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A note on commutativity of rings

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A NOTE ON COMMUTATIVITY OF RINGS

To Professor Kentaro Murata on his 60th birthday

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Let R be a ring with 1. Awtar [3] proved that if R is $n!$-torsion free and satisfies, for example, the identity $x^n y^n = (xy)^n$, then R is commutative. Harmanci [4] showed that if R satisfies the identities $x^n y^n = (xy)^n$ and $x^{n+1}y^{n+1} = (xy)^{n+1}$, then R satisfies the identity $n(n!)^2(xy-yx)=0$. In this note we give a very general theorem including their results (in associative case) by just a simple method.

In §1 we prepare a tool, which is used to prove our theorem in §2. In §3 we discuss the possibility of and the impossibility of extending the theorem.

1. Difference operators. Throughout this note, $\boldsymbol{Z}$ denotes the ring of integers, and $\boldsymbol{Z}\langle x,y\rangle$ the non-commutative polynomial ring in two variables x and y over $\boldsymbol{Z}$. For a set S the cardinality of S is denoted by $|S|$.

Let Δ_x and Δ_y be the difference operators defined on $\boldsymbol{Z}\langle x,y\rangle$ by

$$\Delta_x F(x,y) = F(x+1,y) - F(x,y)$$

and

$$\Delta_y F(x,y) = F(x,y+1) - F(x,y),$$

for $F(x,y) \in \boldsymbol{Z}\langle x,y\rangle$. As is easily seen, the operator Δ_x (resp. Δ_y) decreases the degree of $F(x,y)$ with respect to x (resp. y). The composition of Δ_x and Δ_y is denoted by Δ_{xy}. We have $\Delta_{xy} = \Delta_{yx}$. Let $\Delta_{x^r y^s}$ denote the composition of r Δ_x's and s Δ_y's.

Let $M = x_1 x_2 \cdots x_d$ (x_i is either x or y) be a monic monomial in x and y. Suppose that the numbers of x's and of y's appearing in M are m and n respectively ($m+n=d$). Let $\Phi(M)$ (resp. $\Psi(M)$) be the number of pairs (i,j) such that $1 \leqq i < j \leqq d$ and $x_i = x$, $x_j = y$ (resp. $x_i = y$, $x_j = x$). Clearly we have $\Phi(M) + \Psi(M) = mn$. The functions Φ and Ψ are extended linearly on $\boldsymbol{Z}\langle x,y\rangle$.

Lemma. *Let $M = x_1 x_2 \cdots x_d$ be as above and assume $m,n > 0$. Then the formula*

$$\Delta_{x^{m-1}y^{n-1}} M = (m-1)!(n-1)!(\Phi(M)xy + \Psi(M)yx) + ax + by + c$$

with $a,b,c \in \boldsymbol{Z}$ *holds.*

Proof. Suppose that the set $\{1,\cdots,d\}$ is a disjoint union of subsets I and J such that x_i is equal to x or y according as $i \in I$ or $i \in J$ ($|I| = m$, $|J| = n$). For a polynomial $F \in \boldsymbol{Z}\langle x,y\rangle$, $\overline{F}$ denotes the sum of monomials in F of the highest total degree. Clearly we have

$$\overline{\Delta_x M} = \sum_{i \in I} x_1 x_2 \cdots x_{i-1} x_{i+1} \cdots x_d.$$

By the definition of Φ we obtain

$$\Phi(x_1 x_2 \cdots x_{i-1} x_{i+1} \cdots x_d) = \Phi(M) - l_i$$

for $i \in I$ and

$$\sum_{i \in I} l_i = \Phi(M),$$

where $l_i = |\{j \in J \mid i < j\}|$. From these it follows that

$$\Phi(\overline{\Delta_x M}) = (m-1)\Phi(M).$$

Using this equality $m-1$ times, we get

$$\Phi(\Delta_{x^{m-1}} M) = (m-1)!\,\Phi(M).$$

Since a similar formula holds if we replace x by y or Φ by Ψ, we find

$$\Phi(\Delta_{x^{m-1}y^{n-1}} M) = (m-1)!(n-1)!\,\Phi(M)$$

and

$$\Psi(\Delta_{x^{m-1}y^{n-1}} M) = (m-1)!(n-1)!\,\Psi(M).$$

Since the degrees of $\Delta_{x^{m-1}y^{n-1}} M$ with respect to x and to y are at most 1, the conclusion of the lemma is now clear.

