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Tôru Furukawa*

*Okayama University

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ON ISOMORPHISM INVARIANTS OF INTEGRAL GROUP RINGS

TÔRU FURUKAWA

Let G be a group, $\mathbb{Z}G$ denote its integral group ring. Let $\{\gamma_i(G)\}$, $\{\delta_i(G)\}$ and $\{D_i(G)\}$ denote the lower central series, derived series and integral dimension series of G , respectively. The series $\{\gamma_i(G)\}$ and $\{\delta_i(G)\}$ are defined inductively by $\gamma_1(G) = G$, $\delta_1(G) = G$ and for $i \geq 1$

$$\gamma_{i+1}(G) = (G, \gamma_i(G)), \quad \delta_{i+1}(G) = (\delta_i(G), \delta_i(G)).$$

As usual, for any two subgroups A and B of G , (A, B) is the subgroup generated by all commutators $(a, b) = a^{-1}b^{-1}ab$, $a \in A$, $b \in B$. Denoting by $\mathcal{A}(G)$ the augmentation ideal of $\mathbb{Z}G$, the series $\{D_i(G)\}$ is defined by $D_i(G) = G \cap (1 + \mathcal{A}(G)^i)$. We note here that for every group G , $D_2(G) = \gamma_2(G) = \delta_2(G)$ and $D_3(G) = \gamma_3(G)$ (see e.g. [4]).

In this note we find the subquotients of G which are isomorphism invariants of $\mathbb{Z}G$. It is known [6, p.100, Theorem 4.28] that if G is locally finite, then the successive subquotients $\gamma_i(G)/\gamma_{i+1}(G)$ and $\delta_i(G)/\delta_{i+1}(G)$ are isomorphism invariants of $\mathbb{Z}G$ for all $i \geq 1$. Also, it is known [6, p.102, Theorem 5.3] that if G is torsion, then the quotient group $G/\delta_3(G)$ is an isomorphism invariant of $\mathbb{Z}G$. Extending these results we prove the following two theorems.

Theorem A. *Suppose $\mathbb{Z}G \cong \mathbb{Z}H$ where G is a torsion group. Then, for all $i \geq 1$, we have the following isomorphisms:*

- (i) $\gamma_i(G)/\gamma_{i+j}(G) \cong \gamma_i(H)/\gamma_{i+j}(H)$ and $D_i(G)/D_{i+j}(G) \cong D_i(H)/D_{i+j}(H)$ for all j with $1 \leq j \leq i$,
- (ii) $\delta_i(G)/\delta_{i+1}(G) \cong \delta_i(H)/\delta_{i+1}(H)$.

Theorem B. *Suppose $\mathbb{Z}G \cong \mathbb{Z}H$. If $G/\gamma_2(G)$ is torsion, then*

$$G/\delta_3(G) \cong H/\delta_3(H), \quad G/\gamma_3(G) \cong H/\gamma_3(H), \quad G/\gamma_4(G) \cong H/\gamma_4(H).$$

Moreover if G is torsion, then $G/D_4(G) \cong H/D_4(H)$.

In what follows, unless otherwise stated, G will denote an arbitrary group. Also, for a normal subgroup N of G we denote by $\mathcal{A}(G, N)$ the kernel of the natural homomorphism from $\mathbb{Z}G$ to $\mathbb{Z}(G/N)$ and by $V(\mathbb{Z}G)$ the units of augmentation one of the integral group ring $\mathbb{Z}G$.

In order to prove our results we need some well-known facts which are collected below (see e.g. [3, 5 and 6]).

(I) Let N be a normal subgroup of G . Then

$$N/(G,N) \cong \Delta(G,N)/\Delta(G)\Delta(G,N) + \Delta(G,N)\Delta(G), \text{ and} \\ (G,N) = G \cap (1 + \Delta(G)\Delta(G,N) + \Delta(G,N)\Delta(G)).$$

(II) Let N be a normal subgroup of G . Then

$$N/(N,N) \cong \Delta(G,N)/\Delta(G)\Delta(G,N), \text{ and} \\ (N,N) = G \cap (1 + \Delta(G)\Delta(G,N)).$$

(III) Let N and L be normal subgroups of G such that $(N,N) \subseteq L \subseteq N$. Then

$$N/L \cong \Delta(G,N)/\Delta(G)\Delta(G,N) + \Delta(G,L), \text{ and} \\ L = G \cap (1 + \Delta(G)\Delta(G,N) + \Delta(G,L)).$$

(IV) Let $u = \sum_g u(g)g \in V(\mathbb{Z}G)$ be of finite order with $u(1) \neq 0$. Then $u = 1$.

(V) If G is torsion abelian then $TV(\mathbb{Z}G) = G$, where $TV(\mathbb{Z}G)$ is the torsion subgroup of $V(\mathbb{Z}G)$.

