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ON COPRIMARY DECOMPOSITION THEORY FOR MODULES

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Recently, in his paper [2], D. Kirby introduced the notion of coprimary modules over a commutative ring, and obtained several results on coprimary decompositions for Artinian modules. In this note, by making use of the technique employed in [1] and [3], we shall investigate the s-coprimary decomposition theory for modules over non-commutative rings.

1. Preliminaries. Throughout, R will represent a ring, and M a non-zero left R-module. Given an ideal $\mathfrak{a}$ of R, $M^{\mathfrak{a}}$ is defined to be the intersection $\cap\, \mathfrak{b}M$, where $\mathfrak{b}$ runs over all the finite products of ideals of R not contained in $\mathfrak{a}$. ($M^R=M$ by definition.) As in [3], $p(M)$ will denote the prime radical of $l(M)=\{x\in R\,|\,xM=0\}$. If $l(M')\subseteq p(M)$, or equivalently $p(M')=p(M)$, for every non-zero submodule M' of M, 0 is defined to be a primary submodule of M (cf. [1]). Now, dualizing the notion, M is defined to be *coprimary* if $l(M/M')\subseteq p(M)$, or equivalently $p(M/M')=p(M)$, for every proper submodule M' of M. In case M is coprimary and $\mathfrak{p}=p(M)$, M will be called a $\mathfrak{p}$-*coprimary* module. If M is coprimary and $p(M)$ is nilpotent modulo $l(M)$, M is defined to be *s-coprimary.*

The next is easy, and will be freely used without mention.

Proposition 1. *The following conditions are equivalent:*

(1) *M is coprimary.*

(2) *$\mathfrak{a}M=M$ for every ideal $\mathfrak{a}$ of R not contained in $p(M)$.*

(3) *$M^{p(M)}=M$.*

An idea $\mathfrak{p}$ of R is called a *coassociated ideal* of M if there exists a proper submodule M' such that M/M' is $\mathfrak{p}$-coprimary. The set of all coassociated ideals of M will be denoted by $P^*(M)$. ($P^*(0)=\emptyset$ by definition.) If there exists an ideal $\mathfrak{p}$ in R such that $P^*(M/M')=\{\mathfrak{p}\}$ for every proper submodule M' of M then M is called a P^*-*module.*

Proposition 2. (1) *If M is coprimary, and M' a proper submodule of M, then $l(M/M')$ is a right-primary ideal.*

(2) *Let N and M' be submodules of M. If N is $\mathfrak{p}$-coprimary and not contained in M' then $N+M'/M'$ is $\mathfrak{p}$-coprimary.*

(3) *If N and N' are $\mathfrak{p}$-coprimary submodules of M, then so is $N+N'$.*

Proof. (1) Assume that there exist ideals $\mathfrak{a}$, $\mathfrak{b}$ of R such that $\mathfrak{a}\mathfrak{b}\subseteq l(M/M')$ and $\mathfrak{b}\not\subseteq p(M/M')$. Then, $M'\supseteq\mathfrak{a}\mathfrak{b}M=\mathfrak{a}M$, namely, $\mathfrak{a}\subseteq l(M/M')$.

(2) This is obvious by $N+M'/M'\cong N/N\cap M'$.

(3) Since $l(N+N')=l(N)\cap l(N')$, we have $p(N+N')=p(N)\cap p(N')=\mathfrak{p}$. If $\mathfrak{a}$ is an ideal of R not contained in $\mathfrak{p}$, then $\mathfrak{a}N=N$ and $\mathfrak{a}N'=N'$, and hence $\mathfrak{a}(N+N')=N+N'$.

Proposition 3. (1) *If M is $\mathfrak{p}$-s-coprimary then $\mathfrak{p}$ is prime and $\mathfrak{a}M\neq M$ for every ideal $\mathfrak{a}$ of R contained in $\mathfrak{p}$.*

(2) *Let N and M' be submodules of M. If N is $\mathfrak{p}$-s-coprimary and is not contained in M' then $M+M'/M'$ is $\mathfrak{p}$-s-coprimary.*

(3) *If N and N' are $\mathfrak{p}$-s-coprimary, then so is $N+N'$.*

Proof. (2) and (3) are easy by Prop. 2 (2) and (3).