2. Commutativity theorem. Let $\boldsymbol{N}$ denote the set of all non-negative integers. Let $F(x,y) \in \boldsymbol{Z}\langle x,y\rangle$ and $(m,n) \in \boldsymbol{N} \times \boldsymbol{N}$. The (m,n)-component of F, the sum of all monomials of degree (m,n), that is, of degree m with respect to x and of degree n with respect to y, is denoted by $F_{m,n}$.

Theorem. *Let R be a ring with* 1. *Let $F(x,y)$ be a polynomial in $\boldsymbol{Z}\langle x,y\rangle$ of total degree d. Suppose that the greatest common divisor l of $\{(m-1)!(n-1)!\,\Phi(F_{m,n}) \mid m+n=d,\ m,n>0\}$ is positive. If R satisfies the identity $F(x,y)=0$, then R satisfies the identity $l(xy-yx)=0$. Therefore, if moreover R is l-torsion free, R is commutative.*

Proof. Let m and n be positive integers such that $m+n=d$. Note

that the total degree of $\Delta_{x^{m-1}y^{n-1}}F(x,y)$ is at most 2. By Lemma in the preceding section we have

$$\Delta_{x^{m-1}y^{n-1}}F(x,y) = (m-1)!(n-1)!(\Phi(F_{m,n})xy + \Psi(F_{m,n})yx) + ax^2 + bx + cy^2 + dy + e,$$

with $a,b,c,d,e \in \boldsymbol{Z}$. If R satisfies the identity $F(x,y)=0$, then the right hand side of the equality above is 0 for any $x,y \in R$. Substituting $x=y=0$, we have $e=0$ in R. Substituting $x=0$ or $y=0$ yields the identities $ax^2+bx = cy^2+dy = 0$. Hence we get the identity

$$(m-1)!(n-1)!(\Phi(F_{m,n})xy + \Psi(F_{m,n})yx) = 0.$$

Substituting $x=y=1$ in the last identity, we have

$$(m-1)!(n-1)!(\Phi(F_{m,n}) + \Psi(F_{m,n})) = 0$$

in R. Consequently we obtain the identity

$$(m-1)!(n-1)!\,\Phi(F_{m,n})(xy-yx) = 0.$$

From this the theorem follows.

Example 1. For $F(x,y) = x^n y^n - (xy)^n$ $(n \geq 2)$, we have $\Phi(F) = \frac{n(n-1)}{2}$. Therefore, if $x^n y^n = (xy)^n$ holds in R, then so does

$$((n-1)!)^2 \cdot \frac{n(n-1)}{2}(xy-yx) = 0.$$

It is interesting that Abu-Khuzam [1] has proved the following : If R satisfies the identity $x^n y^n = (xy)^n$ and is $n(n-1)$-torsion free for some $n \geq 2$, then R is commutative (see also [2]).

3. Examples and remarks. Theorem 1 need not be true if we delete the hypothesis that R has the identity element 1. In fact we have

Example 2. Let N be a torsion free non-commutative ring such that $N^3 = 0$. Then, for any non-zero polynomial $F(x,y) \in \boldsymbol{Z}\langle x,y\rangle$ without terms of total degree lower than 3, the identity $F(x,y)=0$ holds in R, but for any $l > 0$, R does not satisfy the identity $l(xy-yx)=0$.

The following example shows that the condition of positivity of l in Theorem is essential.

Example 3. Let N be the same as in Example 2. In the additive

group $R = \boldsymbol{Z} \oplus N$ we define multiplication by

$$(a+u)\cdot(b+v) = ab+av+bu+uv,$$

for $a,b \in \boldsymbol{Z}$ and $u,v \in N$. Then R is a ring with 1. Let $M(x,y) \in \boldsymbol{Z}\langle x,y\rangle$ be a monic monomial of degree $(m,n) \in \boldsymbol{N}\times\boldsymbol{N}$. Let $a,b \in \boldsymbol{Z}$. Then for any $u,v \in N$ we have

$$M(a+u,b+v) = a^m b^n + ma^{m-1}b^n u + na^m b^{n-1}v + \binom{m}{2}a^{m-2}b^n u^2 + \binom{n}{2}a^m b^{n-2}v^2 + \Phi(M)a^{m-1}b^{n-1}uv + \Psi(M)a^{m-1}b^{n-1}vu.$$

