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ON PROJECTIVE KILLING TENSOR IN A RIEMANNIAN MANIFOLD

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Our purpose is to give some conditions for the non-existence of the projective Killing tensor. We also show an integrability condition for the system of the partial differential equations which defines an affine Killing tensor of degree p.

1. Let M^m be an m-dimensional Riemannian manifold with positive definite metric $g=(g_{ab})$[1] with respect to local coordinate system $\{x^a\}$. We identify a skew symmetric covariant tensor $v=(v_{a_1\dots a_p})$ with the differential p-form

$$v=(1/p!)v_{a_1\dots a_p}dx^{a_1}\wedge\dots\wedge dx^{a_p}.$$

dv and δv denote the exterior differentiation and the exterior co-differentiation of v respectively. As is well known, $ddv=0$ and $\delta\delta v=0$ hold, and d and δ are dual each other with respect to the global scalar product.

Following [2], $v=(v_{a_1\dots a_p})$ is called a projective Killing tensor if v is a skew symmetric tensor satisfying

$$\begin{aligned}
&\nabla_c\nabla_b v_{a_1\dots a_p}+(1/2)\sum_{i=1}^{p}R_{cba_i}{}^{e}v_{a_1\dots \underset{(i)}{e}\dots a_p}\\
&-(1/2)(R_{ba_1c}{}^{e}+R_{ca_1b}{}^{e})v_{ea_2\dots a_p}\\
&-(1/2)\sum_{i=2}^{p}(R_{ba_1a_i}{}^{e}v_{ca_2\dots\underset{(i)}{e}\dots a_p}+R_{ca_1a_i}{}^{e}v_{ba_2\dots\underset{(i)}{e}\dots a_p})\\
&=\sum_{i=1}^{p}(-1)^{i-1}(g_{ca_i}\nabla_b\theta_{a_1\dots\hat{a}_i\dots a_p}+g_{ba_i}\nabla_c\theta_{a_1\dots\hat{a}_i\dots a_p})
\end{aligned}\tag{1.1}$$

where $\theta=(\theta_{a_2\dots a_p})$ is a certain skew symmetric tensor and $\hat{a}_i$ means that a_i is deleted. If the left hand side of (1.1) vanishes identically, v is called an affine Killing tensor. As was shown in [2], if v is a Killing tensor then it is an affine one, and if v is an affine Killing tensor then it is a projective one.

If we transvect (1.1) with g^{ba_1} and change the index c to b, we have

1) The indices $a, b, c, \dots$ run on the range $\{1, 2, \dots, m\}$ and the summation convention is used throughout this paper.

$$\nabla_b \theta_{a_2\ldots a_p} = -\{1/(m-p+2)\}\left[\nabla_b(\delta v)_{a_2\ldots a_p} + \sum_{i=2}^{p}(-1)^i g_{ba_i}(\delta\theta)_{a_2\ldots\hat{a}_i\ldots a_p}\right],$$

that is,

$$\nabla_b \theta_{a_1\ldots\hat{a}_i\ldots a_p} = -\{1/(m-p+2)\}\left[\nabla_b(\delta v)_{a_1\ldots\hat{a}_i\ldots a_p} + \sum_{1\leq j<i}(-1)^{j+1} g_{ba_j}(\delta\theta)_{a_1\ldots\hat{a}_j\ldots\hat{a}_i\ldots a_p} + \sum_{i<j\leq p}(-1)^j g_{ba_j}(\delta\theta)_{a_1\ldots\hat{a}_i\ldots\hat{a}_j\ldots a_p}\right].$$

From this, it follows by the straightforward computation that

$$\sum_{i=1}^{p}(-1)^{i-1}(g_{ca_i}\nabla_b\theta_{a_1\ldots\hat{a}_i\ldots\hat{a}_p} + g_{ba_i}\nabla_c\theta_{a_1\ldots\hat{a}_i\ldots a_p}) = -\{1/(m-p+2)\}\sum_{i=1}^{p}(-1)^{i-1}(g_{ca_i}\nabla_b(\delta v)_{a_1\ldots\hat{a}_i\ldots a_p} + g_{ba_i}\nabla_b(\delta v)_{a_1\ldots\hat{a}_i\ldots a_p}).$$