(1) If $\mathfrak{a}$ is an ideal of R contained in $\mathfrak{p}$ then there exists a positive integer h such that $\mathfrak{a}^hM=0$, which means $\mathfrak{a}M\neq M$. Next, we shall prove that $\mathfrak{p}$ is prime. Let $\mathfrak{b}$, $\mathfrak{c}$ be ideals of R such that $\mathfrak{b}\mathfrak{c}\subseteq\mathfrak{p}$. As was shown just above, there holds $\mathfrak{b}\mathfrak{c}M\neq M$. If $\mathfrak{c}\not\subseteq\mathfrak{p}$, then $M\supset\mathfrak{b}\mathfrak{c}M=\mathfrak{b}M$, and hence $\mathfrak{b}\subseteq\mathfrak{p}$.

Proposition 4. *If N is a submodule of M then $P^*(M/N)\subseteq P^*(M)\subseteq P^*(N)\cup P^*(M/N)$.*

Proof. Let S be a proper submodule of M such that M/S is $\mathfrak{p}$-coprimary. If $S+N\neq M$ then $M/S+N$ *is* $\mathfrak{p}$-coprimary and $\mathfrak{p}\in P^*(M/N)$. On the other hand, if $S+N=M$ then $N/N\cap S\cong M/S$ is $\mathfrak{p}$-coprimary and $\mathfrak{p}\in P^*(N)$. The inclusion $P^*(M/N)\subseteq P^*(M)$ is almost evident.

2. Coprimary decompositions. A finite set $\{M_i\,|\,i\in I\}$ of coprimary (resp. s-coprimary) submodules of M is called a *coprimary* (resp. *s-coprimary*) *decomposition* of M if $M=\sum_{i\in I}M_i$, $M\neq\sum_{i\in I'}M_i$ for every proper subset I' of I, and $p(M_i)\neq p(M_j)$ for every $i\neq j$. If $\{N_j\,|\,j\in J\}$ is a finite set of coprimary (resp. s-coprimary) submodules of M with $M=\sum_{j\in J}N_j$, then Prop. 2 (3) (resp. Prop. 3 (3)) secures the existence of a coprimary (resp. s-coprimary) decomposition of M.

Proposition 5. *Let $\{M_i\,|\,i=1,\cdots,k\}$ be an s-coprimary decomposition of M, and $\mathfrak{p}_i=p(M_i)$ $(i=1,\cdots,k)$. Let $\mathfrak{a}$ be an ideal of R.*

(1) *If $\mathfrak{a}M=M$ then $\mathfrak{a}\not\subseteq\mathfrak{p}_i$ $(i=1,\cdots,k)$, and conversely.*

(2) *$M^{\mathfrak{a}}=\sum_{\mathfrak{p}_i\subseteq\mathfrak{a}}M_i$. If $\mathfrak{a}$ does not contain all $\mathfrak{p}_i$'s then $M^{\mathfrak{a}}=\mathfrak{b}M$ with a finite product $\mathfrak{b}$ of ideals of R not contained in $\mathfrak{a}$.*

Proof. (1) If $\mathfrak{a}$ is contained in some $\mathfrak{p}_i$ then $\mathfrak{a}^h M_i = 0$ for some positive integer h. Accordingly, we have $\mathfrak{a}^h M \neq M$, whence it follows $\mathfrak{a}M \neq M$. The converse is obvious.

(2) Without loss of generality, we may assume that $\mathfrak{p}_1, \cdots, \mathfrak{p}_l \subseteq \mathfrak{a}$ and $\mathfrak{p}_{l+1}, \cdots, \mathfrak{p}_k \not\subseteq \mathfrak{a}$. In case $l=k$, our assertion is evident by (1). Henceforth, we assume that $(0 \leqq)\ l < k$. There exists a positive integer h such that $\mathfrak{p}_j^h M_j = 0$ $(j = l+1, \cdots, k)$. Since every $\mathfrak{p}_i$ is prime by Prop. 3 (1), $\mathfrak{b} = (\mathfrak{p}_{l+1} \cdots \mathfrak{p}_k)^h \not\subseteq \mathfrak{p}_i$ $(i=1, \cdots, l)$. There holds then $M^{\mathfrak{a}} \supseteq \sum_{i=1}^{l} M_i^{\mathfrak{a}} \supseteq \sum_{i=1}^{l} M_i^{\mathfrak{p}_i} = \sum_{i=1}^{l} M_i = \mathfrak{b}M \supseteq M^{\mathfrak{a}}$, namely, $M^{\mathfrak{a}} = \sum_{i=1}^{l} M_i = \mathfrak{b}M$.

