

Mathematical Journal of Okayama University

Volume 22, Issue 1

1980

Article 5

JUNE 1980

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ON THE BLUM-HANSON THEOREM IN L_p

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1. Introduction and the theorem. In this paper the following will be proved.

Theorem. Let $(X, \mathcal{F}, \mu)$ be a finite measure space and $1 < p < p_1 \leq \infty$. Let T be a positive linear operator on $L_p(\mu) = L_p(X, \mathcal{F}, \mu)$ and simultaneously on $L_{p_1}(\mu)$ such that

$$\sup_n \|T^n\|_p < \infty \text{ and } \sup_n \|T^n\|_{p_1} < \infty.$$

Then the two following assertions are equivalent :

- (a) T^n converges weakly in $L_p(\mu)$.
- (b) For any uniformly regular matrix (a_{ni}) , $A_n = \sum_i a_{ni} T^i$ converges strongly in $L_p(\mu)$.

(We recall that a matrix (a_{ni}) is uniformly regular if $\sup_n \sum_i |a_{ni}| < \infty$, $\lim_n \sum_i a_{ni} = 1$ and $\lim_n \max_i |a_{ni}| = 0$.)

Corollary. Let $(X, \mathcal{F}, \mu)$ be a finite measure space and $1 < p < \infty$. Let T be a positive linear operator on $L_p(\mu)$ such that there exists a strictly positive function h in $L_p(\mu)$ and a constant M satisfying $T^n h \leq Mh$ for all $n \geq 1$. Then the assertions (a) and (b) in the Theorem are equivalent.

For an operator T induced by a measure preserving point transformation on $(X, \mathcal{F}, \mu)$, the Theorem is due to Blum and Hanson [3]. Later, Akcoglu and Sucheston [2] extended the Blum-Hanson theorem as follows: For any positive contraction T on $L_p(\mu)$, (a) and (b) are equivalent. Akcoglu and Sucheston [1] also showed that (a) and (b) are still equivalent, with $p = 1$, if T is a (not necessarily positive) contraction on $L_1(\mu)$. Recently, Millet [5] examined a positive operator T on $L_1(\mu)$ such that $\sup_n \|T^n\|_1 < \infty$, and showed that for such a T the equivalence of (a) and (b) holds, with $p = 1$. His method of proof chiefly depends upon Akcoglu-Sucheston's theorem [1] and a decomposition theorem due to Sucheston [9]. (A similar argument appears in Sato [8].) In this paper we shall apply Millet's technique together with Hölder's inequality to obtain the above Theorem. The Corollary generalizes Proposition 1 in Millet [5].

Here I would like to express my hearty thanks to the referee for helpful comments; in particular, Lemma 1 is due to him.

2. Proofs. Throughout this section we may and will assume without loss of generality that $\mu(X)=1$. To prove the Theorem we need two lemmas; the first one is somewhat general.

Lemma 1. *Let each S_n be a linear operator on $L_p(\mu)$ and simultaneously on $L_{p_1}(\mu)$ for some $p_1 > p$ such that $\sup_n \|S_n\|_{p_1} < \infty$. Then for $f \in L_{p_1}(\mu)$ the sequence $S_n f$ converges in $L_p(\mu)$ if (and only if) $S_n f$ converges in measure.*

Proof. Under the convergence in measure, the convergence of $S_n f$ in $L_p(\mu)$ is equivalent to the uniform p -integrability (see [6], Chapter II), i. e. $\sup_n \|S_n f\|_p < \infty$ and

$$\sup_n \|1_E \cdot S_n f\|_p \longrightarrow 0 \text{ as } \mu(E) \longrightarrow 0.$$

But by Hölder's inequality we have

$$\begin{aligned} \|1_E \cdot S_n f\|_p &\leq \mu(E)^{\frac{p_1-p}{p_1 p}} \|S_n f\|_{p_1} \\ &\leq \mu(E)^{\frac{p_1-p}{p_1 p}} \|f\|_{p_1} \cdot \sup_k \|S_k\|_{p_1}. \end{aligned}$$

Lemma 2. *Let T be as in the Theorem. If $T1=1$, then (a) implies (b).*

Proof. By [7] X is decomposed into two disjoint measurable sets Y and Z such that

- (i) $Y = \{x : e(x) > 0\}$ for some $0 \leq e \in L_q(\mu)$ with $T^*e = e$, where $1/p + 1/q = 1$,
- (ii) $L_p(Z, \mu)$ is invariant for T and $\frac{1}{n} \sum_{i=0}^{n-1} T^i$ converges strongly to 0 on $L_p(Z, \mu)$.

Using this decomposition we divide the proof into several steps.

Step I. $A_n f$ converges to 0 in $L_p(\mu)$ for $f \in L_{p_1}(Z, \mu)$.