Thus we have

Lemma 1. *A skew symmetric tensor v is projective if and only if it satisfies*

$$\begin{aligned}
\nabla_c\nabla_b v_{a_1\ldots a_p} &+ (1/2)\sum_{i=1}^{p} R_{cba_i}{}^{e} v_{a_1\ldots\underset{(i)}{e}\ldots a_p} \\
&- (1/2)(R_{ba_1c}{}^{e} + R_{ca_1b}{}^{e})v_{ea_2\ldots a_p} \\
&- (1/2)\sum_{i=2}^{p}(R_{ba_1a_i}{}^{e} v_{ca_2\ldots\underset{(i)}{e}\ldots a_p} + R_{ca_1a_i}{}^{e} v_{ba_2\ldots\underset{(i)}{e}\ldots a_p}) \\
= -\{1/(m-p+2)\}&\sum_{i=1}^{p}(-1)^{i-1}(g_{ca_i}\nabla_b(\delta v)_{a_1\ldots\hat{a}_i\ldots a_p} + g_{ba_i}\nabla_c(\delta v)_{a_1\ldots\hat{a}_i\ldots a_p}).
\end{aligned}\tag{1.2}$$

If we denote the covariant differentiation by ∇, we have

Corollary 2. *A projective Killing tensor v is affine if and only if $\nabla\delta v = 0$ holds.*

Proof. The if part is obvious. Let v be affine. The left hand side of (1.2) vanishes. Transvecting this with g^{ba_1}, we have $\nabla_c(\delta v)_{a_2\ldots a_p} = 0$. Q. E. D.

Lemma 3. *In M^m, a projective Killing tensor v of degree p satisfies*

$$(\delta dv)_{a_1\cdots a_p}+\{(m-p)/(m-p+2)\}(d\delta v)_{a_1\cdots a_p}$$

$$=\{(p+2)/2\}\Big[R_{a_1}{}^{e}v_{ea_2\cdots a_p}-\sum_{i=2}^{p}R^{d}{}_{a_1a_i}{}^{e}v_{da_2\cdots \underset{(i)}{e}\cdots a_p}\Big] \tag{1.3}$$
$$+(1/2)\Big[\sum_{i=1}^{p}R_{a_i}{}^{e}v_{a_1\cdots \underset{(i)}{e}\cdots a_p}-\sum_{\substack{1\le i,j\le p\\ i\ne j}}R^{d}{}_{a_ia_j}{}^{e}v_{a_1\cdots \underset{(j)}{e}\cdots \underset{(i)}{d}\cdots a_p}\Big].$$

In particular, if M^m is of constant curvature k, v satisfies

$$k(m-p)(p+1)v=\delta dv+\{(m-p)/(m-p+2)\}d\delta v. \tag{1.4}$$

(If $k\neq 0$, then (1.4) gives us a decomposition of v.)

Proof. It holds that

$$(\delta dv)_{a_1\cdots a_p}=-\nabla^b\nabla_b v_{a_1\cdots a_p}+\sum_{i=1}^{p}\nabla^b\nabla_{a_i}v_{a_1\cdots \underset{(i)}{b}\cdots a_p}. \tag{1.5}$$

From (1.2), we have

$$-\nabla^b\nabla_b v_{a_1\cdots a_p}=R_{a_1}{}^{e}v_{ea_2\cdots a_p}-\sum_{j=2}^{p}R^{d}{}_{a_1a_j}{}^{e}v_{da_2\cdots \underset{(j)}{e}\cdots a_p} \tag{1.6}$$
$$+\{2/(m-p+2)\}(d\delta v)_{a_1\cdots a_p}$$

and

$$\sum_{i=1}^{p}\nabla^b\nabla_{a_i}v_{a_1\cdots \underset{(i)}{b}\cdots a_p}$$
$$=(p/2)\Big[R_{a_1}{}^{e}v_{ea_2\cdots a_p}-\sum_{i=2}^{p}R^{d}{}_{a_1a_i}{}^{e}v_{da_2\cdots \underset{(i)}{e}\cdots a_p}\Big] \tag{1.7}$$
$$+(1/2)\Big[\sum_{i=1}^{p}R_{a_i}{}^{e}v_{a_1\cdots \underset{(i)}{e}\cdots a_p}-\sum_{\substack{1\le i,j\le p\\ i\ne j}}R^{d}{}_{a_ia_j}{}^{e}v_{a_1\cdots \underset{(j)}{e}\cdots \underset{(i)}{d}\cdots a_p}\Big]$$
$$-(d\delta v)_{a_1\cdots a_p}.$$

The desired result follows from (1.5), (1.6) and (1.7).

