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IDENTIFICATION OF THE RATIO ERGODIC LIMIT FOR AN INVERTIBLE POSITIVE ISOMETRY ON L_1

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1. Introduction. Let T be an invertible positive isometry on L_1 of a σ-finite measure space. It is proved that if f and p are in L_1 and p is nonnegative, then the ratios $\left(\sum_{i=m}^{n} T^i f\right)\Big/\left(\sum_{i=m}^{n} T^i p\right)$ converge almost everywhere on the set $\left\{\sum_{-\infty}^{+\infty} T^i p > 0\right\}$ as $m \to -\infty$ and $n \to +\infty$, independently ; and the identification of the limit is obtained.

Let $(X, \mathscr{F}, \mu)$ be a σ-finite measure space and T a linear operator from $L_1 = L_1(X, \mathscr{F}, \mu)$ into itself. T is called *positive* if $f \geq 0$ implies $Tf \geq 0$, a *contraction* if $\|Tf\|_1 \leq \|f\|_1$ for all $f \in L_1$, and an *isometry* if $\|Tf\|_1 = \|f\|_1$ for all $f \in L_1$. For f and p in L_1, with $p \geq 0$, we write

$$R_m^n(f, p)(x) = \left(\sum_{i=m}^{n} T^i f(x)\right)\Big/\left(\sum_{i=m}^{n} T^i p(x)\right).$$

It follows from the Chacon-Ornstein theorem [2] that if T is a positive contraction on L_1 then the pointwise limit

$$\lim_{n\to+\infty} R_0^n(f, p)(x)$$

exists and is finite a.e. on the set $\left\{x : \sum_{0}^{+\infty} T^i p(x) > 0\right\}$; furthermore if C denotes the conservative part of T (i.e. $C = \left\{x : \sum_{0}^{+\infty} T^i g(x) = +\infty\right\}$ for some $g \in L_1$ with $g > 0$ a.e. on X), then the identification of the limit can be done, on C, by the Chacon identification theorem [1]. However, in this paper, we assume T to be an invertible positive isometry on L_1 and consider the ratios $R_m^n(f, p)(x)$, with $m < 0 < n$. Noticing that the conservative parts of T and T^{-1} coincide, we may apply these two theorems to infer that the pointwise limit

$$R_{-\infty}^{+\infty}(f, p)(x) = \lim_{m\to-\infty, n\to+\infty} R_m^n(f, p)(x)$$

exists and is finite a.e. on the set $\left\{x : \sum_{-\infty}^{+\infty} T^i p(x) > 0\right\}$. But the Chacon identification theorem can be applied only on the conservative part C, not on

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the whole set X. This is the starting point for the study in this paper. We shall obtain the identification of the limit $R_{-\infty}^{+\infty}(f, p)(x)$ on the whole set X. In the process of doing this, the almost everywhere existence of the limit $R_{-\infty}^{+\infty}(f, p)(x)$ is proved as a by-product ; we do not use the Chacon-Ornstein theorem. The method is chiefly dependent upon the argument given in Garsia ([4], pp. 39-41) for the identification of the limit function in the Chacon-Ornstein theorem.

2. Identification of the ratio ergodic limit. Let T be an invertible positive isometry on L_1. Since $\|T^{-1}f\|_1 = \|f\|_1$ for any $f \in L_1$, if $f \geq 0$ then we must have $T^{-1}f \geq 0$. Thus T^{-1} is also a positive isometry on L_1. For any nonnegative function h on $(X, \mathscr{F}, \mu)$, we define $Th = \lim_n Tf_n$, where $f_n \in L_1$ and $0 \leq f_n \uparrow h$. Clearly, this definition is independent of the choice of the sequence. Similarly, $T^{-1}h$ is defined.

$A \in \mathscr{F}$ is called *invariant* if

$$A = \text{supp } T1_A = \{x : T1_A(x) \neq 0\},$$

where 1_A denotes the indicator function of A. It is easy to check that $A \in \mathscr{F}$ is invariant if and only if $TL_1(A) = L_1(A)$, where $L_1(A) = \{f \in L_1 : \text{supp} f \subset A\}$. Therefore, if $0 \leq p \in L_1$ then the set $E(p) = \left\{x : \sum_{-\infty}^{+\infty} T^i p(x) > 0\right\}$ is invariant. The class of all invariant sets is denoted by $\mathscr{I}$. Since T is invertible, $\mathscr{I}$ is a sub-σ-field of $\mathscr{F}$.

We are now in a position to state the theorem.

Theorem. *If T is an invertible positive isometry on L_1, then for any f and p in L_1, with $p \geq 0$, the pointwise limit*

$$\lim_{m \to -\infty, n \to +\infty} R_m^n(f, p)(x) = R_{-\infty}^{+\infty}(f, p)(x)$$

exists and is finite a.e. on the set $E(p) = \left\{x : \sum_{-\infty}^{+\infty} T^i p(x) > 0\right\}$; furthermore, the limit function $R_{-\infty}^{+\infty}(f, p)$ is measurable with respect to $\mathscr{I}$ and satisfies

$$\int_A R_{-\infty}^{+\infty}(f, p) \cdot p \, d\mu = \int_A f \, d\mu$$

for all $A \in \mathscr{I}$ with $A \subset E(p)$.

