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ON THE FIXED POINT SET OF S^1-ACTIONS ON $CP^m \times CP^n$

Dedicated to Professor Yasuo Nasu on his 60th birthday

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Introduction. In this paper we investigate an action by the circle group S^1 on the product space of two complex projective spaces. We introduce in § 1 two polynomials F and G from the equivariant cohomology ring under an action. We show that the number of common zero points of F and G is equal to that of the connected components of the fixed point set and the cohomology of the fixed point set is also determined by these two polynomials.

1. Let G be the circle group S^1, and X a complex projective product space $CP^m \times CP^n$ with a G-action. We use F, F_j, $j=1, \cdots, k$, for the fixed point set and its connected components. Let $E_G \longrightarrow B_G$ be a universal principal G-bundle. We denote by $H_G^*(X)$ the equivariant cohomology ring of X with rational coefficient $H^*(X \times_G E_G; Q)$.

Lemma 1.1. *$H_G^*(X) \cong Q[t, \alpha, \beta]/(F(t, \alpha, \beta),\ G(t, \alpha, \beta))$. $F(w, x, y) = x^{m+1} - wp(w, x, y)$ and $G(w, x, y) = y^{n+1} - wp(w, x, y)$, where p and q are homogeneous polynomials of degree m and n respectively.*

Proof. Since $G(=S^1)$ acts freely on $X \times E_G$, we have a principal S^1 bundle $X \times E_G \longrightarrow X \times_G E_G$. Let ξ be the associated complex line bundle with the total space $X \times E_G \times C^1/S^1$, where S^1 acts on C^1 by the multiplication, and define η similarly with respect to the principal bundle $E_G \longrightarrow B_G$. Let π be the natural projection of $X \times_G E_G$ onto B_G. Now, we consider the exact sequence of Gysin:

$$\longrightarrow H^i(X; Q) \longrightarrow H_G^{i-1}(X) \xrightarrow{\cup w} H_G^{i+1}(X) \longrightarrow H^{i+1}(X; Q) \longrightarrow,$$

where w is the Euler class of ξ. Since X is totally non-homologous to zero in $X \times_G E_G$ we have $H_G^{odd}(X) = 0$ and π^* is a monomorphism. Let t be the Euler class of η. Then $H^*(B_G; Q) = Q[t]$. We may identify w with t, since ξ is the induced bundle $\pi^*(\eta)$. Thus the above sequence splits;

$$\text{(A)} \qquad 0 \longrightarrow H_G^{2i}(X) \xrightarrow{\cup t} H_G^{2i+2}(X) \xrightarrow{\varphi} H^{2i+2}(X; Q) \longrightarrow 0.$$

Let a, b be a basis of $H^2(X;Q)$ such that $a^{m+1}=0$ and $b^{n+1}=0$, and take α, $\beta\in H_G^2(X)$ satisfying $\varphi(\alpha)=a$ and $\varphi(\beta)=b$. Then $H_G^*(X)$ is generated by t, α and β. By (A) we have two relations $F(t,\alpha,\beta)=0$ and $G(t,\alpha,\beta)=0$, where $F(w,x,y)=x^{m+1}-wp(w,x,y)$, $G(w,x,y)=y^{n+1}-wp(w,x,y)$ and w, x and y are indeterminates. Let $f(t,\alpha,\beta)=0$. We must show that $f(w,x,y)=AF(w,x,y)+BG(w,x,y)$ for some A, $B\in Q[w,x,y]$. We put

$$f(w,x,y)=g_1(x,y)x^{m+1}+g_2(x,y)y^{n+1}+wh(w,x,y)+k(x,y),$$

where k has degree $\leqq m$ with respect to x and $\leqq n$ with respect to y. Then, by (A), it follows that $k=0$ and thus we have

$$f(t,\alpha,\beta)=t(g_1(\alpha,\beta)p(t,\alpha,\beta)+g_2(\alpha,\beta)q(t,\alpha,\beta)+h(t,\alpha,\beta)).$$

Since the cup product by t is isomorphism, we have

$$g_1(\alpha,\beta)p(t,\alpha,\beta)+g_2(\alpha,\beta)q(t,\alpha,\beta)+h(t,\alpha,\beta)=0.$$

It is shown by the induction with respect to deg f that

$$\begin{aligned}&g_1(x,y)p(w,x,y)+g_2(x,y)q(w,x,y)+h(w,x,y)\\&=A'F(w,x,y)+B'G(w,x,y),\quad A',B'\in Q[w,x,y].\end{aligned}$$

Thus, $f-wA'F-wB'G=g_1F+g_2G$, and hence $f=AF+BG$, where $A=wA'+g_1$ and $B=wB'+g_2$. q. e. d.