1. Proof of Theorem A. Using (IV) the same argument as in Lemma 3.1 of [1] gives us the following result, whose proof will be omitted.

Lemma 1.1. *If X is a periodic subgroup of $V(\mathbb{Z}G)$, then the elements of X are linearly independent over $\mathbb{Z}$, the ring of rational integers.*

A series $G = N_1 \supseteq N_2 \supseteq \cdots \supseteq N_n \supseteq \cdots$ of subgroups of G is called an N -series of G if $(N_n, N_m) \subseteq N_{n+m}$ for all $n, m \geq 1$. The familiar examples of N -series of G are its lower central series $\{\gamma_i(G)\}$ and integral dimension series $\{D_i(G)\}$.

Lemma 1.2. *Let $\{N_i\}$ be an N -series of G . Then*

$$N_i/N_{i+j} \cong \Delta(G, N_i)/\Delta(G)\Delta(G, N_i) + \Delta(G, N_{i+j})$$

for all $i \geq 1$ and all j with $1 \leq j \leq i$.

Proof. Since $(N_i, N_i) \subseteq N_{2i} \subseteq N_{i+j}$, the result is an immediate consequence of (III).

We recall that a (ring) homomorphism f from $\mathbf{Z}G$ to the integral group ring $\mathbf{Z}H$ is called *normalized* if the sum of all the coefficients of $f(g)$ is 1 for all $g \in G$.

Lemma 1.3. *Let G be a torsion group. Then any normalized epimorphism $f : \mathbf{Z}G \rightarrow \mathbf{Z}H$ satisfying $G \cap (1 + \text{Ker } f) = \{1\}$ is an isomorphism.*

Proof. It is easy to see that the restriction of f to G is an imbedding into $V(\mathbf{Z}H)$. Hence the result follows from Lemma 1.1.

We now adapt the method of Cohn-Livingstone [1] to prove Theorem A. Suppose that we have a normalized isomorphism $\theta : \mathbf{Z}G \rightarrow \mathbf{Z}H$. Then, following [1], for any normal subgroup N of G the normal subgroup ϕN of H is defined by $\phi N = H \cap (1 + \theta(\Delta(G, N)))$. (Note that we do not identify $\mathbf{Z}H$ with $\mathbf{Z}G$.) Similarly, for any normal subgroup K of H we define $\phi^* K = G \cap (1 + \theta^{-1}(\Delta(H, K)))$. From the definition of ϕ we readily see that $\theta(\Delta(G, N)) \cong \Delta(H, \phi N)$. Furthermore we have

Lemma 1.4. (1) *If N is a normal subgroup of G such that $H/\phi N$ is torsion, then $\theta(\Delta(G, N)) = \Delta(H, \phi N)$.*

(2) *Let I be an ideal of $\mathbf{Z}G$ and set $N = G \cap (1 + I)$, $K = H \cap (1 + \theta(I))$. If $G/\phi^* K$ is torsion, then $\phi N = K$.*

Proof. (1) Since $\theta(\Delta(G, N)) \cong \Delta(H, \phi N)$, there is a natural epimorphism $\mathbf{Z}(H/\phi N) \rightarrow \mathbf{Z}H/\theta(\Delta(G, N))$, which induces a normalized epimorphism $f : \mathbf{Z}(H/\phi N) \rightarrow \mathbf{Z}(G/N)$ such that $H/\phi N \cap (1 + \text{Ker } f) = \{1\}$. Hence by Lemma 1.3, f is an isomorphism, which implies that $\theta(\Delta(G, N)) = \Delta(H, \phi N)$.

(2) Since $\Delta(G, N) \subseteq I$ and $\Delta(H, K) \subseteq \theta(I)$, it follows that $\phi N \subseteq K$ and $\phi^* K \subseteq N$. On the other hand, since $G/\phi^* K$ is torsion, applying (1) of this lemma to θ^{-1} and ϕ^* we get $\theta^{-1}(\Delta(H, K)) = \Delta(G, \phi^* K)$. This means that $K = \phi\phi^* K$ and $K \subseteq \phi N$. Thus $\phi N = K$.

