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A NOTE ON COMMUTATIVE SEPARABLE ALGEBRAS

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In this note, we prove that separability descends by faithful flatness and hence is a local property. We also prove that separability is a punctual property over a semi-local ring.

Throughout this paper, rings and algebras are commutative with identity and ring homomorphisms carry the identity to the identity. In what follows, A denotes a ring with identity 1 and B an A-algebra. B is a separable A-algebra if and only if there exists an element e in $B \otimes_A B$ such that $(b \otimes 1)e = (1 \otimes b)e$ for all $b \in B$ and $p(e) = 1$, where p is the multiplication map from $B \otimes_A B$ to B ([3, p. 40]). It is easily seen that e is idempotent and unique. This element is called the separability idempotent of B over A, and is invariant under the switch map $B \otimes_A B \to B \otimes_A B$ given by $b_i \otimes b_j \mapsto b_j \otimes b_i$.

Now, it is well known that if B is separable over A then for any A-algebra C, the C-algebra $B \otimes_A C$ is again separable. Moreover, by [4, Prop. 2. 2 (c)], it is known that if C is a faithfully flat A-algebra and $B \otimes_A C$ is separable over C then B is separable over A, provided that B is finitely generated as an A-algebra. Our main result is that the hypothesis on B is not necessary.

Theorem 1. *Let B be an A-algebra and C a faithfully flat A-algebra. If the C-algebra $B \otimes_A C$ is separable, then B is separable over A.*

Proof. Let ε_0 (resp. ε_1) : $C \to V = C \otimes_A C$ denote the A-algebra homomorphism defined by $c \mapsto 1 \otimes c$ (resp. $c \to c \otimes 1$). Then, the homomorphisms

$$1 \otimes \varepsilon_i : U = B \otimes_A C \longrightarrow W = B \otimes_A C \otimes_A C \quad (i = 0, 1)$$

give rise to the homomorphisms

$$(1 \otimes \varepsilon_i) \otimes (1 \otimes \varepsilon_i) : U \otimes_C U \longrightarrow W \otimes_V W \quad (i = 0, 1).$$

Moreover, we have the homomorphisms

$$U \xrightarrow{\mu} U \otimes_{\mathrm{Im}(\varepsilon_i)} V \xrightarrow{\nu_i} \mathrm{Im}\,(1 \otimes \varepsilon_i) \cdot V = W \quad (i = 0, 1)$$

where $\mu(u) = u \otimes 1$, and $\nu_i(u \otimes v) = (1 \otimes \varepsilon_i)(u)v$. Now, in general, if U is separable over T and V is any T-algebra, then the separability idempotent for U over T goes to the separability idempotent for any homomorphic image of $U \otimes_T V$ over V. Applying this to our case, we see that the separability idempotent e' of U over C (which is an element of $U \otimes_C U$) must be sent to 0 under the difference $d = (1 \otimes \varepsilon_0) \otimes (1 \otimes \varepsilon_0) - (1 \otimes \varepsilon_1) \otimes (1 \otimes \varepsilon_1)$. Since C is faithfully flat over A, it follows from [2, Lemma 3. 8] that the sequence

$$0 \longrightarrow B \otimes_A B \xrightarrow{\rho} B \otimes_A B \otimes_A C \xrightarrow{1 \otimes 1 \otimes \varepsilon_0 - 1 \otimes 1 \otimes \varepsilon_1} B \otimes_A B \otimes_A C \otimes_A C$$

where $\rho(m) = m \otimes 1$, is exact. From this, one will easily see that the sequence

$$0 \longrightarrow B \otimes_A B \xrightarrow{\sigma} U \otimes_C U \xrightarrow{d} W \otimes_{C} W$$

where $\sigma(b_1 \otimes b_2) = (b_1 \otimes 1) \otimes (b_2 \otimes 1)$, is also exact. Hence there exists an element e in $B \otimes_A B$ so that $\sigma(e) = e'$. Obviously, $p(e) = 1$, and $(b \otimes 1)e = (1 \otimes b)e$ for all $b \in B$. Thus, B is separable over A, completing the proof.

An application of Th. 1 is the following corollary which shows that separability is a local property.

Corollary 2. *Let B be an A-algebra and let $\{f_1, \cdots, f_n\}$ be a family of elements of A which generates the unit ideal of A. Then B is separable over A if and only if for all i, B_{f_i} is separable over A_{f_i}, where A_{f_i} is the ring of fractions of B having denominators equal to some power of f, and $B_{f_i} = B \otimes_A A_{f_i}$.*

Proof. By the result of [1, Ch. II, Prop. 5. 1. 3], $C = \Pi_{i=1}^{n} A_{f_i}$ is a faithfully flat A-algebra. From this, the assertion follows immediately.

As a second application of Th. 1, we have the following corollary which shows that separablility is a punctual property provided that the base ring has only a finite number of maximal ideals.

Corollary 3. *Let A be a semi-local ring and B an A-algebra. Then the following conditions are equivalent.*

i) *B is separable over A.*
ii) *$B_{\mathfrak{p}}$ is separable over $A_{\mathfrak{p}}$ for each prime ideal $\mathfrak{p}$ of A.*
iii) *$B_{\mathfrak{m}}$ is separable over $A_{\mathfrak{m}}$ for each maximal ideal $\mathfrak{m}$ of A.*

Proof. Only iii) ⇒ i) needs proof. Let Ω denote the set of maximal ideals of A. Since Ω is finite, $\Pi_{\mathfrak{m}\in\Omega} A_{\mathfrak{m}}$ is faithfully flat over A by [1, Ch. II, Prop. 3. 3. 10].

Remark 1. By virtue of Corollary 2, we easily see that an A-algebra B is separable if and only if for every prime ideal $\mathfrak{p}$ of A, there exists an element t in $A-\mathfrak{p}$ (the complement of $\mathfrak{p}$ in A) such that B_t is separable over A_t. Moreover, the result of Corollary 3 is a partial generalization of [4, Prop. 2. 5].

Remark 2. By Theorem 1, we see that for A-algebras B, C, if $B\otimes_A C$ is separable over C then B is separable over A, provided that $\{A, C\}$ is one of (1), (2) and (3) :

(1) A = a Noetherian ring, C = the I-adic completion of A where I is an ideal of A contained in the Jacobson radical of A ([1, p. 206]).

(2) A = a local ring, C = the Henselization of A ([3, p. 73]).

(3) A = a coherent ring (e. g., a Noetherian ring), $C = A[[X_1, \cdots, X_n]]$, a formal power series ring over A ([1, p. 49]).

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REFERENCES

[1] N. BOURBAKI : Commutative Algebra (Translated from the French. Actualités Sci. Ind. 1290, Hermann, Paris), Addison-Wesley, 1972.

[2] S. U. CHASE and A. ROSENBERG : Amitsur cohomology and the Brauer group, Mem. A. M. S. **52** (1965). 34—79 (Reprinted. 1968). MR **33** #4119.

[3] F. DEMEYER and E. INGRAHAM : Separable Algebras over Commutative Rings. Lecture Notes in Math. **181** (1971), Springer. MR **43** # 6199.

[4] M. A. KNUS and M. OJANGUREN : Théorie de la Descente et Algèbres d' Azumaya, Lecture Notes in Math. **389** (1974). Springer. MR **53** #5209.

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