with common identity 1 of B . In [4], a sequence of additive A -endomorphisms $\{f_0 = 1, f_1, \dots, f_n\}$ of B is said to be a relative sequence of homomorphisms with Ψ if it satisfies the following conditions: for every $j \in N = \{1, 2, \dots, n\}$,

- (1) $f_j f_k = f_k f_j$ and $f_k(1) = 0$ for all $k \in N$.
- (2) There exists $\Psi_j = \{g(f_j, f_i); 0 \leq i \leq j\} \subset \text{End}(B_A)$ such that
 - (i) $f_j(xy) = \sum_{i=0}^j g(f_j, f_i)(x)f_i(y)$ for $x, y \in B$,
 - (ii) $g(f_j, f_0) = f_j$,
 - (iii) $g(f_j, f_j)$ is a ring isomorphism.

As is easily seen, for an A -automorphism σ of B , if we put $D = \sigma - 1$, then $\Phi = \{D^0 = 1, D, \dots, D^n\}$ becomes a relative sequence of homomorphisms with Ψ such that

$$\Psi_j = \{g(D^j, D^i) = \binom{j}{i} \sigma^i D^{j-i}; 0 \leq i \leq j\}.$$

A subset $\Phi = \{d_0 = 1, d_1, \dots, d_n\}$ of $\text{End}(B_A)$ is said to be an A -higher derivation of B if $d_j(xy) = \sum_{i=0}^j d_{j-i}(x)d_i(y)$ for $x, y \in B$. Then Φ becomes a relative sequence of homomorphisms with Ψ such that $\Psi_j = \{g(d_j, d_i) = d_{j-i}; 0 \leq i \leq j\}$.

Now, for a relative sequence of homomorphisms $\Phi = \{f_0 = 1, f_1, \dots, f_n\}$ with Ψ , we consider the multiplication subsemigroup L of $\text{End}(B_A)$ which is generated by Φ , and assume the following conditions on L :

- (1) There exists a positive integer q such that $(f_k)^q = 0$ and $(f_k)^s \neq 0$ for all $k \in N$ and $0 \leq s \leq q-1$.
- (2) $\prod_{i=1}^n f_i^{s_i} \neq 0$ if $0 \leq s_i \leq q-1$ for all $i \in N$.
- (3) If $\Omega = \prod_{i=1}^n f_i^{s_i}$ and $\Lambda = \prod_{i=1}^n f_i^{r_i}$ ($0 \leq s_i, r_i \leq q-1$), then $\Omega = \Lambda$ if and only if $s_i = r_i$ for all $i \in N$.

Then $L = U \cup \{0\}$ where $U = \{\prod_{i=1}^n f_i^{s_i}; 0 \leq s_i \leq q-1\}$ becomes a

commutative finite multiplicative subsemigroup of $\text{End}(B_A)$, and U becomes a modular lattice as is shown in Theorem 6.

Further, in this case, we can see that

$$(\cdot)^* : \Omega(xy) = \sum_{\Gamma \leq \Omega} g(\Omega, \Gamma)(x)\Gamma(y)$$

for $x, y \in B$ where $g(\Omega, \Gamma)$ is obtained as a sum of products of $g(f_{j_s}, f_{i_s})$'s with $\prod_s f_{j_s} = \Omega$ and $\prod_s f_{i_s} = \Gamma$. One of the authors made a study on Galois theory of B/A where $A = B^L = \{b \in B; \Lambda(b) = 0 \text{ for all } \Lambda (\neq 1) \in U\}$ in [4]. In that paper, $(\cdot)^*$ and the uniqueness of $|\Omega > 1|$ play important rôles. One of motivations of this paper comes from this study of Galois theory.

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