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A NOTE ON SEPARABLE POLYNOMIALS IN SKEW POLYNOMIAL RINGS OF AUTOMORPHISM TYPE

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Throughout, B will mean a ring with identity element 1. By $B[X;\rho]$, we denote a skew polynomial ring $\sum_{i=0}^{\infty} X^i B$ whose multiplication is given $bX = X\rho(b)$ $(b \in B)$ where ρ is an automorphism of B. A monic polynomial $f \in B[X;\rho]$ is called to be separable (resp. Frobenius) if $fB[X;\rho] = B[X;\rho]f$ and the factor ring $B[X;\rho]/fB[X;\rho]$ is separable (resp. Frobenius) over B. Moreover, an element a of B is said to be π-regular (resp. left π-regular (resp. right π-regular)) if there exists an element c in B and an integer $t > 0$ such that $a^t c a^t = a^t$ (resp. $ca^t = a^{t-1}$ (resp. $a^t c = a^{t-1}$)). If every element of B is π-regular then B will be called to be π-regular (cf. [4]).

The main purpose of this note is to present a generalization of the result of S. Ikehata [1, Th. 1 (a), (b)] which is as follows: For $f = \sum_{i=0}^{n} X^i a_i \in B[X;\rho]$, if f is separable and one of coefficients $\{a_0, a_1\}$ is π-regular (or left (or right) π-regular) then f is Frobenius (Lemma 2 and Th. 3). Moreover, some applications of the above will be given (Cor. 4 and Th. 5).

First, we shall prove the following lemma which is useful in our study.

Lemma 1. *Let $f = X^n - \sum_{i=0}^{n-1} X^i a_i \in B[X;\rho]$ and $fB[X;\rho] = B[X;\rho]f$. Then*

(i) $\alpha\rho^t(a_i) = \rho^t(a_i)\rho^{n-i}(\alpha)$ $(n-1 \geq i \geq 0,\ t \geq 0)$ *for all* $\alpha \in B$.

(ii) $\rho^{n-1-i}(a_i) = a_i$ *and* $\rho(a_i^2) = a_i^2$ $(n-1 \geq i \geq 0)$.

Proof. We consider the factor ring $B[X;\rho]/fB[X;\rho]$ and set $x = X + fB[X;\rho]$. Then $x^n = \sum_{i=0}^{n-1} x^i a_i$. Hence we have

$$\begin{aligned} x^n x &= x^n \rho(a_{n-1}) + \textstyle\sum_{i=0}^{n-2} x^{i+1}\rho(a_i) \\ &= \textstyle\sum_{i=1}^{n-1} x^i(a_i\rho(a_{n-1}) + \rho(a_{i-1})) + a_0\rho(a_{n-1}), \\ xx^n &= \textstyle\sum_{i=1}^{n-1} x\,(a_i a_{n-1} + a_{i-1}) + a_0 a_{n-1}. \end{aligned}$$

Since $\{x^i \mid n-1 \geq i \geq 0\}$ is a right free B-basis of $B[X;\rho]/fB[X;\rho]$, it follows that

$$(1) \qquad a_i\rho(a_{n-1}) + \rho(a_{i-1}) = a_i a_{n-1} + a_{i-1} \quad (n-1 \geq i \geq 1),$$
$$a_0\rho(a_{n-1}) = a_0 a_{n-1}.$$

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Moreover, for any $\alpha \in B$, we have $\alpha x^n = x^n \rho^n(\alpha)$ which implies that $\alpha a_i = a_i \rho^{n-i}(\alpha)$ $(n-1 \geq i \geq 0)$, and so,

$$(2) \qquad \alpha \rho^t(a_i) = \rho^t(a_i)\rho^{n-i}(\alpha) \quad (n-1 \geq i \geq 0,\ t \geq 0).$$