In the above, it is further noticed that if we take other $\bar{\alpha}$, $\bar{\beta}\in H_G^2(X)$ such that $H_G^*(X)\cong Q[t,\bar{\alpha},\bar{\beta}]/(\bar{F},\bar{G})$, where $\bar{F}=x^{m+1}-w\bar{p}(w,x,y)$ and $\bar{G}=y^{n+1}-w\bar{q}(w,x,y)$, then we get $\bar{\alpha}=A\alpha+Ct$, $\bar{\beta}=B\beta+Dt$, $\bar{F}=F(w,x-Ct/A,y-Dt/B)$ and $\bar{G}=G(w,x-Cy/A,y-Dt/B)$.

Since X is totally non-homologous to zero in $X\times_G E_G$, there is an exact sequence:

$$\text{(B)}\qquad 0\longrightarrow H_G^*(X)\xrightarrow{j^*}H_G^*(F)\longrightarrow H^*(X/G,F;Q)\longrightarrow 0,$$

where j is the inclusion of F into X [1]. We let $j^*(\alpha)=\sum_{j=1}^k(a_j+C_jt)$ and $j^*(\beta)=\sum_{j=1}^k(b_j+D_jt)$, where a_j, $b_j\in H^2(F_j;Q)$ and C_j, D_j are rational numbers. There is an integer N such that $H^i(X/G,F;Q)=0$ for $i>N$. So from (B) and the fact that $H_G^*(F)=H^*(B^G;Q)\otimes_Q H^*(F;Q)$, we see that $H^*(F_j;Q)$ is generated by a_j and b_j. We consider the polynomials

$$\begin{aligned}F(w,x+C_jw,y+D_jw)&=P_{m+1}^j(x,y)w^{m+1}+P_m^j(x,y)+\cdots+P_0^j(x,y),\\G(w,x+C_jw,y+D_jw)&=Q_{n+1}^j(x,y)w^{n+1}+Q_n^j(x,y)+\cdots+Q_0^j(x,y),\end{aligned}$$

where $P^j_{m+1} = F(1, C_j, D_j)$, $Q^j_{n+1} = G(1, C_j, D_j)$, $P^j_0 = x^{m+1}$ and $Q^j_0 = y^{n+1}$. Since $F(t, \alpha, \beta) = 0$ and $G(t, \alpha, \beta) = 0$, we have $P^j_i(a_j, b_j) = 0$ and $Q^j_h(a_j, b_j) = 0$ and in particular $F(1, C_j, D_j) = 0$ and $G(1, C_j, D_j) = 0$. We denote by I_j ($\subseteqq Q[x, y]$) the ideal generated by P^j_i and Q^j_h, $i = 1, \cdots, m$; $h = 1, \cdots, n$. By the fact that j^* is onto in high degrees it is clear that $(C_i, D_i) \neq (C_j, D_j)$ if $i \neq j$.

Proposition 1.2. $H^*(F_j ; Q) \cong Q[x, y]/I_j$.

Proof. If we map x and y to a_j and b_j respectively, we have a well-defined homomorphism from $Q[x, y]/I_j$ onto $H^*(F_j ; Q)$. Let $f(x, y)$ be a homogeous polynomial such that $f(a_j, b_j) = 0$. For $i \neq j$, we put $L_i = x - C_i w$ if $C_i \neq C_j$, and $L_i = y - D_i w$ otherwise. Put $g(w, x, y) = f(x - C_j w, y - D_j w) \prod_{i \neq j} L_i(w, x, y)^r$, where $r > \max(n, m) + 1$. Clearly $j^*(g(t, \alpha, \beta)) = 0$ and thus we have $g(t, \alpha, \beta) = 0$. Then, by Lemma 1.1, $g(w, x, y) = AF(w, x, y) + BG(w, x, y)$ for some $A, B \in Q[w, x, y]$. The coefficient of the highest degree with respect to w in the polynomial $g(w, x + C_j w, y + D_j w)$ is $Cf(x, y)$, where C is a non-zero constant. This implies $f(x, y) \in I_j$.

2. In this section we investigate the ideals I_j by the method of lifting actions. Let Y be the product space of spheres of S^{2m+1} and S^{2n+1}, and q the natural projection of Y onto X. Then $q : Y \longrightarrow X$ is a principal $T^2 = S^1 \times S^1$ bundle. By [3, 4] there is a lifted G-action on Y, i. e.,

(1) the G-action and T^2-action on Y commute each other,

(2) the following diagram is commutative

$$\begin{array}{ccc} G \times Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ G \times X & \longrightarrow & X \end{array} .$$

The isotropy subgroup of $G \times T^2$-action on Y is a finite cyclic or circle subgroup. Let $T_1, \cdots, T_h$ be the circle subgroups in the isotropy subgroups. It is known that the fixed point set $F(T_j, Y)$ has the cohomology ring with rational coefficients which is isomorphic to that of $S^{2s_j+1} \times S^{2t_j+1}$ [1]. In particular, $F(T_j, Y)$ is connected. Since $F(T_j, Y)$, $j = 1, \cdots, h$, are disjoint, it is clear that $h = k$ and $q(F(T_j, Y)) = F_j$ after suitable renumbering.