First of all, we remark that $L_{p_1}(Z, \mu)$ is invariant for T , as well. Since T is positive,

$$\frac{1}{n} \sum_{i=0}^{n-1} \|T^i f\|_1 \leq \left\| \frac{1}{n} \sum_{i=0}^{n-1} T^i |f| \right\|_p \longrightarrow 0$$

by (ii). Then there exists a subsequence $T^{k_n} f$ such that $\|T^{k_n} f\|_1 \longrightarrow 0$ as $k_n \longrightarrow \infty$; hence $T^{k_n} f$ converges in measure, and we can apply Lemma 1 with $S_n = T^{k_n}$ to obtain that $\|T^{k_n} f\|_p \longrightarrow 0$. This yields

$$\limsup_n \|T^n f\|_p \leq \sup_n \|T^n\|_p \cdot \limsup_n \|T^{k_n} f\|_p = 0,$$

that is, $T^n f$ converges to 0 in $L_p(Z, \mu)$. Then clearly $A_n f$ converges to 0 in $L_p(Z, \mu)$.

Step II. Suppose That T^n converges weakly in $L_p(\mu)$. Then $1_Y \cdot A_n f$ converges in $L_p(\mu)$ for $f \in L_{p_1}(Y, \mu)$.

To see this, let $R(eg) = eTg$ for $g \in L_p(Y, \mu)$. Since $L_p(Y, \mu)$ is dense in $L_1(Y, \mu)$, and by (i)

$$\begin{aligned} \|R(eg)\|_1 &\leq \int eT|g| d\mu \\ &= \int |g| T^* e d\mu = \|eg\|_1, \end{aligned}$$

R is well defined and is uniquely extended to a positive contraction on $L_1(Y, \mu)$. Since T^n converges weakly in $L_p(\mu)$, and by definition

$$R^n(eg) = eT^n g,$$

R^n converges weakly in $L_1(Y, \mu)$. Therefore by [1], if $f \in L_{p_1}(Y, \mu)$ and if (a_{ni}) is a uniformly regular matrix, then the sequence

$$B_n(ef) := \sum_i a_{ni} R^i(ef)$$

converges in $L_1(Y, \mu)$. Since

$$B_n(ef) = eA_n f,$$

this implies the measure convergence of the sequence $1_Y \cdot A_n f$. Now we apply Lemma 1 with $S_n = 1_Y \cdot A_n$ to conclude that $1_Y \cdot A_n f$ converges in $L_p(Y, \mu)$.

Step III. Suppose that $T1 = 1$. If $f \in L_{p_1}(Y, \mu)$ and $|f| \leq \alpha 1_Y$ for some $\alpha \geq 0$, then $1_Z \cdot A_n f$ converges in $L_p(Z, \mu)$.

To prove this, let

$$U = 1_Y \cdot T \text{ and } W = 1_Z \cdot T.$$

Then it follows from (ii) that $1_r \cdot T^n = U^n$, $UW = 0$ and

$$T^n = U^n + WU^{n-1} + \cdots + W^n;$$

hence

$$1_z \cdot T^n = WU^{n-1} + \cdots + W^n.$$

Since the assumption $T1 = 1$ implies $U1_r = 1_r$, the above identity shows that the positive function $h = W1_r$ satisfies

$$h + \cdots + W^{n-1}h = 1_z \cdot T^n 1_r,$$

and hence

$$\begin{aligned} \sum_{i=0}^{\infty} \|W^i h\|_1 &= \left\| \sum_{i=0}^{\infty} W^i h \right\|_1 \leq \left\| \sum_{i=0}^{\infty} W^i h \right\|_p \\ &\leq \sup_n \|T^n\|_p < \infty. \end{aligned}$$

Therefore the sequence $\sum_{i=m+1}^{\infty} W^i h$ converges to 0 in measure as $m \rightarrow \infty$. Since for each i ,

$$\begin{aligned} \left| \sum_{k=0}^{\infty} a_{n,i+k} U^k f \right| &\leq \alpha \left(\sum_{k=0}^{\infty} |a_{n,i+k} U^k 1_r| \right) \\ &\leq \alpha \beta 1_r \end{aligned}$$

where we put

$$\beta = \sup_n \sum_i |a_{ni}|,$$

it follows that

$$\begin{aligned} 0 &\leq \sum_{i=m+1}^{\infty} W^i \left| \sum_{k=0}^{\infty} a_{n,i+k} U^k f \right| \\ &\leq \alpha \beta \sum_{i=m}^{\infty} W^i h \rightarrow 0 \text{ in measure as } m \rightarrow \infty. \end{aligned}$$