In particular, there holds that

$$(3) \qquad a_i a_{n-1} = a_{n-1}\rho(a_i) \quad (n-1 \geq i \geq 0).$$

Now, we assume that $a_m a_{n-1} = a_{n-1} a_m$ for some $(n-1 \geq)\, m \geq 1$. Then by (1) and (3), we have

$$\begin{aligned} a_{n-1}a_m a_{n-1} + a_{n-1}a_{m-1} &= a_{n-1}(a_m a_{n-1} + a_{m-1}) = a_{n-1}(a_m\rho(a_{n-1}) + \rho(a_{m-1})) \\ &= a_m a_{n-1}\rho(a_{n-1}) + a_{n-1}\rho(a_{m-1}) = a_m a_{n-1} a_{n-1} + a_{m-1}a_{n-1}. \end{aligned}$$

and hence $a_{n-1}a_{m-1} = a_{m-1}a_{n-1}$. Therefore, by induction method, we obtain

$$(4) \qquad a_i a_{n-1} = a_{n-1} a_i \quad (n-1 \geq i \geq 0).$$

Now, by (1), (2) and (4), we have

$$\begin{aligned} a_{n-1}a_i + a_{i-1} &= a_i a_{n-1} + a_{i-1} = a_i\rho(a_{n-1}) + \rho(a_{i-1}) \\ &= \rho(a_{n-1})\rho(a_i) + \rho(a_{i-1}) = \rho(a_{n-1}a_i + a_{i-1}) \quad (n-1 \geq i \geq 1) \end{aligned}$$

which is ρ-invariant. We assume here that $\rho^{n-1-m}(a_m) = a_m$ for some $(n-1 \geq)\, m \geq 1$. Then, by (2) and the above, we see that

$$\begin{aligned} a_m\rho^{n-m}(a_{n-1}) + a_{m-1} &= a_{n-1}a_m + a_{m-1} \\ &= \rho^{n-1-m}(a_m\rho(a_{n-1}) + \rho(a_{m-1})) = a_m\rho^{n-m}(a_{n-1}) + \rho^{n-m}(a_{m-1}). \end{aligned}$$

This implies that $a_{m-1} = \rho^{n-m}(a_{m-1})$. Hence, by induction method, we obtain $\rho^{n-1-i}(a_i) = a_i$ for all $(n-1 \geq)\, i \geq 0$. Therefore, it follows from (2) that

$$\begin{aligned} a_i a_i &= a_i\rho^{n-i}(a_i) = \rho^{n-i}(a_i)\rho^{n-i}(a_i) \\ &= \rho(\rho^{n-i-1}(a_i)\rho^{n-i-1}(a_i)) = \rho(a_i a_i) \quad (n-1 \geq i \geq 0). \end{aligned}$$

This completes the proof.

Next, we shall prove the following

Lemma 2. *Let* $f = X^n - \sum_{i=0}^{n-1} X^i a_i \in B[X;\rho]$ *and* $fB[X;\rho] = B[X;\rho]f$. *Then, for a coefficient* a_m $(n-1 \geq m \geq 0)$ *of* f, *the following conditions are equivalent.*

(a) a_m *is π-regular.*
(b) a_m *is left π-regular.*
(c) a_m *is right π-regular.*
(c)′ $a_m^t B = a_m^{t+1} B$ *for some integer* $t \geq 0$.

Proof. Given an integer $t \geq 0$, Lemma 1(i) shows that for any $\alpha \in B$, $\alpha a_m^t = a_m^t \rho^{t(n-m)}(\alpha)$ and $a_m^t \alpha = \rho^{t(m-n)}(\alpha) a_m^t$. Using this fact, the assertion will be easily seen.

Now, we shall prove the following theorem which is our main result.