Hence, for $F(x,y) \in \boldsymbol{Z}\langle x,y\rangle$ we find

$$F(a+u,b+v) = \sum_{m,n}\{F_{m,n}(1,1)G^{m,n}(u,v) + c_{m,n}(\Phi(F_{m,n})uv + \Psi(F_{m,n})vu)\},$$

where $G^{m,n}(x,y) \in \boldsymbol{Z}\langle x,y\rangle$ and $c_{m,n} \in \boldsymbol{Z}$. Therefore, R satisfies the identity $F(x,y) = 0$ for any $F(x,y) \in \boldsymbol{Z}\langle x,y\rangle$ satisfying $F_{m,n}(1,1) = \Phi(F_{m,n}) = 0$ for all $(m,n) \in \boldsymbol{N}\times\boldsymbol{N}$ (note that $\Psi(F_{m,n})$ is also 0, because $\Phi(F_{m,n}) + \Psi(F_{m,n}) = mnF_{m,n}(1,1)$). However, R does not satisfy the identity $l(xy-yx) = 0$, for every $l > 0$.

Example 4. Let R be the same as in Example 3. Let $F(x,y) \in \boldsymbol{Z}\langle x,y\rangle$ and assume that the greatest common divisor q of $\{\Phi(F_{m,n}) \mid m,n > 0\}$ is positive. Suppose that $F_{m,n}(1,1) \equiv 0 \pmod{q}$ for all $(m,n) \in \boldsymbol{N}\times\boldsymbol{N}$. Set $R_q = R \otimes \boldsymbol{Z}_q$, where $\boldsymbol{Z}_q$ is the residue class ring of $\boldsymbol{Z}$ modulo q. Then, R_q satisfies the identity $F(x,y) = 0$, but the identity $l(xy-yx) = 0$ is not satisfied unless $l \equiv 0 \pmod{q}$.

Example 4 shows that in our theorem, l cannot be replaced by a proper divisor of the greatest common divisor q of $\{\Phi(F_{m,n}) \mid m,n > 0\}$. There is a gap between l and q. This gap is filled for division rings and for algebras over infinite fields, though it is impossible in general.

Proposition. *Let R be a division ring or an algebra over an infinite field. Let $p \geqq 0$ be the characteristic of R. Let $F(x,y) \in \boldsymbol{Z}\langle x,y\rangle$ and q the greatest common divisor of $\{\Phi(F_{m,n}) \mid m,n > 0\}$. If R satisfies the identity $F(x,y) = 0$ and $q \not\equiv 0 \pmod{p}$, then R is commutative.*

Proof. First let R be an algebra over an infinite field K. Then by the proof of Kaplansky [5, Lemma 4] we may assume that F is homogeneous and $q = \Phi(F)$. Suppose that F is of degree (m,n) with $m,n > 0$. Then we have

$$F(a+x,b+y)=\sum_{i,j}a^{m-i}b^{n-j}F^{i,j}(x,y)=0$$

for all $x,y\in R$ and for infinitely many $a,b\in K$, where $F^{i,j}$ is a homogeneous polynomial in $\boldsymbol{Z}\langle x,y\rangle$ of degree (i,j). It follows that $F^{1,1}(x,y)=\Phi(F)xy+\Psi(F)yx=0$. Substituting $x=y=1$, we obtain $\Phi(F)+\Psi(F)=0$ in R. Thus we get $q(xy-yx)=0$. Since $q\not\equiv 0 \pmod{p}$, $xy=yx$ follows.

Next assume that R is a division ring. Then, by [5, Theorem 1] R is finite dimensional over its center Z. If Z is finite, then so is R, and therefore R is commutative. If Z is infinite, the commutativity of R follows from the assertion proved first.

The proposition need not be true even for primitive rings.

Example 5. Let $R=M(2,\boldsymbol{Z}_2)$ be the total matrix algebra of degree 2 over $\boldsymbol{Z}_2$. Then R satisfies the identity

$$F(x,y)=xy(x^3y^3)^3xy-xy(y^3x^3)^3xy=0.$$

Note that $\Phi(F)=27\not\equiv 0 \pmod 2$, but R is not commutative.

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