Lemma 2.1. *The ideal I_j is generated by homogeneous polynomials $f_j(x, y)$ and $g_j(x, y)$ of degree $s_j + 1$ and $t_j + 1$ respectively. Moreover, dim $H^*(F_j, Q) = (s_j + 1)(t_j + 1)$.*

Proof. We consider the spectral sequence of the principal T^2 bundle $F(F_j, Y) \longrightarrow F^j$. The E_2 terms are as follows: $E_2^{p,q} = H^p(F_j\,;\,Q) \otimes_Q H^q(T^2\,;\,Q)$. By Proposition 1. 2, $H^{odd}(F^j\,;\,Q) = 0$ and hence we have $d_3 = d_4 = \cdots = 0$. Thus we have the exact sequences:

(C) $\quad 0 \longrightarrow E_2^{2i-2,2} \longrightarrow E_2^{2i,1} \longrightarrow E_2^{2i+2,0} \longrightarrow 0$, where $i \neq -2$, s_j, t_j and $s_j + t_j + 2$.

Let a_j, b_j be generators of $H^*(F_j\,;\,Q)$ as in Proposition 1. 2, and a, b a basis of $H^1(T^2\,;\,Q)$ such that $d_2(a) = a_j$, $d_2(b) = b_j$. We suppose $0 < s_j \leqq t_j$, since the other cases are treated analogously. Let $i \leqq s_j$, and suppose a_j^{i-1}, $a_j^{i-2}b_j$, $\cdots$, b_j^{i-1} is a basis of $E_2^{2i-2,0}$. So, $a_j^{i-2}, \cdots, b_j^{i-2}$ are a basis of $E_2^{2i-4,0}$, dim $E_2^{2i-2,1} = 2i$ and dim $E_2^{2i-4,2} = i-1$. By (C), dim $E_2^{2i,0} =$ dim $E_2^{2i-2,1} -$ dim $E_2^{2i-4,2} = 2i-(i-1) = i+1$, and thus $a_j^i, \cdots, b_j^i$ are a basis of $E_2^{2i,0}$. It is shown by the induction on i that a_j^{s}j, $\cdots$, b_j^{s}j are a basis of $E_2^{2s_j,0}$. Since $E_3^{2s_j,1} \cong E_\infty^{2s_j,1} \cong H^{2s_j+1}(F(T_j, Y)\,;\,Q) = Q$, we have dim $E_2^{2s_j-2,2} =$ dim $Z_2^{2s_j,1} - 1$, where $Z_2^{2s_j,1} = Ker\ d_2^{2s_j,1}$, and dim $E_2^{3s_j+2,0} =$ dim $E_2^{2s_j,1} -$ dim $Z_2^{3s_j,1} = 2(s_j+1)-(s_j+1) = s_j+1$. Thus there is only one relation $f_j(a_j, b_j) = 0$ in $E_2^{2s_j+2,0}$, where f_j is a homogeneous polynomial of degree s_j+1. Let $s_j+1 < i \leqq t_j$. Then by (C), dim $E_2^{2i,0} = s_j+1$, and hence there are $(i-s_j)$ relations in $E_2^{2i,0}$, i. e., $f_j a_j^{i-s_j-1} = 0, \cdots, f_j b_j^{i-s_j-1} = 0$. Similarly as in $E_2^{2s\ +2,0}$, dim $E_2^{2t_j+2,0} = s_j$, and hence we have only one relation $g_j(a_j, b_j) = 0$ independent of $f_j a^{t_j-s_j} = 0, \cdots, f_j b^{t_j-s_j} = 0$, where g_j is a homogeneous polynomial of degree t_j+1. Let $t_j+1 < i \leqq t_j+s_j$. By (C), dim $E_2^{2i,0} = s_j+t_j-i$. Thus, there exist $2i-s_j-t_j$ relations in $E_2^{2i,0}$. It remains therefore to show that $g_j a_j^{i-t_j-1} = 0, \cdots, g_j b_j^{i-t_j-1} = 0, \cdots, f_j a_j^{i-s_j-1} = 0, \cdots, f_j b_j^{i-s_j-1} = 0$ are independent. But this follows from the fact $g_j = 0$, $f_j a_j^{t_j-s_j} = 0, \cdots, f_j b_j^{t_j-s_j} = 0$ are independent. Since $H^*(F_j\,;\,Q) \cong \sum_{i=0}^{s_j+t_j} E_2^{2i,0}$ as ring, it is now clear that dim $H^*(F_j\,;\,Q) = (s_j+1)(t_j+1)$, which completes the proof.

