

Mathematical Journal of Okayama University

Volume 23, Issue 1

1981

Article 12

JUNE 1981

On the Euler-Poincaré characteristic of 6-dimensional harmonic manifolds

Masanori Kôzaki*

*Saga University

Copyright ©1981 by the authors. *Mathematical Journal of Okayama University* is produced by
The Berkeley Electronic Press (bepress). <http://escholarship.lib.okayama-u.ac.jp/mjou>

ON THE EULER-POINCARÉ CHARACTERISTIC OF 6-DIMENSIONAL HARMONIC MANIFOLDS

MASANORI KÔZAKI

1. Introduction. An analytic Riemannian manifold M is said to be harmonic at a point m of M , if Δs is a function of s only, where Δ denotes the Laplacian of M and s the geodesic distance from m to the point in a normal neighborhood of m . If M is harmonic at any point, M is said to be harmonic. For a harmonic manifold, it is known that $\Delta \Omega = f(\Omega)$ is a function of Ω only and does not depend on the reference point m , where $\Omega := s^2/2$. Then $f(\Omega)$ is called the characteristic function of M .

The purpose of this note is to estimate the Euler-Poincaré characteristic $\chi(M)$ of a 6-dimensional compact harmonic Riemannian manifold M in terms of some curvature tensors and the volume $\text{vol } M$ of M . We denote by $R = (R_{jkl}^i)^{11}$, $\rho = (R_{jkr}^r) = (R_{jk})$ and τ the Riemannian curvature tensor, the Ricci tensor and the scalar curvature, respectively. The results are stated as follows:

Theorem 1. *Let M be a 6-dimensional compact orientable harmonic Riemannian manifold, and f the characteristic function of M . Then the following inequality holds*

$$\chi(M) \leq -\frac{54 \text{ vol } M}{384\pi^3} (5\dot{f}(0)^3 + 82\dot{f}(0)\ddot{f}(0) + 140\ddot{f}(0)),$$

and the equality holds if and only if M is locally symmetric.

Theorem 2. *Let M be a 6-dimensional compact harmonic Kaehler manifold.*

- (1) *If $\tau > 0$, then M is of constant holomorphic sectional curvature.*
- (2) *If $\tau = 0$, then $\chi(M) = 0$.*
- (3) *If $\tau < 0$, then the following inequality holds*

$$\chi(M) \geq \frac{\tau^3 \text{ vol } M}{12 \cdot 384 \pi^3},$$

1) We follow the definition of the Riemannian curvature tensor in [11], which is different from that of [2] and [9] in sign.

and the equality holds if and only if M is of constant holomorphic sectional curvature.

Our theorems are related to the conjecture that any harmonic Riemannian manifold is locally symmetric (see [2], [8]).

Example. Let S^6 be a 6-dimensional sphere with constant sectional curvature k , and PC^3 a 6-dimensional complex projective space with constant holomorphic sectional curvature k . Then the characteristic function of S^6 and PC^3 are respectively given by $f(\varrho) = 1 + 5\sqrt{2k}\varrho \cot\sqrt{2k}\varrho$ and $f(\varrho) = 1 + 5l \cot l - l \tan l$, where $l := \sqrt{k}s/2$ ([8], [10], [12] and also §4). By a straightforward calculation, we obtain the extremal case in Theorem 1.

2. Proof of Theorem 1. By the harmonicity of M , M is Einstein and the following equalities hold ([2], [8], [11]):

$$(2.1) \quad \tau = -9\dot{f}(0), \quad |R|^2 = -9\{\dot{f}(0)^2 + 20\ddot{f}(0)\},$$

$$(2.2) \quad 27|\nabla R|^2 - \frac{8}{9}\tau^3 - 24\tau|R|^2 + 112\beta - 32\gamma = 151200\ddot{f}(0),$$

where $\nabla R := (R_{jkl,h}^i)$, $\beta := R^{ijkl}R_{ij}{}^{uv}R_{kluv}$, $\gamma := R^{ijkl}R_i{}^u{}_k R_{julv}$ and $|\cdot|$ denotes the norm. By making use of $\Delta|R|^2 = 0$ and the well-known Lichnerowicz formula, we have

$$(2.3) \quad |\nabla R|^2 + \frac{1}{3}\tau|R|^2 + \beta + 4\gamma = 0.$$

In the proofs of our theorems, we shall use the following expression for Gauss-Bonnet theorem ([9]).

