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ON RIEMANNIAN MANIFOLDS OF NON-POSITIVE SECTIONAL CURVATURE ADMITTING A KILLING VECTOR FIELD

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Let M be an n -dimensional connected and complete Riemannian manifold of non-positive sectional curvature. Let X be a Killing vector field on M and φ_t , $t \in \mathbb{R}$, the 1-parameter subgroup of isometries generated by X . In this note, supposing that the length of X is bounded, we shall study some properties of M .

1. Killing vector field with bounded length. We consider the function $f = \frac{1}{2} \langle X, X \rangle$ on M . Let $y \in M_p$ be a tangent vector at $p \in M$ and γ the geodesic with initial velocity y . Since X is a Jacobi field along γ and M is of non-positive sectional curvature, we have

$$(1) \quad \frac{d^2}{dt^2} f \circ \gamma(0) = \|\nabla_y X\|^2 - \langle R(y, X)X, y \rangle \geq 0.$$

Proposition 1. *Let M be an n -dimensional connected and complete Riemannian manifold of non-positive sectional curvature. Let X be a Killing vector field on M . If the length of X is bounded, then X is a parallel vector field on M . In particular, the orbit $\{\varphi_t(p); t \in \mathbb{R}\}$ is a geodesic for each point $p \in M$.*

Proof. Since the length of X is bounded, by (1) we see that f must be constant. Thus, by (1) we have $\nabla_y X = 0$ for any point $p \in M$ and any tangent vector $y \in M_p$. Hence, X is parallel.

Remark 1. In Proposition 1 we have $\langle R(Y, X)X, Y \rangle = 0$ for any tangent vector to M . If $\dim M = 2$, then M is flat.

In the following sections, we assume that M is always an n -dimensional connected and complete Riemannian manifold of non-positive sectional curvature admitting a Killing vector field X with bounded length.

By Proposition 1 we may assume that X is unit.

2. An isometric immersion of the euclidean two-plane.

Proposition 2. *For any plane section σ at $p \in M$ such that $X_p \in \sigma$,*

the euclidean plane E^2 can be isometrically immersed in M so that its image is tangent to σ at p .

Proof. Let p be a given point of M and $\tau: R \rightarrow M$ a geodesic such that $\tau(0)=p$, $\langle \dot{\tau}(0), X \rangle = 0$, and $\|\dot{\tau}(s)\|=1$. We define a mapping $g: E^2 \rightarrow M$ by $g(s, t) = \varphi_t(\tau(s))$. We set $Y = \partial g / \partial s$. Clearly $X = \partial g / \partial t$. Then, we have $\nabla_Y Y = 0$, because φ_t is isometric. Clearly $\nabla_X X = 0$. Since $\nabla_Y X = 0$ and $T_Y X = T_X Y$, where T_Y is the shape operator of $W = g(E^2)$ in M , we see that W is totally geodesic. By Remark 1 and the equation of Gauss, W is flat.

Corollary 1. *If M is compact and the orbit $\{\varphi_t(p); t \in R\}$ is not periodic for any point p of M , then the euclidean cylinder $S^1 \times R$ can be isometrically immersed in M .*

Proof. Since M is compact, there exists a closed geodesic $\rho: S^1 \rightarrow M$. Then, we see that the mapping $h: S^1 \times R \rightarrow M$ defined by $h(s, t) = \varphi_t(\rho(s))$ is an isometric immersion.

3. Decomposition of M . We set $D_p = \{Y \in M_p; \langle Y, X_p \rangle = 0\}$. Since X is parallel, we can easily prove the following

Lemma 1. *The $(n-1)$ -dimensional distribution $D: p \rightarrow D_p$ is completely integrable.*

We denote by $S(p)$ the $(n-1)$ -dimensional maximal connected integral submanifold of D containing a point p of M . Since X is parallel, we have the following

Lemma 2. *$S(p)$ is a totally geodesic hypersurface of M for each point p of M .*

By Lemma 2, M is locally isometric to $S(p) \times R$ in a neighbourhood of $p \in M$.

Theorem 1. *If M is non-compact and $S(p_0)$ is compact for some point p_0 of M , then M is isometric to $S(p_0) \times R$.*

To prove Theorem 1, we shall prepare some lemmas. Since M is non-compact and $S(p_0)$ is compact, we can take a minimal divergent geodesic ray $\gamma: [0, \infty) \rightarrow M$ such that $\gamma(0) = p' \in S(p_0)$, $\dot{\gamma}(0) = \varepsilon X_{p'}$, $\varepsilon = \pm 1$, and $\|\dot{\gamma}(t)\|=1$. Then $\gamma(t) = \varphi_{\varepsilon t}(p')$, $t \in [0, \infty)$, because the orbit $\{\varphi_t(p'); t \in R\}$ is geodesic. We may assume therefore that $\gamma(t) = \varphi_t(p')$, $t \in [0, \infty)$.