Theorem 3. *Let* $f = X^n - \sum_{i=0}^{n-1} X^i a_i \in B[X; \rho]$ *be separable and one of coefficients* $\{a_0, a_1\}$ *π-regular. Then* f *is Frobenius.*

Proof. First, we shall prove the assertion in case a_0 is π-regular. By Lemma 2, there exists an integer $t \geq 0$ such that $a_0^t B = a_0^{t+1} B$. In case either $a_0^t B = B$ or $a_0^t B = \{0\}$, there holds the assertion by the result of [1, Th. 1 (a), (b)]. Now, let $B \supsetneqq a_0^t B \supsetneqq \{0\}$. Then, we have $t > 0$ and $a_0^t B = a_0^{2t} B = a_0^{4t} B$. Moreover, by Lemma 1, we have $a_0^2 B = B a_0^2$ and $\rho(a_0^2) = a_0^2$. Hence, as in [3, Lemma 1], there exist central idempotents e_1, e_2 in B such that $e_1 e_2 = 0$, $e_1 + e_2 = 1$, $e_1 B = a_0^{2t} B$, and $\rho(e_i) = e_i (i = 1, 2)$. Further, each $e_i f$ is separable in $e_i B[X; \rho \mid e_i B]$. Since $e_1 a_0$ is inversible in $e_1 B$, it follows from [1, Th. 1(a)] that $e_1 f$ is Frobenius over $e_1 B$. Now, by [1, Lemma 1], there exist elements d_0, d_1 in B such that $a_0 d_0 + a_1 d_1 = 1$. Then, by Lemma 1(i), we easily see that $a_0^{2t} u_0 + a_1 u_1 = 1$ for some elements u_0, u_1 in B. Since $e_2 a_0^{2t} u_0 = 0$, we have $e_2 a_1 u_1 = e_2$. This implies that $e_2 a_1$ is inversible in $e_2 B$. Hence by [1, Th. 1(b)], $e_2 f$ is Frobenius over $e_2 B$. Therefore, again by [3, Lemma 1], it follows that f is Frobenius over B. A similar argument applies to case a_1 is π-regular, completing the proof.

As a direct consequence of Lemma 2 and Th. 3, we obtain the following

Corollary 4. *Let* B *be π-regular. Then, any separable polynomial in* $B[X; \rho]$ *is Frobenius.*

Next, we shall prove the following theorem which contains a generalization of the results of Ikehata [1, Cor. 3] and Miyashita [2, Cor. to Th. 3. 5] to non-commutative artinian rings.

Theorem 5. *Let* B *satisfy the descending chain condition on two-sided ideals. Then, any separable polynomial in* $B[X; \rho]$ *is Frobenius.*

Proof. Let $f \in B[X; \rho]$ be a monic polynomial with constant term a_0, and $fB[X; \rho] = B[X; \rho]f$. Then by Lemma 1, we have $a_0^s B = Ba_0^s$ for any integer $s \geq 0$, Hence, by the assumption, there exists an integer $t \geq 0$ such that $a_0^t B = a_0^{t+1} B$. Therefore, it follows from Lemma 2 that a_0 is π-regular. Combining this fact with Th. 3, we obtain the assertion, completing the proof.

We shall conclude our study with the following remark.

Remark. As in Th. 5, let B satisfy the descending chain condition on two-sided ideals. Then, B is a direct sum of a finite number of (directly) indecomposable rings B_i. Now let Z_i be the center of B_i, and α an element of Z_i. Since B_i satisfies the descending chain condition on two-sided ideals, it follows that $\alpha^t B_i = \alpha^{t+1} B_i$ for some integer $t \geq 0$. Hence B_i is the direct sum of $\alpha^t B_i$ and the annihilator of α^t in B_i. Since B_i is indecomposable, there holds that either $\alpha^t B_i = B_i$ or $\alpha^t B_i = \{0\}$. If $\alpha^t B_i = B_i$ then α is inversible in B_i and so is in Z_i. Hence Z_i is a local ring. Thus, the center of B is a semi-local ring. Combining this fact with the results of Ikehata [1, Cor. 2 and Lemma 2], we see that any separable polynomial in $B[X; \rho]$ is Frobenius. We have therefore proved Th. 5 alternatively.

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