1.5. Proof of Theorem A. Let $\theta : \mathbf{Z}G \rightarrow \mathbf{Z}H$ be the given isomorphism; then we may assume here that θ is normalized. Let ϕ be defined as above. In view of (II) and Lemma 1.2, to complete the proof of the theorem it remains only to prove that for all $i \geq 1$

$$(1.6) \quad \theta(\Delta(G, \gamma_i(G))) = \Delta(H, \gamma_i(H)),$$

$$(1.7) \quad \theta(\Delta(G, D_i(G))) = \Delta(H, D_i(H)),$$

$$(1.8) \quad \theta(\Delta(G, \delta_i(G))) = \Delta(H, \delta_i(H)).$$

First, we use induction on i to see (1.6), the case $i = 1$ being obvious. Assume that for some $i \geq 1$, we have

$$\theta(\Delta(G, \gamma_k(G))) = \Delta(H, \gamma_k(H)) \text{ for all } k \text{ with } 1 \leq k \leq i.$$

Then it follows by (I) that $\gamma_k(G)/\gamma_{k+1}(G) \cong \gamma_k(H)/\gamma_{k+1}(H)$ for all k with $1 \leq k \leq i$. From this we deduce that $H/\gamma_{i+1}(H)$ is torsion. Noting here that

$$\gamma_{i+1}(G) = G \cap (1 + \Delta(G)\Delta(G, \gamma_i(G)) + \Delta(G, \gamma_i(G))\Delta(G))$$

by (I), it follows from Lemma 1.4 (2) and the induction hypothesis that

$$\begin{aligned} \phi\gamma_{i+1}(G) &= H \cap (1 + \theta\{\Delta(G)\Delta(G, \gamma_i(G)) + \Delta(G, \gamma_i(G))\Delta(G)\}) \\ &= H \cap (1 + \Delta(H)\Delta(H, \gamma_i(H)) + \Delta(H, \gamma_i(H))\Delta(H)) \\ &= \gamma_{i+1}(H) \end{aligned}$$

and so, $H/\phi\gamma_{i+1}(G)$ is torsion. Hence by Lemma 1.4 (1), we have

$$\theta(\Delta(G, \gamma_{i+1}(G))) = \Delta(H, \phi\gamma_{i+1}(G)) = \Delta(H, \gamma_{i+1}(H))$$

which completes the induction on i . Thus (1.6) is established. Next we shall prove (1.7). By Lemma 1.4 (2), we obtain

$$\phi D_i(G) = H \cap (1 + \theta(\Delta(G)^i)) = H \cap (1 + \Delta(H)^i) = D_i(H)$$

for all $i \geq 1$. Since each $H/\gamma_i(H)$ is torsion as seen by using (1.6), so is each $H/\phi D_i(G)$ because $\gamma_i(H) \subseteq D_i(H) = \phi D_i(G)$. Hence by Lemma 1.4 (1), $\theta(\Delta(G, D_i(G))) = \Delta(H, D_i(H))$ proving (1.7). Finally, the proof of (1.8) being similar to that of (1.6) (using (II)), we omit the details.

2. Proof of Theorem B. Given an ideal I of ZG , we mean by $G+I$ the subset (of ZG) of all elements of the form $g+a$, $g \in G$, $a \in I$. The following lemma can be easily verified.

Lemma 2.1 (see [2, Lemma 1]). *Let $\theta : ZG \rightarrow ZH$ be an isomorphism. Let I be an ideal of ZG such that $\theta(G+I) = H + \theta(I)$. Then we have the following isomorphism:*

$$G/G \cap (1+I) \cong H/H \cap (1+\theta(I)).$$

We recall that for any normal subgroup N of G , $\Delta(G, N)$ is the ideal of ZG generated by all $x-1$, $x \in N$.

Lemma 2.2. *Let N and L be normal subgroups of G , with $N \supseteq L$. Then, $\Delta(G)\Delta(G, N) + \Delta(G, L)\Delta(G) = \Delta(G)\Delta(G, N) + \Delta(G, (G, L))$.*

Proof. The equality follows easily from the identity $(l-1)(g-1) = (g-1)(l-1) - lg((g,l)-1)$ for $l \in L$ and $g \in G$.

We now define the ideals $I(G)$, $I_3(G)$, $I_4(G)$ and $J_4(G)$ of $\mathbf{Z}G$ by setting

$$\begin{aligned} I(G) &= \mathcal{A}(G)\mathcal{A}(G, \gamma_2(G)), \quad I_3(G) = I(G) + \mathcal{A}(G, \gamma_3(G)), \\ I_4(G) &= I(G) + \mathcal{A}(G, \gamma_4(G)) \quad \text{and} \quad J_4(G) = I(G) + \mathcal{A}(G, D_4(G)). \end{aligned}$$

Using Lemma 2.2 we have

$$\begin{aligned} I(G) + \mathcal{A}(G, \gamma_2(G))\mathcal{A}(G) &= I(G) + \mathcal{A}(G, \gamma_3(G)) = I_3(G), \\ I(G) + \mathcal{A}(G, \gamma_3(G))\mathcal{A}(G) &= I(G) + \mathcal{A}(G, \gamma_4(G)) = I_4(G), \end{aligned}$$

and moreover

$$\begin{aligned} I(G) + \mathcal{A}(G, \gamma_3(G))\mathcal{A}(G) &= I(G) + \{I(G) + \mathcal{A}(G, \gamma_3(G))\}\mathcal{A}(G) = \\ I(G) + \{I(G) + \mathcal{A}(G, \gamma_2(G))\mathcal{A}(G)\}\mathcal{A}(G) &= I(G) + \mathcal{A}(G, \gamma_2(G))\mathcal{A}(G)^2. \end{aligned}$$