Lemma 1. Let M be a 6-dimensional compact orientable Riemannian manifold. Then

$$(2.4) \quad \chi(M) = \frac{1}{384\pi^3} \int_M \{\tau^3 - 12\tau|\rho|^2 + 3\tau|R|^2 + 16R_i{}^j R_j{}^k R_k{}^i \\ - 24R^{ik} R^{jl} R_{ijkl} - 24R^{rs} R_{rjkl} R_i{}^{jkl} + 8\gamma - 4\beta\} dM,$$

where dM denotes the volume element of M .

Since M is Einstein, (2.4) takes the following form

$$(2.5) \quad \chi(M) = \frac{1}{384\pi^3} \int_M \left\{ \frac{\tau^3}{9} - \tau|R|^2 + 8\gamma - 4\beta \right\} dM.$$

Eliminating β , γ from (2.2), (2.3) and (2.5), we have

$$\begin{aligned}
 (2.6) \quad & 20 \cdot 384 \pi^3 \chi(M) - \int_M \left\{ \frac{4}{3} \tau^3 - \frac{164}{3} \tau |R|^2 - 151200 \ddot{f}(0) \right\} dM \\
 & = -5 \int_M |R|^2 dM.
 \end{aligned}$$

From (2.1) and (2.6) we have

$$\begin{aligned}
 & \chi(M) + \frac{54}{384 \pi^3} \int_M \{5 \dot{f}(0)^3 + 82 \dot{f}(0) \ddot{f}(0) + 140 \ddot{f}(0)\} dM \\
 & = -\frac{1}{4 \cdot 384 \pi^3} \int_M |R|^2 dM,
 \end{aligned}$$

whence it follows Theorem 1.

3. Proof of Theorem 2. Let M be a $2n$ -dimensional compact Kaehler manifold, and $\theta^1, \dots, \theta^n$ a local field of unitary frames. The Chern form γ_k of M is given by

$$\gamma_k := \frac{(-1)^k}{(2\pi\sqrt{-1})^k k!} \sum \partial_{\beta_1}^{\alpha_1} \cdots \partial_{\beta_k}^{\alpha_k} \mathcal{Q}_{\alpha_1}^{\beta_1} \wedge \cdots \wedge \mathcal{Q}_{\alpha_k}^{\beta_k},$$

where $\mathcal{Q}_{\beta}^{\alpha} := -\sum R_{\beta\bar{\gamma}}^{\alpha\bar{\gamma}} \theta^{\gamma} \wedge \bar{\theta}^{\bar{\gamma}}$. Let A be the operator of the interior product by the fundamental 2-form $\omega = \frac{\sqrt{-1}}{2} \sum \theta^{\alpha} \wedge \bar{\theta}^{\bar{\alpha}}$. Then, after calculations, we obtain the following

$$\begin{aligned}
 (3.1) \quad & A^3 \gamma_1 \wedge \gamma_2 = \frac{3}{64 \pi^3} \{ \tau^3 - 8 \tau |\rho|^2 + \tau |R|^2 + 16 \sum R_{\alpha\bar{\beta}} R_{\bar{\gamma}}^{\beta} R_{\gamma}^{\alpha} \\
 & - 32 \sum R_{\alpha\bar{\lambda}} R_{q\bar{p}} R_{p\lambda\bar{\alpha}}^q - 64 \sum R_{\alpha\bar{\gamma}} R_{q\bar{\lambda}}^p R_{p\lambda\bar{\alpha}}^q \}.
 \end{aligned}$$