3. We recall here some facts concerning the intersection numbers of plane algebraic curves [2]. Let F, $G \in C[x, y]$, and $P \in C^2$. The intersection number $I(P, F \cap G)$ of F and G at P is characterized by seven properties, and two of them are the following:

(I) $I(P, F \cap G) = I(P, F \cap (G+AF))$ for every $A \in C[x, y]$.

(II) $I(P, F \cap G) \geqq m_P(F)m_P(G)$ with equality occuring if and only if F and G have no tangent lines in common at P. $m_P(F)$ is the multiplicity of F at P.

We note further that the polynomials $F(w, x, y,)$ and $G(w, x, y)$ in Lemma 1. 1 have no irreducible components in common.

Theorem 3.1. *There is a* 1–1 *correspondence between the common zero* *points P_j of the polynomials $F(w, x, y)$ and $G(w, x, y)$ and the connected components F_j of the fixed point set F in $CP^m \times CP^n$ in such a way that dim $H^*(F_j\,;\,Q)$ coincides with the intersection number $I(P_j, F\cap G)$ of $F(w, x, y)$ and $G(w, x, y)$ at P_j.*

Proof. Since X is totally non-homologous to zero in $X\times_G E_G$, we obtain $\sum_{j=1}^{k}$ dim $H^*(F_j\,;\,Q) =$ dim $H^*(X\,;\,Q) = (n+1)(m+1)$ [1]. Let P_{k+i}, $i=1, \cdots, h$, be the common zero points of $F(1, x, y)$ and $G(1, x, y)$ other than $P_j=(C_j, D_j)$, $j=1, \cdots, k$. Noting that there exist no common zero points of $F(w, x, y)$ and $G(w, x, y)$ on the line $w=0$, Bezout's Theorem implies that $\sum_{i=1}^{k+h} I(P_i, F(1, x, y)\cap G(1, x, y)) = (n+1)(m+1)$ [2]. To end the proof, it suffices to show that dim $H^*(F_j\,;\,Q)\leqq I(P_j, F(1, x, y)\cap G(1, x, y))$, since $I(P_j, F\cap G)\geqq 1$. We identify $P_j = (0, 0)$ with the translation $x' = x + C_j$, $y' = y + D_j$. Then, by making use of the notations in §1, we have

$$F = P_n^j(x, y) + \cdots + P_0^j(x, y),$$
$$G = Q_m^j(x, y) + \cdots + Q_0^j(x, y),$$

where $P_i^j(x, y)$ and $Q_i^j(x, y)$ are homogeneous polynomials of degree $m+1-i$ and $n+1-i$ respectively. Let $P_{m_0}^j(x, y)$ and $Q_{n_0}^j(x, y)$ be the first non-zero terms of F and G respectively. Then, we may suppose that $m+1-m_0\leqq n+1-n_0$. By Proposition 1. 2 Lemma 2. 1, we have deg $P_{m0}^j =$ deg $f_j = s_j + 1$. If $P_{m_0}^j \nmid Q_{n_0}^j$ then deg $g_j = t_j+1 \leqq$ deg Q_{n0}^j, and hence we have dim $H^*(F_j\,;\,Q) = (s_j+1)(t_j+1) \leqq$ deg $P_{m_0}^j\cdot$ deg $Q_{n_0}^j = m_{P_j}(F)m_{P_j}(G)$. Thus dim $H^*(F_j\,;\,Q)\leqq I(P_j, F\cap G)$ by the property (II). Now, suppose $P_{m0}^j \mid Q_{n_0}^j$. Then $Q_{n_0}^j(x, y) = R(x, y) P_{m_0}^j(x, y)$ for some homogeneous polynomial $R \in Q[x, y]$. Now, we can replace G by $\bar{G}=G-RF$, since I_j coincides with the ideal generated by the homogenous parts of F and $\bar{G}$ and we have $I(P_j, F\cap G) = I(P_j, F\cap \bar{G})$ by the property (I). We put

$$\bar{G} = \bar{Q}_u(x, y) + \cdots,\ \bar{Q}_u(x, y)\neq 0\,.$$

Then we have deg $\bar{Q}_u \geqq$ deg $Q_{n_0}^j$. If $P_{m_0}^j \nmid \bar{Q}_u$ then we are finished, and if not so, we continue the above process. This enables us to complete the proof.

Finally, as can easily be seen, if we get the above two polynomials F and G then the cohomology ring of the fixed point set of S^1-space $CP^m\times CP^n$ is completely determined by Proposition 1. 2 and Theorem 3. 1.

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