If M^m is of constant curvature k, (1.3) gives us (1.4). Q. E. D.

2. We deal with non-existence problem of v.

Theorem 4. *In a Riemannian manifold of constant curvature $k\neq 0$, there exists no closed affine Killing tensor of degree p ($<m$) other than the zero tensor.*

Proof. Let v be closed and affine. From Corollary 2, we can regard v as a projective Killing tensor satisfying $\nabla\delta v=0$. Substituting this and $dv=0$ into (1.4), we have $v=0$. Q. E. D.

Next, we treat the problem more generally. First, we see

$$\sum_{i=1}^{p} R_{a_i}{}^{e} v_{a_1\cdots \underset{(i)}{e}\cdots a_p} v^{a_1\cdots a_p} = pR_{de}v^{e}{}_{a_2\cdots a_p}v^{da_2\cdots a_p}, \tag{2.1}$$

$$\sum_{i=2}^{p} R^{d}{}_{a_1 a_i}{}^{e} v_{da_2\cdots \underset{(i)}{e}\cdots a_p} v^{a_1\cdots a_p} = -\{(p-1)/2\}R_{fgde}v^{fg}{}_{a_3\cdots a_p}v^{dea_3\cdots a_p}, \tag{2.2}$$

and

$$\sum_{\substack{1\le i,j\le p\\ i\ne j}} R^{d}{}_{a_i a_j}{}^{e} v_{a_1\cdots \underset{(j)}{e}\cdots \underset{(i)}{d}\cdots a_p} v^{a_1\cdots a_p} = p\sum_{i=2}^{p} R^{d}{}_{a_1 a_i}{}^{e} v_{da_2\cdots \underset{(i)}{e}\cdots a_p} v^{a_1\cdots a_p}. \tag{2.3}$$

If we transvect (1.3) with $v^{a_1\cdots a_p}$ and use the identities (2.1), (2.2) and (2.3), we get

$$\begin{aligned}&(\delta dv)_{a_1\cdots a_p}v^{a_1\cdots a_p}+\{(m-p)/(m-p+2)\}(d\delta v)_{a_1\cdots a_p}v^{a_1\cdots a_p}\\ &=(p+1)F(v,v)\end{aligned} \tag{2.4}$$

where we have put

$$\begin{aligned}F(v,v) &= R_{fd}v^{f}{}_{a_2\cdots a_p}v^{da_2\cdots a_p}\\ &\quad+\{(p-1)/2\}R_{fgde}v^{fg}{}_{a_3\cdots a_p}v^{dea_3\cdots a_p}.\end{aligned} \tag{2.5}$$

If M^m is compact and orientable, (2.4) gives us

$$p\,!(dv,dv)+p\,!\{(m-p)/(m-p+2)\}(\delta v,\delta v)=(p+1)\int_{M^m}F(v,v)^*1 \tag{2.6}$$

where (,) and ^{*1} denote the global scalar product and the volume element respectively.

Theorem 5. *In a compact orientable Riemannian manifold M^m, there exists no projective Killing tensor v of degree $p\ (<m)$ which satisfies $F(v,v)\leqq 0$, other than the parallel tensor.*

Especially, if $F(v,v)$ is negative definite, then there exists no projective Killing tensor of degree $p\ (<m)$ other than the zero tensor.

Proof. The latter part is obvious from (2.6). If $F(v,v)\leqq 0$, we have from (2.6) that $dv=0$ and $\delta v=0$. From Corollary 2, we see v is an affine Killing tensor. As was shown in [2], an affine Killing tensor in a compact orientable Riemannian manifold is a Killing tensor. Thus, v satisfies

$$(dv)_{a_1\dots a_{p+1}} = (p+1)\nabla_{a_1} v_{a_2\dots a_{p+1}}.$$

From $dv = 0$, we have $\nabla v = 0$. Q. E. D.

Corollary 6. *In a compact orientable Riemannian manifold M^m, there exists no Killing tensor of degree $p\,(< m)$ which satisfies $F(v, v) \leqq 0$, other than the parallel tensor.*

Especially, if $F(v, v)$ is negative definite, then there exists no Killing tensor of degree $p\,(< m)$ other than the zero tensor.

(In the case $p \leqq m/2$, this was proved by T. Kashiwada [1].)