In particular, if M is Einstein, then we have

$$(3.2) \quad \int_M \gamma_1 \wedge \gamma_2 \wedge \omega^{n-3} = \frac{3(n-1)!}{384 \pi^3 n(n+1)} \int_M \tau \left(\tau^2 + \frac{n+1}{n-1} A \right) dM, \quad \text{where}$$

$A := |R|^2 - 2\tau^2/n(n+1)$. The case $n=3$ in (3.2) is given by

$$\begin{aligned}
 (3.3) \quad & c_1 c_2 [M] - \frac{\tau^3 \text{vol } M}{2 \cdot 384 \pi^3} = \frac{1}{384 \pi^3} \int_M \tau \left(|R|^2 - \frac{\tau^2}{6} \right) dM, \quad \text{where} \\
 & c_1 c_2 [M] := \int_M \gamma_1 \wedge \gamma_2.
 \end{aligned}$$

In order to prove Theorem 2, we use the following

Lemma 2 ([4], [7]). *Let M be a $2n$ -dimensional compact Kaehler manifold. Then the inequality $A \geq 0$ holds and the equality holds if and only if M is of constant holomorphic sectional curvature.*

By (3.2) and Lemma 2 we have

Lemma 3. *Let M be a $2n$ -dimensional compact Einstein Kaehler manifold. If $\tau > 0$ ($\tau < 0$), then $384\pi^3 n(n+1) \int_M \tilde{r}_1 \wedge \tilde{r}_2 \wedge \omega^{n-3} - 3(n-1)! \tau^3 \text{vol } M \geq 0$ (≤ 0) and the equality holds if and only if M is of constant holomorphic sectional curvature.*

Remark. For the case $n = 3$ in Lemma 3, putting $a(M) := c_1 c_2 [M]/24$, we obtain a generalization of Proposition 2 in [7]. The arithmetic genus of M is defined by $a(M) := 1 - h^{1,0} + h^{2,0} - h^{3,0}$, where $h^{p,0}$ denotes the dimension of the space of holomorphic p -forms of M . If $\tau > 0$, then M is algebraic (Kodaira-Spencer, see [5]), and consequently $a(M) = c_1 c_2 [M]/24$ (Riemann-Roch-Hirzebruch, see [5]).

Proof of Theorem 2. Let M be a 6-dimensional compact harmonic Kaehler manifold. Then the equality $4\tau - 2\beta = -45\tau\ddot{f}(0)$ holds ([11]). Making use of this equality and (2.5), we obtain

$$(3.4) \quad \chi(M) = \frac{1}{2 \cdot 384 \pi^3} \int_M \tau \left(\frac{\tau^2}{3} - |R|^2 \right) dM.$$

From (3.3) and (3.4) we have

$$(3.5) \quad 12 \left\{ a(M) - \frac{\tau^3 \text{vol } M}{48 \cdot 384 \pi^3} \right\} = \frac{\tau^3 \text{vol } M}{12 \cdot 384 \pi^3} - \chi(M).$$

Now, from the above remark and (3.5) we obtain the assertion (3) of Theorem 2. The assertion (2) is obvious.

Finally, we prove the assertion (1). Assume $\tau > 0$. As M is Einstein, (i) M is simply connected ([6]), and (ii) $a(M) = 1$ because of $h^{p,0} = 0$ for $1 \leq p \leq 3$ ([13]). By (i) and the harmonicity of M , for every point p in M , all geodesics starting from p are simply closed and of the same length, and consequently $\chi(M) = 4$ ([1], [2]). By $a(M) = 1$, $\chi(M) = 4$ and (3.5), we have

$$a(M) = \frac{\tau^3 \text{vol } M}{48 \cdot 384 \pi^3}.$$

Therefore, again by the above remark, M is of constant holomorphic sectional curvature.