Therefore, we observe that

$$(2.3) \quad I_3(G) = I(G) + \mathcal{A}(G, \gamma_2(G))\mathcal{A}(G) \quad \text{and} \quad I_4(G) = I(G) + \mathcal{A}(G, \gamma_2(G))\mathcal{A}(G)^2.$$

Also, from (II) and (III) we obtain:

$$(2.4) \quad \begin{aligned} G \cap (1 + I(G)) &= \delta_3(G), \quad G \cap (1 + I_3(G)) = \gamma_3(G), \\ G \cap (1 + I_4(G)) &= \gamma_4(G) \quad \text{and} \quad G \cap (1 + J_4(G)) = D_4(G). \end{aligned}$$

We are now in a position to prove Theorem B.

2.5. Proof of Theorem B. We may assume that the given isomorphism $\theta : \mathbf{Z}G \rightarrow \mathbf{Z}H$ is normalized, and therefore $\theta(\mathcal{A}(G)) = \mathcal{A}(H)$. Since $\mathcal{A}(G, \gamma_2(G))$ is the smallest ideal I such that $\mathbf{Z}G/I$ is commutative, it follows that $\theta(\mathcal{A}(G, \gamma_2(G))) = \mathcal{A}(H, \gamma_2(H))$, and hence $\theta(I(G)) = I(H)$. Further, we deduce by (2.3) that $\theta(I_3(G)) = I_3(H)$ and $\theta(I_4(G)) = I_4(H)$. Now, suppose that $G/\gamma_2(G)$ is torsion. Then, also $H/\gamma_2(H)$ is torsion, since we have $G/\gamma_2(G) \cong H/\gamma_2(H)$. We claim

$$(2.6) \quad \theta(G + I(G)) = H + I(H).$$

As a matter of fact, the proof of (2.6) can be found in [7]. However, for the sake of completeness, we give its proof. Let $g \in G$, and let $\pi : \mathbf{Z}H \rightarrow \mathbf{Z}(H/\gamma_2(H))$ be the natural homomorphism. Then $\pi(\theta(g))$ is a torsion element of $V(\mathbf{Z}(H/\gamma_2(H)))$ and thus by (V), $\pi(\theta(g)) = \pi(h_1)$ for some $h_1 \in H$, so $\theta(g) = h_1 + \alpha$ where α is in $\mathcal{A}(H, \gamma_2(H))$. By (II), $\gamma_2(H)/\delta_3(H) \cong \mathcal{A}(H, \gamma_2(H))/I(H)$ and, this isomorphism being given by $h\delta_3(H) \rightarrow h-1+I(H)$, there exists $h_2 \in \gamma_2(H)$ such that $\alpha \equiv h_2-1 \pmod{I(H)}$.

So, $\theta(g) \equiv h_1 + h_2 - 1 = h_1 h_2 - (h_1 - 1)(h_2 - 1) \equiv h_1 h_2 \pmod{I(H)}$, which implies that $\theta(g) \in H + I(H)$. We have therefore proved that $\theta(G) \subseteq H + I(H)$. As seen above, $\theta(I(G)) = I(H)$, so $\theta(G + I(G)) \subseteq H + I(H)$. The same argument shows that $\theta^{-1}(H + I(H)) \subseteq G + I(G)$, and hence (2.6) has been proved. It is clear from (2.6) that $\theta(G + I_3(G)) = H + I_3(H)$ and $\theta(G + I_4(G)) = H + I_4(H)$. Hence by Lemma 2.1 and (2.4) we conclude that

$$G/\delta_3(G) \cong H/\delta_3(H), \quad G/\gamma_3(G) \cong H/\gamma_3(H), \quad G/\gamma_4(G) \cong H/\gamma_4(H).$$

Finally, if G is torsion then $\theta(\Delta(G, D_4(G))) = \Delta(H, D_4(H))$ by (1.7), so that $\theta(J_4(G)) = J_4(H)$. Therefore, we have $\theta(G + J_4(G)) = H + J_4(H)$ and hence $G/D_4(G) \cong H/D_4(H)$, which completes the proof of the theorem.

Corrigendum. In Theorem in [2] the inequalities $1 \leq i \leq j+1$ are false and should be replaced by $1 \leq j \leq i+1$.

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DEPARTMENT OF MATHEMATICS
OKAYAMA UNIVERSITY

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