To prove the theorem, we need some lemmas.

Lemma 1. *Let $h \in L_\infty = L_\infty(X, \mathscr{F}, \mu)$. Then $T^*h = h$ if and only if h is measurable with respect to $\mathscr{I}$.*

Proof. Suppose h is measurable with respect to $\mathscr{I}$. An easy approximation argument shows that for the proof of $T^*h = h$, it suffices to prove that $T^*1_A = 1_A$ for all $A \in \mathscr{I}$. But, $A \in \mathscr{I}$ implies $T^*1_A = 0$ on $X\backslash A$, because $\langle f, T^*1_A\rangle = \langle Tf, 1_A\rangle = \int_A Tf\, d\mu = 0$ for all $f \in L_1(X\backslash A)$. Similarly, $T^*1_{X\backslash A} = 0$ on A. Thus $T^*1_A = 1_A$, since $T^*1 = 1$.

Conversely, suppose $T^*h = h$. (Here we may and will assume without loss of generality that $0 \leq h \leq 1$.) Given an $\alpha > 0$, write $A = \{x : h(x) > \alpha\}$, $h_1(x) = \min\{h(x), \alpha\}$ and $h_2(x) = h(x) - h_1(x)$. Then $h = h_1 + h_2 = T^*h = (T^{-1})^*h = (T^{-1})^*h_1 + (T^{-1})^*h_2$. Since $(T^{-1})^*h_1 \leq \alpha$ and $h > \alpha$ on A, it follows that $(T^{-1})^*h_2 > 0$ on A. Since $\operatorname{supp} h_2 = A$, we then have

$$(T^{-1})^*1_A > 0 \text{ on } A.$$

By this and the fact that $T^*(T^{-1})^*1_A = 1_A$, we see that

$$T^*1_A = 0 \text{ on } X\backslash A.$$

Hence $T^*1_A \leq 1_A$, and by a similar argument, $(T^{-1})^*1_A \leq 1_A$. Consequently,

$$1_A = T^*(T^{-1})^*1_A \leq T^*1_A \leq 1_A,$$

which implies $1_A = T^*1_A$ and hence $A \in \mathscr{I}$. The proof is complete.

Lemma 2. *If $h \in L_\infty$ satisfies $T^*h = h$, then for any $f \in L_1$*

$$T(hf) = h(Tf).$$

Proof. If $A \in \mathscr{I}$ then, clearly, $T(1_A f) = 1_A(Tf)$ for all $f \in L_1$. This, together with Lemma 1 and an easy approximation argument, completes the proof.

Lemma 3. *If f and p are in L_1 and $p > 0$ a.e. on X, define*

$$M(f, p)(x) = \sup_{m \leq 0 \leq n} |R_m^n(f, p)(x)|.$$

Then, for any $\lambda > 0$, $\int_{\{M(f,p)>\lambda\}} p\, d\mu \leq \frac{4}{\lambda}\|f\|_1$.

Proof. Put

$$M_+(f, p)(x) = \sup_{0\leq n} |R_0^n(f, p)(x)|$$

and

$$M_-(f, p)(x) = \sup_{m\leq 0} |R_m^0(f, p)(x)|.$$

Then we have $M(f, p) \leq M_+(f, p)+M_-(f, p)$, and so

$$\{M(f, p) > \lambda\} \subset \left\{M_+(f, p) > \frac{\lambda}{2}\right\} \cup \left\{M_-(f, p) > \frac{\lambda}{2}\right\}.$$

Since $\left\{M_+(f, p) > \frac{\lambda}{2}\right\} \subset \left\{\sup_{0\leq n} \sum_{i=0}^{n} T^i\left(|f|-\frac{\lambda}{2}p\right) > 0\right\}$, the Hopf maximal ergodic theorem (see e.g. [4], p. 23) gives

$$\int_{\{M_+(f,p)>\frac{\lambda}{2}\}} p\, d\mu \leq \frac{2}{\lambda}\|f\|_1.$$

Similarly, $\int_{\{M_-(f,p)>\frac{\lambda}{2}\}} p\, d\mu \leq \frac{2}{\lambda}\|f\|_1$, and hence the proof is completed.

Proof of the Theorem. We can easily show that we need only check the validity of the Theorem when $p > 0$ a.e. on X. Thus in the following proof we will assume that $p > 0$ a.e. on X.