We can rewrite the condition concerning $F(v, v)$, variously. For instance, if we substitute Weyl projective curvature tensor

$$W_{dcb}{}^{a} = R_{dcb}{}^{a} + \{1/(m-1)\}(R_{db}\delta_c^a - R_{cb}\delta_d^a)$$

into (2. 5), we have

(2. 7) $F(v, v) = [\{(p-1)/2\}\, W_{fgde} + \{(m-p)/(m-1)\}\, R_{fd}\, g_{gc}]\, v^{fg}{}_{a_3\dots a_p} v^{dca_3\dots a_p}.$

Now, assume that Ricci tensor is negative definite and let λ_0 be the least upper bound of the greatest eigen values on M^m of Ricci tensor. We then have

$$R_{fd} v^{f}{}_{a_2\dots a_p} v^{da_2\dots a_p} \leqq p!\ \lambda_0\ \langle v, v\rangle \qquad (\leqq 0),$$

where $\langle\, ,\, \rangle$ denotes the local scalar product. If we put

$$2W = \sup\{\,|W_{fgde}\xi^{fg}\xi^{de}|/\langle \xi, \xi\rangle\,;\ \text{for bivector } \xi\}$$

we then have

$$W_{fgde} v^{fga_3\dots a_p} v^{de}{}_{a_3\dots a_p} \leqq p!\ W\langle v, v\rangle.$$

Thus, from (2. 7) we have

$$F(v, v) \leqq [\{(p-1)/2\}\, W + \{(m-p)/(m-1)\}\,\lambda_0]\, p!\,\langle v, v\rangle.$$

Consequently, we have

Theorem 7. *In a compact orientable Riemannian manifold M^m with negative definite Ricci tensor, if*

(2. 8) $$\{(p-1)/2\}\, W + \{(m-p)/(m-1)\}\,\lambda_0 \leqq 0$$

for some $p\,(< m)$, then there exists no projective Killing tensor of degree

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p other than the parallel tensor.

Especially, if strict inequality holds in (2. 8), *there exists no projective Killing tensor of degree p ($< m$) other than the zero tensor.*

Corollary 8. (i) *If M^m is a compact orientable locally flat Riemannian manifold, there exists no projective Killing tensor of degree p ($< m$) other than the parallel tensor.*
(ii) *If M^m is a compact orientable Riemannian manifold of negative constant curvature, there exists no projective Killing tensor of degree p ($< m$) other than the zero tensor.*

Proof. (i) follows from Theorem 5. Since a projectively flat Riemannian manifold M^m $(m > 1)$ is of constant curvature, (ii) follows from Theorem 7. Q. E. D.

In the case $p \leqq m/2$, this corollary (ii) was proved by S. Tachibana [2]. This corollary (ii) can be obtained from (1. 4) directly.

3. We deal with an integrability condition for the system of the partial differential equations which defines an affine Killing tensor of degree p ($< m$).

Theorem 9. *A necessary and sufficient condition for a Riemannian manifold M^m $(m > 2)$ to be a space of locally flat is that for any point Q and any constant $C_{a_1\dots a_p}$ (skew symmetric in all indices) and $C_{a_0 a_1 \dots a_p}$ (skew symmetric in $a_1, \dots, a_{p-1}$ and a_p) there exists locally an affine Killing tensor $v_{a_1\dots a_p}$ of degree p ($< m$) satisfying $v_{a_1\dots a_p}(Q) = C_{a_1\dots a_p}$ and $(\nabla_{a_0} v_{a_1\dots a_p})(Q) = C_{a_0 a_1\dots a_p}$.*

Proof. To consider the equation, the left hand side of (1. 1) equal to zero, is equivalent to consider the system of the partial differential equations with unknown functions $v_{a_1\dots a_p}$ and $v_{b a_1\dots a_p}$ as follows ;

$$\nabla_b v_{a_1\dots a_p} = v_{b a_1 \dots a_p} \tag{3. 1}$$

$$\begin{aligned}\nabla_c v_{b a_1\dots a_p} = &-(1/2)\Big[\sum_{i=1}^{p} R_{c b a_i}{}^e \delta_{a_1}^{b_1} \dots \delta_{e}^{b_i} \dots \delta_{a_p}^{b_p} \\ &- (R_{b a_1 c}{}^e + R_{c a_1 b}{}^e)\, \delta_e^{b_1} \delta_{b_2}^{a_2} \dots \delta_{a_p}^{b_p} \\ &- \sum_{i=2}^{p} R_{b a_1 a_i}{}^e \delta_c^{b_1} \delta_{a_2}^{b_2} \dots \delta_e^{b_i} \dots \delta_{a_p}^{b_p} \\ &- \sum_{i=2}^{p} R_{c a_1 a_i}{}^e \delta_b^{b_1} \delta_{b_2}^{a_2} \dots \delta_e^{b_i} \dots \delta_{a_p}^{b_p}\Big] v_{b_1\dots b_p}\end{aligned} \tag{3. 2}$$

and

$$v_{a_1\cdots a_i\cdots a_j\cdots a_p} = -v_{a_1\cdots \underset{(i)}{a_j}\cdots \underset{(j)}{a_i}\cdots a_p} \quad (1 \leqq i < j \leqq p). \tag{3.3}$$

The necessary part is obvious.