Lemma 3. *The orbit $\{\varphi_t(p'); t \in R\}$ is a minimal geodesic line.*

Proof. For $t, t_2 \in R$ such that $t < 0, t_1 < t_2$, let ρ be a shortest geodesic segment from $\varphi_{t_1}(p')$ to $\varphi_{t_2}(p')$. Since $\varphi_{|t_1|}$ is isometric, $\varphi_{|t_1|} \circ \rho$ is a shortest geodesic segment from p' to $\varphi_{|t_1|+t_2}(p')$. Then $\varphi_{|t_1|} \circ \rho \subset \{\varphi_t(p'); t \in [0, \infty)\}$, because $\{\varphi_t(p'); t \in [0, \infty)\}$ is a minimal geodesic ray. Thus $\rho \subset \{\varphi_t(p'); t \in R\}$.

Lemma 4. $\varphi_t(q) \notin S(p_0)$ holds for each point $q \in S(p_0)$ and $t \in (0, \infty)$.

Proof. Suppose $\varphi_{t'}(q) \in S(p_0)$ for some point $q \in S(p_0)$ and some $t' > 0$. Since $S(p_0)$ is maximal, $\varphi_{t'}(S(p_0)) \subset S(p_0)$, and so is $\varphi_{t'}(p') \in S(p_0)$. This is contradictory to Lemma 3.

From Lemma 4 the following is immediate.

Lemma 5. $\varphi_s(q) \notin S(p_0)$ holds for each point $q \in S(p_0)$ and $s \in (-\infty, 0)$.

Proof of Theorem 1. Since $S(p_0)$ is compact, there exists a perpendicular from each point of M to $S(p_0)$. It is contained in some orbit $\{\varphi_t(p); t \in R\}$, $p \in S(p_0)$. Thus, by Lemmas 4 and 5, we see that the orbit $\{\varphi_t(p); t \in R\}$ is a minimal geodesic line for each point p of $S(p_0)$. Therefore, we can define an isometry $\phi: S(p_0) \times R \rightarrow M$ by $\phi(p, t) = \varphi_t(p)$. This completes the proof of Theorem 1.

In the above theorem, since $S(p_0)$ is compact and totally geodesic, there exists a closed geodesic $\tau: S^1 \rightarrow S(p_0)$ which is also a closed geodesic of M . Hence, we have the following

Corollary 2. The euclidean cylinder $S^1 \times R$ can be isometrically imbedded in M .

Theorem 2. If the orbit $\Gamma = \{\varphi_t(p_0); t \in R\}$ is periodic and $\Gamma \cap S(p_0) = \{p_0\}$ for some point p_0 of M , then M is the bundle space of a fibre bundle whose base space is S^1 and fibre $S(p_0)$.

Proof. Let t_0 be the minimal period of Γ . We set $\varphi_t(S(p_0)) = S_t$, $0 \leq t < t_0$. Let q be a point of $M \setminus \Gamma$ and τ a perpendicular from q to Γ . Then $\tau \subset S_t$ for some $t \in [0, t_0)$, and clearly $q \in S_t$. Since each S_t is maximal and $\Gamma \cap S_0 = \{p_0\}$, we see that $S_{t_1} \cap S_{t_2} = \emptyset$, $t_1 \neq t_2 \in [0, t_0)$. Thus, we have

$$(2) \quad M = \bigcup_{0 \leq t < t_0} S_t, \quad S_{t_1} \cap S_{t_2} = \emptyset \quad \text{for } t_1 \neq t_2 \in [0, t_0).$$

Suppose $q \in S_t$. Then we can express a coordinate neighbourhood of q in M as $U_s \times (t - \varepsilon, t + \varepsilon)$, where U_s is a coordinate neighbourhood of q in S_t and $0 < \varepsilon < t_0/2$. Then the projection $\pi: M \rightarrow S^1 = R/t_0Z$ can be

defined by $\pi(q')=t'$, $q' \in S_{t'}$, $t' \in (t-\varepsilon, t+\varepsilon)$. π is a Riemannian submersion and by (2) $\pi^{-1}(t')=S_{t'}$. The mapping $\phi_k: (t-\varepsilon, t+\varepsilon) \times S(p_0) \longrightarrow \pi^{-1}(t-\varepsilon, t+\varepsilon)$ defined by $\phi_k(t', p) = \varphi_{\kappa_0+t'}(p)$, $k \in \mathbb{Z}$, is an isometry. If we fix t' , then $\phi_k^{-1} \circ \phi_j$ is an isometry of $S(p_0)$. Now we set $\phi = \varphi_{t_0}$. Then, ϕ is an isometry of $S(p_0)$ and $\phi(p_0)=p_0$. Let $G = \{\phi^k\}$, $k \in \mathbb{Z}$, $\phi^0 = \text{id.}$, be the discrete group generated by ϕ . Then G is an effective topological group acting on $S(p_0)$. Therefore, we see that M is the bundle space of a fibre bundle whose base space is S^1 , fibre $S(p_0)$, and structure group is G .

Corollary 3. $\pi_1(M)/\pi_1(S(p_0)) \cong \mathbb{Z}$.

Corollary 4. If $\varphi_{t_0}(p)=p$ for any point p of $S(p_0)$, then M is isometric to $S(p_0) \times S^1$.

Remark 2. If M is compact, $S(p)$ is compact for any point p of M .

Remark 3. If $\dim M=2$, then $\phi(q)=q$ or $\phi^2(q)=q$ for any point q of $S(p_0)$.

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