Theorem 1. *Let* $\{M_i \mid i=1, \cdots, k\}$ *be an s-coprimary decomposition of* M, *and* $\mathfrak{p}_i = p(M_i)$ $(i=1, \cdots, k)$. *Then there holds the following*:

(1) $P^*(M) = \{\mathfrak{p}, \cdots, \mathfrak{p}_k\}$.

(2) *A prime divisor* $\mathfrak{p}$ *of* $l(M)$ *is contained in* $P^*(M)$ *if and only if* $\mathfrak{p}M^{\mathfrak{p}} \neq M^{\mathfrak{p}}$. *Every minimal prime divisor of* $l(M)$ *is contained in* $P^*(M)$, *and if* $\mathfrak{p}_i$ *is minimal in* $P^*(M)$ *then* $M^{\mathfrak{p}_i} = M_i$.

Proof. (1) Evidently, $p(M)$ is nilpotent modulo $l(M)$. Next, we claim that if M is s-coprimary then $k=1$. Since $\mathfrak{p} = p(M) = \bigcap_{i=1}^{k} \mathfrak{p}_i$ is prime by Prop. 3 (1), without loss of generality, we may assume that $\mathfrak{p}_1, \cdots, \mathfrak{p}_m \subseteq \mathfrak{p}$ $(m \geqq 1)$ and $\mathfrak{p}_{m+1}, \cdots, \mathfrak{p}_k \not\subseteq \mathfrak{p}$. Then, by Prop. 5, $M = M^{\mathfrak{p}} = \sum_{i=1}^{m} M_i$, whence it follows $m=k$. Combining this with $\mathfrak{p} = \bigcap_{i=1}^{k} \mathfrak{p}_i$, we obtain $k=1$. Now, we shall proceed into the proof of (1). Obviously, $M/\sum_{j \neq i} M_j$ is $\mathfrak{p}_i$-coprimary as a non-zero homomorphic image of M_i, and so $P^*(M) \supseteq \{\mathfrak{p}_1, \cdots, \mathfrak{p}_k\}$. Conversely, assume that M/N is $\mathfrak{p}$-coprimary. Then, $M/N = \sum_{i=1}^{k} (M_i + N)/N$, where $(M_i+N)/N$ is either 0 or $\mathfrak{p}_i$-s-coprimary by Prop. 3 (2). Accordingly, by Prop. 3 (3), M/N has an s-coprimary decomposition $\{M_j'/N \mid j=1. \cdots, l\}$ such that $\{\mathfrak{p}(M_j'/N) \mid j=1, \cdots, l\} \subseteq \{\mathfrak{p}_1, \cdots, \mathfrak{p}_k\}$. Then, as was mentioned above, we obtain $l=1$ and $\mathfrak{p} \in \{\mathfrak{p}_1, \cdots, \mathfrak{p}_k\}$.

(2) If $\mathfrak{p}$ is contained in $P^*(M) = \{\mathfrak{p}_1, \cdots, \mathfrak{p}_k\}$, then we may assume that $\mathfrak{p}_1, \cdots, \mathfrak{p}_{m-1} \subseteq \mathfrak{p} = \mathfrak{p}_m$ and $\mathfrak{p}_{m+1}, \cdots, \mathfrak{p}_k \not\subseteq \mathfrak{p}$. Then, $M^{\mathfrak{p}} = M_1 + \cdots + M_m \neq \mathfrak{p}M^{\mathfrak{p}}$ by Prop. 5. Next, we shall prove the converse. Since $\mathfrak{p} \supseteq \bigcap_{i=1}^{k} \mathfrak{p}_i$, we may assume that $\mathfrak{p}_1, \cdots, \mathfrak{p}_m \subseteq \mathfrak{p}$ $(m \geqq 1)$ and $\mathfrak{p}_{m+1}, \cdots, \mathfrak{p}_k \not\subseteq \mathfrak{p}$. (If $\mathfrak{p}$ is a minimal prime divisor of $l(M)$ then it is obviously in $P^*(M)$.) Since $M^{\mathfrak{p}} \neq \mathfrak{p}M^{\mathfrak{p}}$, we obtain $\sum_{i=1}^{m} M_i \neq \mathfrak{p}(\sum_{i=1}^{m} M_i)$ by Prop. 5 (2), and hence $\mathfrak{p} \subseteq \mathfrak{p}_i$, namely, $\mathfrak{p} = \mathfrak{p}_i$, for some $i \leqq m$ (Prop. 5 (1)). The final assertion is evident by Prop. 5 (2).