Thus we have proved that $\sum_{i=m+1}^{\infty} W^i \left[\sum_{k=0}^{\infty} a_{n,i+k} U^k f \right]$ converges to 0 in measure (uniformly in n) as $m \rightarrow \infty$. On the other hand, for each fixed i , the matrix $(a_{n,i+k})_{n,k}$ is uniformly regular, so that by Step II the sequence

$$\sum_{k=0}^{\infty} a_{n,i+k} U^k f = 1_r \cdot \left(\sum_{k=0}^{\infty} a_{n,i+k} T^k f \right)$$

converges in $L_p(\mu)$ as $n \rightarrow \infty$, and so does

$$W^i \left[\sum_{k=0}^{\infty} a_{n,i+k} U^k f \right] \text{ as } n \rightarrow \infty.$$

Now, the convergence of $1_z \cdot A_n f$ in $L_p(Z, \mu)$ results from Lemma 1

with $S_n = 1_Z \cdot A_n$, because the relation

$$\begin{aligned} 1_Z \cdot A_n f &= \sum_{i=1}^{\infty} a_{ni} 1_Z \cdot T^i f \\ &= \sum_{i=1}^{\infty} a_{ni} [WU^{i-1}f + \cdots + W^i f] \\ &= \sum_{i=1}^m W^i \left[\sum_{k=0}^{\infty} a_{n, i+k} U^k f \right] \\ &\quad + \sum_{i=m+1}^{\infty} W^i \left[\sum_{k=0}^{\infty} a_{n, i+k} U^k f \right] \end{aligned}$$

implies that the sequence $1_Z \cdot A_n f$ is a Cauchy sequence for convergence in measure and hence converges in measure.

Step IV. Suppose that $T1 = 1$ and that T^n converges weakly in $L_p(\mu)$. Then A_n converges strongly in $L_p(\mu)$.

In fact, by Steps I, II and III, $A_n f$ converges in $L_p(\mu)$ whenever $f \in L_{p_1}(\mu)$ and $1_Y \cdot |f| \leq \alpha 1_Y$ for some $\alpha \geq 0$. But since the set of those f is dense in $L_p(\mu)$, and since $\sup_n \|A_n\|_p < \infty$, the sequence $A_n g$ converges for every $g \in L_p(\mu)$.

This completes the proof.

Proof of the Theorem. (a) $\Rightarrow$ (b): We may choose an $h \in L_p(\mu)$, with $0 \leq h = Th$, such that if $g \in L_p(\mu)$ and $Tg = g$ then $\{x : g(x) \neq 0\} \subset \{x : h(x) > 0\}$. Put $X^+ = \{x : h(x) > 0\}$ and $X^0 = \{x : h(x) = 0\}$. It can be readily seen that $L_p(X^+, \mu)$ is invariant for T , and that if we set $Sg = 1_{X^0} \cdot Tg$ for $g \in L_p(X^0, \mu)$, then S^n converges to 0 in $L_p(X^0, \mu)$. Hence, as in Step I of the proof of Lemma 2, we have

$$\lim_n \|S^n g\|_p = \lim_n \|1_{X^0} \cdot T^n g\|_p = 0$$

for every $g \in L_p(X^0, \mu)$. On the other hand, since $L_p(X^+, \mu)$ is isometrically isomorphic to $L_p(X^+, h^p d\mu)$ by the mapping $g \rightarrow g/h$, and since $(Th)/h = 1$, Lemma 2 shows that if $g \in L_p(X^+, \mu)$ and (a_{ni}) is a uniformly regular matrix then the sequence $A_n g = \sum_i a_{ni} T^i g$ converges in $L_p(X^+, \mu)$.

Now, fix any $f \in L_p(\mu)$. Given an $\varepsilon > 0$, take an N with $\|1_{X^0} \cdot T^i f\|_p \leq \varepsilon / \sup_n \|T^n\|_p$ for all $i \geq N$. Then we get

$$\|T^i f - T^{i-N}(1_{X^+} \cdot T^N f)\|_p \leq \varepsilon \quad (i \geq N).$$

Since the sequence

$$C_n f := \sum_{i=N}^{\infty} a_{ni} T^{i-N}(1_{X^+} \cdot T^N f)$$

converges in $L_p(X^+, \mu)$, the above inequality implies that the sequence $A_n f = \sum_i a_{ni} T^i f$ is a Cauchy sequence in $L_p(\mu)$. Hence $A^n f$ converges in $L_p(\mu)$.

(b) $\Rightarrow$ (a): This is straightforward and is omitted.

Proof of the Corollary. Considering S on $L_p(h^p d\mu)$ representing T on $L_p(\mu)$, the Corollary follows immediately from the Theorem, because the assumption $T^n h \leq Mh$ for all $n \geq 1$ implies $\|S^n\|_\infty \leq M$ for all $n \geq 1$.

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(Received July 17, 1979)