For the system to be completely integrable, it is necessary that

$$\nabla_c v_{ba_1a_2a_3\cdots a_p} + \nabla_c v_{ba_2a_1a_3\cdots a_p} = 0 \tag{3.4}$$

and

$$\begin{aligned} &\nabla_d\nabla_c v_{ba_1\cdots a_p} - \nabla_c\nabla_d v_{ba_1\cdots a_p} \\ &= -\Big[\sum_{i=1}^{p} R_{dca_i}{}^e \delta_b^{b_0}\delta_{a_1}^{b_1}\cdots\delta_e^{b_i}\cdots\delta_{a_p}^{b_p} + R_{dcb}{}^e\delta_e^{b_0}\delta_{a_1}^{b_1}\cdots\delta_{a_p}^{b_p}\Big] v_{b_0b_1\cdots b_p}. \end{aligned} \tag{3.5}$$

It follows from (3. 4) and (3. 2) that

$$[B_{cba_1a_2a_3\cdots a_p}{}^{b_1\cdots b_p} + B_{cba_2a_1a_3\cdots a_p}{}^{b_1\cdots b_p}]\, v_{b_1\cdots b_p} = 0 \tag{3.6}$$

where we have put

$$\begin{aligned} B_{cba_1\cdots a_p}{}^{b_1\cdots b_p} = &\sum_{i=1}^{p} R_{cba_i}{}^e \delta_{a_1}^{b_1}\cdots\delta_e^{b_i}\cdots\delta_{a_p}^{b_p} \\ &- (R_{ba_1c}{}^e + R_{ca_1b}{}^e)\delta_e^{b_1}\delta_{a_2}^{b_2}\cdots\delta_{a_p}^{b_p} \\ &- \sum_{i=2}^{p} R_{ba_1a_i}{}^e\delta_c^{b_1}\delta_{a_2}^{b_2}\cdots\delta_e^{b_i}\cdots\delta_{a_p}^{b_p} \\ &- \sum_{i=2}^{p} R_{ca_1a_i}{}^e\delta_b^{b_1}\delta_{a_2}^{b_2}\cdots\delta_e^{b_i}\cdots\delta_{a_p}^{b_p}. \end{aligned}$$

Since $v_{b_1\cdots b_p}$ are arbitrary and (3. 3) holds, it follows from (3. 6) that

$$\sum_{\tau\in\mathfrak{S}_p} \varepsilon_\tau B_{cba_1a_2\cdots a_p}{}^{\tau(b_1)\cdots\tau(b_p)} + \sum_{\tau\in\mathfrak{S}_p} \varepsilon_\tau B_{cba_2a_1\cdots a_p}{}^{\tau(b_1)\cdots\tau(b_p)} = 0$$

where ε_τ denotes the sign of the permutation $\tau \in \mathfrak{S}_p$. If we put

$$\delta_{a_1\cdots a_p}^{b_1\cdots b_p} = \sum_{\tau\in\mathfrak{S}_p} \varepsilon_\tau \delta_{a_1}^{\tau(b_1)}\cdots\delta_{a_p}^{\tau(b_p)} = det(\delta_a^{b_j}),$$