Now, let $\{M_i \mid i=1, \cdots, k\}$ be an s-coprimary decomposition of M. A subset P^* of $\{\mathfrak{p}_i = p(M_i) \mid i=1, \cdots, k\}$ is called an *isolated subset* of $\{\mathfrak{p}_i \mid i = 1, \cdots, k\}$ if every $\mathfrak{p}_i$ contained in one of the members of P^* belongs to P^*. For an isolated subset P^* of $\{\mathfrak{p}_i \mid i=1, \cdots, k\}$ we set $M^{P^*} = \sum_{\mathfrak{p}_i \in P^*} M_i$.

which coincides with $\sum_{\mathfrak{p}\in P^*} M^{\mathfrak{p}}$ by Prop. 5 (2) and is called a *coisolated component* of M. By Th. 1, we readily obtain the following:

Theorem 2. *Suppose that M has an s-coprimary decomposition. Then, the set of coisolated components of M does not depend on the choice of s-coprimary decompositions of M.*

Finally, we shall examine cases in which every s-coprimary decomposition is direct.

Theorem 3. *Suppose R contains 1 and M is unital. Let $\{M_i \mid i=1, \cdots, k\}$ be a finite set of s-coprimary submodules of M such that $M=\sum_{i=1}^k M_i$ and $(R \neq)\ \mathfrak{p}_i = p(M_i)$ $(i=1, \cdots, k)$. If $\mathfrak{p}_i$'s are pairwise comaximal, then $M=\bigoplus_{i=1}^k M_i$ and this is the unique s-coprimary decompsition of M.*

Proof. Since $\mathfrak{p}_i$'s are comaximal, so are $l(M_i)$'s, and so $l(M_i)+l(\sum_{j\neq i} M_j)=R$. Hence, $M_i \cap \sum_{j\neq i} M_j = (l(M_j)+l(\sum_{j\neq i} M_j))(M_i \cap \sum_{j\neq i} M_j) = 0$, which means $M=\bigoplus_{i=1}^k M_i$. Obviously, the last is an s-coprimary decomposition of M and $P^*(M)=\{\mathfrak{p}_1, \cdots, \mathfrak{p}_k\}$ by Th. 1. Further, every $\mathfrak{p}_i$ is minimal in $P^*(M)$ and $M_i = M^{\mathfrak{p}_i}$ by Th. 1 (2), which means the uniqueness of the s-coprimary decompositions.

Corollary. *Let R be a left Artinian ring with 1. If M is a completely reducible module with a finite number of homogeneous components, then the idealistic decomposition of M is the unique s-coprimary decomposition of M.*

Proof. If N is an arbitrary irreducible submodule of M then $l(N)=p(N)$ is a maximal ideal of R and N is isomorphic to a minimal left ideal of $R/l(N)$. We have seen therefore that if N' is another irreducible submodule of M non-isomorphic to N then $p(N') \neq p(N)$. Further, to be easily seen, the homogeneous component of M containing N is $p(N)$-s-coprimary. Now, our assertion is a consequence of Th. 3.

3. Coprimary decomposition theory and AR^*-modules. When every non-zero submodule of M has a coprimary (resp. s-coprimary) decomposition, M is said to have the *coprimary* (resp. *s-coprimary*) *decomposition theory*. In case M has the coprimary (resp. s-coprimary) decomposition theory, every non-zero factor submodule of M has a coprimary (resp. s-coprimary) decomposition by Prop. 2 (resp. Prop. 3), and if N is a primary submodule of M then M/N is coprimary. Conversely, in case M has the primary decomposition theory, if M/N is coprimary then N is primary. (Cf. [3].)

If M satisfies one of the following equivalent conditions (I) and (II), it is called an *AR^*-module*:

(I) For each submodule N of M and each ideal $\mathfrak{a}$ of R, there exists a positive integer h such that $N+\mathfrak{a}^{-h}0\supseteq\mathfrak{a}^{-1}N\ (=\{x\in M\mid \mathfrak{a}x\subseteq N\})$.

(II) For each submodule N of M and each ideal $\mathfrak{a}$ of R, there exists a positive integer h such that $\mathfrak{a}N+(\mathfrak{a}^{-h}0\cap N)=N$.

One may remark here that if M is an AR^*-module, then so is every non-zero factor submodule of M. Finally, M is said to be *p^*-worthy* if $P^*(M^*)$ is finite and non-empty for every non-zero factor submodule M^* of M.