Let M be the class of all functions f of the form

$$f = hp+g-Tg, \text{ where } h \in L_\infty,\ g \in L_1 \text{ and } T^*h = h.$$

Making use of Lemma 2, if $f = hp+g-Tg \in M$ then

$$R_m^n(f, p)(x) = h(x)+\frac{T^m g(x) - T^{n+1}g(x)}{\sum_{i=m}^{n} T^i p(x)}.$$

Since $p > 0$ a.e. on X, the Chacon-Ornstein lemma (see e.g. Theorem 2.4.2 in [4]) shows that

$$\lim_{m\to-\infty, n\to+\infty} R_m^n(f, p) = h \text{ a.e. on } X.$$

Next, to prove that M is dense in L_1, let $k \in L_\infty$ be such that $\langle f, k\rangle = 0$ for all $f \in M$. Then we have $\langle g - Tg, k\rangle = \langle g, k - T^*k\rangle = 0$ for all $g \in L_1$. Thus $k = T^*k$, and $\langle kp, k\rangle = \int k^2 p \, d\mu = 0$. It follows that $k = 0$ a.e. on X, which proves the denseness of M in L_1.

For $f = hp + g - Tg \in M$, put

$$Hf = hp = R_{-\infty}^{+\infty}(f, p) \cdot p.$$

Then

$$\|Hf\|_1 = \int (\operatorname{sgn} h) hp \, d\mu = \int (\operatorname{sgn} h)[f - g + Tg] \, d\mu$$

where $\operatorname{sgn} h(x) = h(x)/|h(x)|$ if $h(x) \neq 0$, and is 0 if $h(x) = 0$. Since $T^*(\operatorname{sgn} h) = \operatorname{sgn} h$ by Lemma 1, it follows that

$$\int (\operatorname{sgn} h)[f - g + Tg] \, d\mu = \int (\operatorname{sgn} h) f \, d\mu \leq \|f\|_1.$$

Thus $\|Hf\|_1 \leq \|f\|_1$ $(f \in M)$. Since M is dense in L_1, H can be uniquely extended to a contraction operator on L_1. We will denote this extension by the same letter H. Clearly, if $A \in \mathscr{I}$ then

$$\int_A Hf \, d\mu = \int_A f \, d\mu$$

for all $f \in M$ and thus for all $f \in L_1$.

Now, to finish the proof of the Theorem, it suffices to show that

$$\lim_{m \to -\infty, n \to +\infty} R_m^n(f, p) = (1/p) Hf \text{ a.e. on } X$$

for all $f \in L_1$. To do this, we notice that if $f \in L_1$ and $e \in M$ then

$$\begin{aligned} &|R_m^n(f, p) - (1/p)Hf| \\ &\quad \leq |R_m^n(f - e, p) - (1/p)H(f - e)| + |R_m^n(e, p) - (1/p)He| \\ &\quad \leq M(f - e, p) + |(1/p)H(f - e)| + |R_m^n(e, p) - (1/p)He|. \end{aligned}$$

Since $R_m^n(e, p) \to (1/p)He$ a.e. on X as $m \to -\infty$ and $n \to +\infty$, independently, if we let

$$f^*(x) = \lim_{N \to +\infty} \sup_{m \leq -N, n \geq N} \left| R_m^n(f, p)(x) - \frac{Hf(x)}{p(x)} \right|$$

then, for any $\varepsilon > 0$,

$$\{f^* > 2\varepsilon\} \subset \{M(f-e, p) > \varepsilon\} \cup \{|(1/p)H(f-e)| > \varepsilon\}.$$

By Lemma 3,

$$\int_{\{M(f-e,p)>\varepsilon\}} p \, d\mu \leq \frac{4}{\varepsilon} \|f-e\|_1.$$

On the other hand,

$$\int_{\{|(1/p)H(f-e)|>\varepsilon\}} p \, d\mu \leq \frac{1}{\varepsilon} \|H(f-e)\|_1 \leq \frac{1}{\varepsilon} \|f-e\|_1.$$

Here $\|f-e\|_1$ can be arbitrarily small. Thus $\int_{\{f^*>2\varepsilon\}} p \, d\mu = 0$. Since $\varepsilon > 0$ is arbitrary, it follows that $f^* = 0$ a.e. on X, and this completes the proof.

Remark. If $\{T^n : -\infty < n < +\infty\}$ is a group of positive linear operators on L_1 satisfying $\sup_n \|T^n\| = K < +\infty$, then the convergence result in the Theorem holds. In fact, if L denotes a Banach limit (cf. [3]) and if we define $\lambda(A) = \mathrm{L}\left(\int T^n 1_A \, d\mu\right)$ for $A \in \mathscr{F}$ with $\mu(A) < +\infty$ and $\lambda(A) = \sup \{\lambda(B) : B \in \mathscr{F}$ with $B \subset A$ and $\mu(B) < +\infty\}$ for $A \in \mathscr{F}$ with $\mu(A) = +\infty$, then, as is easily seen, $(X, \mathscr{F}, \lambda)$ is a σ-finite measure space such that $K^{-1}\mu \leq \lambda \leq K\mu$ and T is an invertible positive isometry on $L_1(X, \mathscr{F}, \lambda)$. Since $f \in L_1(X, \mathscr{F}, \mu)$ if and only if $f \in L_1(X, \mathscr{F}, \lambda)$, the convergence result follows from the Theorem.

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