4. Appendix. In this section we calculate the characteristic functions of compact symmetric spaces of rank one. Let M be an n -dimensional Riemannian manifold ($n \geq 2$), and m a point of M . We denote by s

the geodesic distance from m to the point in a normal neighborhood of m . Let $(s, u) := (s, u_2, \dots, u_n)$ be a system of geodesic polar coordinates at m . Then the Riemannian metric is given by the form (g_{ij}) , $g_{1j} = \delta_{1j}$. Fix $s > 0$, and put $\gamma(t) := \exp_m(tu)$. We take linearly independent vectors $A_2, \dots, A_n$ on the tangent space to M at m orthogonal to u such that $Y_i(t) := (\exp_m)_* (tA_i)$ ($i=2, \dots, n$) are Jacobi fields along the geodesic γ and $Y_2(s), \dots, Y_n(s)$ are orthonormal. Put $\theta(tu) := |Y_2(t) \wedge \dots \wedge Y_n(t)| / t^{n-1} |A_2 \wedge \dots \wedge A_n|$ and $g(s, u) := \det(g_{ij})$. Then we have the following equalities ([3]):

$$\Delta s = \frac{1}{\sqrt{g(s, u)}} \frac{\partial}{\partial s} \sqrt{g(s, u)} = \frac{\theta'}{\theta} + \frac{n-1}{s} = \sum_{i=2}^n \langle Y'_i, Y_i \rangle(s).$$

Using the well-known Jacobi fields of compact symmetric spaces of rank one (see, for instance [2]) and the formula mentioned above, we obtain the following equalities:

1. The sphere S^n (with constant sectional curvature k):

$$f(\Omega) = 1 + 2(n-1)l \cot 2l, \quad \text{where } l := \sqrt{k}s/2.$$

2. The complex projective space PC^n (with constant holomorphic sectional curvature k):

$$f(\Omega) = 1 + (2n-1)l \cot l - l \tan l.$$

3. The quaternion projective space PQ^n (with maximal sectional curvature k):

$$f(\Omega) = 1 + (4n-1)l \cot l - 3l \tan l.$$

4. The Cayley projective plane PC_a^2 (with maximal sectional curvature k):

$$f(\Omega) = 1 + 15l \cot l - 7l \tan l.$$

REFERENCES

- [1] A. ALLAMIGEON: Propriétés globales des espaces de Riemann harmoniques, Ann. Inst. Fourier **15** (1965), 91—135.
- [2] A. L. BESSE: Manifolds all of whose Geodesics are Closed, Springer-Verlag, 1978.
- [3] R. L. BISHOP and R. J. CRITTENDEN: Geometry of Manifolds, Academic Press, 1964.
- [4] B.-Y. CHEN and K. OGIUE: On compact Einstein Kaehler manifolds, Proc. Amer. Math. Soc. **53** (1975), 176—178.
- [5] F. HIRZEBRUCH: Topological Methods in Algebraic Geometry, Springer-Verlag, 1966.
- [6] S. KOBAYASHI: On compact Kähler manifolds with positive definite Ricci tensor, Ann. of Math. **74** (1961), 570—574.
- [7] M. KÔZAKI: The spectrum of Kaehler manifolds, Tensor (N.S.) **33** (1979), 33—36.

- [8] H. S. RUSE, A. G. WALKER and T. J. WILLMORE : Harmonic Spaces, Edzioni Cremonese, Rome, 1961.
- [9] T. SAKAI : On eigen-values of Laplacian and curvature of Riemannian manifold, Tôhoku Math. J. **23** (1971), 589—603.
- [10] S. TACHIBANA : On the characteristic function of spaces of constant holomorphic curvature, Colloq. Math. **26** (1972), 149—155.
- [11] Y. WATANABE : On the characteristic function of harmonic Kählerian spaces, Tôhoku Math. J. **27** (1975), 13—24.
- [12] Y. WATANABE : On the characteristic functions of quaternion Kählerian spaces of constant Q-sectional curvature, Kōdai Math. Sem. Rep. **28** (1977), 284—299.
- [13] K. YANO and S. BOCHNER: Curvature and Betti Numbers, Ann. of Math. Studies **32**, Princeton, 1953.

SAGA UNIVERSITY

(Received March 17, 1980)