Proposition 6. *If M has the s-coprimary decomposition theory, then there holds the following*:

(1) *M is an s-module, that is, $p(M^*)$ is nilpotent modulo $l(M^*)$ for every non-zero factor submodule M^* of M.*

(2) *For every submodule N of M, if $N^{\mathfrak{a}_1}\supset(N^{\mathfrak{a}_1})^{\mathfrak{a}_2}\supset\cdots\supset(\cdots(N^{\mathfrak{a}_1})\cdots)^{\mathfrak{a}_n}$* then $n< s(N)$ with a positive integer $s(N)$ depending solely on N.

(3) *M is p^*-worthy.*

(4) *M is an AR^*-module.*

Proof. (1)–(3) are easy by Props. 3 and 5 and Th. 1.

(4) It suffices to consider non-zero N. Let $\{N_i \mid i=1,\cdots,k\}$ be an s-coprimary decomposition of N. We may assume then $\mathfrak{a}\subseteq p(N_1),\cdots,p(N_l)$ and $\mathfrak{a}\not\subseteq p(N_{l+1}),\cdots,p(N_k)$. There exists a positive integer h such that $\mathfrak{a}^hN_i=0$ $(i=1,\cdots,l)$. Since $N_1+\cdots+N_l\subseteq\mathfrak{a}^{-h}0\cap N$ and $N_{l+1}+\cdots+N_k\subseteq\mathfrak{a}N$, it follows $\mathfrak{a}N+(\mathfrak{a}^{-h}0\cap N)=N$.

Proposition 7. *Let M be an AR^*-module and an s-module.*

(1) *If N is a P^*-submodule of M then N is s-coprimary.*

(2) *If M is Artinian, then M has the s-coprimary decomposition theory.*

Proof. (1) Let N' be an arbitrary proper submodule of N. Since $P^*(N/N')=P^*(N)=\{\mathfrak{p}\}$, there exists a proper submodule N'' of N containing N' such that N/N'' is $\mathfrak{p}$-s-coprimary. Now, let W be an arbitrary proper submodule of N, and choose a proper submodule W' of N containing W such that N/W' is $\mathfrak{p}$-s-coprimary. Since $l(N/N')\subseteq l(N/N'')\subseteq\mathfrak{p}$, Prop. 3 (1) yields $l(N/N')N+W'\subset N$, which means that $l(N/N')N$ is small in N. By the condition (II), there exists a positive integer h such that $l(N/N')N+((l(N/N'))^{-h}0\cap N)=N$. It follows then $(l(N/N'))^{-h}0\cap N=N$, namely, $l(N/N')^hN=0$. This means that $l(N/N')\subseteq p(N)$, that is, N is s-coprimary.

(2) Since every non-zero submodule of M is a finite sum of sum-

irreducible submodules, it remains only to show that if a non-zero submodule N of M is not s-coprimary then N is not sum-irreducible. There exists a proper submodule N' of N such that $\mathfrak{a}=l(N/N')\not\subseteq p(N)$, or $\mathfrak{a}^n N\neq 0$ for every positive integer n. By the condition (II), there exists a positive integer h such that $\mathfrak{a}N+(\mathfrak{a}^{-h}0\cap N)=N$. It is obvious that $\mathfrak{a}N\subseteq N'\subset N$ and $\mathfrak{a}^{-h}0\cap N\subset N$.

Combining Prop. 6 with Prop. 7, we obtain at once

Theorem 4. *Let M be an Artinian module. In order that M have the s-coprimary decomposition theory, it is necessary and sufficient that M be an AR^*-module and an s-module.*

In [2], D. Kirby has proved that every unital Artinian s-module over a commutative ring with 1 has the s-coprimary decomposition theory. However, the following example will show that it is not the case for non-commutative rings.

Example. Let $R=\begin{pmatrix} F & 0 & 0 \\ F & F & 0 \\ F & F & F \end{pmatrix}$, where F is a field, and M the left R-module R. To be easily seen, $\mathfrak{a}=\begin{pmatrix} 0 & 0 & 0 \\ F & 0 & 0 \\ F & F & F \end{pmatrix}$ is an ideal of R and $\mathfrak{a}\supset\mathfrak{a}^2=\mathfrak{a}^3=\cdots=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ F & F & F \end{pmatrix}$. Moreover, $\mathfrak{a}^{-2}0\cap\mathfrak{a}=0$, and we have $\mathfrak{a}\cdot\mathfrak{a}+(\mathfrak{a}^{-2}0\cap\mathfrak{a})\neq\mathfrak{a}$, which means that M is not an AR^*-module.

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