we get from the above that

$$\begin{aligned} &(R_{ba_1c}{}^e + R_{ca_1b}{}^e)\delta_{e\,a_2\cdots a_p}^{b_1b_2\cdots b_p} + (R_{ba_2c}{}^e + R_{ca_2b}{}^e)\delta_{e\,a_1a_3\cdots a_p}^{b_1b_2b_3\cdots b_p} \\ &+ (R_{ba_1a_2}{}^e + R_{ba_2a_1}{}^e)\delta_{c\,e\,a_3\cdots a_p}^{b_1b_2b_3\cdots b_p} + \sum_{i=3}^{p} R_{ba_1a_i}{}^e\delta_{c\,a_2\cdots e\cdots a_p}^{b_1b_2\cdots b_i\cdots b_p} + \sum_{i=3}^{p} R_{ba_2a_i}{}^e\delta_{c\,a_1a_3\cdots e\cdots a_p}^{b_1b_2b_3\cdots b_i\cdots b_p} \\ &+ (R_{ca_1a_2}{}^e + R_{ca_2a_1}{}^e)\delta_{b\,e\,a_3\cdots a_p}^{b_1b_2b_3\cdots b_p} + \sum_{i=3}^{p} R_{ca_1a_i}{}^e\delta_{b\,a_2\cdots e\cdots a_p}^{b_1b_2\cdots b_i\cdots b_p} \\ &+ \sum_{i=3}^{p} R_{ca_2a_i}{}^e\delta_{b\,a_1a_3\cdots e\cdots a_p}^{b_1b_2b_3\cdots b_i\cdots b_p} = 0. \end{aligned}$$

Contracting this with respect to a_j and b_j $(2 \leqq j \leqq p)$, and taking account of the identity

$$\delta^{b_1\cdots b_k b_{k+1}\cdots b_p}_{a_1\cdots a_k b_{k+1}\cdots b_p} = \{(m-k)!/(m-p)!\}\,\delta^{b_1\cdots b_k}_{a_1\cdots a_k},$$

we can get $W_{a_1 bc}{}^{b_1} + W_{a_1 cb}{}^{b_1} = 0$. It follows from this that

$$W_{dcb}{}^{a} = 0. \tag{3.7}$$

On the other hand, from (3.5), (3.2) and (3.1) we have

$$\begin{aligned}
&[B_{cba_1\cdots a_p}{}^{b_1\cdots b_p}\delta^{b_0}_{d} - B_{dba_1\cdots a_p}{}^{b_1\cdots b_p}\delta^{b_0}_{c} \\
&- 2\sum_{i=1}^{p} R_{dca_i}{}^{e}\delta^{b_0}_{b}\delta^{b_1}_{a_1}\cdots\delta^{b_i}_{e}\cdots\delta^{b_p}_{a_p} - 2R_{dcb}{}^{e}\delta^{b_0}_{e}\delta^{b_1}_{a_1}\cdots\delta^{b_p}_{a_p}]\,v_{b_0 b_1\cdots b_p} \\
&+ [\nabla_d B_{cba_1\cdots a_p}{}^{b_1\cdots b_p} - \nabla_c B_{dba_1\cdots a_p}{}^{b_1\cdots b_p}]\,v_{b_1\cdots b_p} = 0.
\end{aligned}$$

From this and the assumption of the theorem, we have

$$\begin{aligned}
&\sum_{\tau\in\mathfrak{S}_p}\varepsilon_\tau B_{cba_1\cdots a_p}{}^{\tau(b_1)\cdots\tau(b_p)}\delta^{b_0}_{d} - \sum_{\tau\in\mathfrak{S}_p}\varepsilon_\tau B_{dba_1\cdots a_p}{}^{\tau(b_1)\cdots\tau(b_p)}\delta^{b_0}_{c} \\
&- 2\sum_{i=1}^{p} R_{dca_i}{}^{e}\delta^{b_0}_{b}\delta^{b_1\cdots b_i\cdots b_p}_{a_1\cdots e\cdots a_p} - 2R_{dcb}{}^{b_0}\delta^{b_1\cdots b_p}_{a_1\cdots a_p} = 0.
\end{aligned}$$

If we contract this with respect to b_0 and d, b_1 and c, and b_j and a_j $(2 \leqq j \leqq p)$, we can get $R_{ba_1} = 0$. Substituting this into (3.7), we have $R_{dcb}{}^{a} = 0$. Q. E. D.

Last of all, we see the corresponding problem on projective Killing tensor of degree p. If in this case we consider the system (3.1), (3.3) and

$$\begin{aligned}
\nabla_c v_{ba_1\cdots a_p} &= -(1/2)B_{cba_1\cdots a_p}{}^{b_1\cdots b_p}v_{b_1\cdots b_p} \\
&\quad + \{1/(m-p+2)\}\sum_{i=1}^{p}(-1)^{i-1}(g_{ca_i}\nabla_b v^{e}{}_{ea_1\cdots\hat{a}_i\cdots a_p} \\
&\quad + g_{ba_i}\nabla_c v^{e}{}_{ea_1\cdots\hat{a}_i\cdots a_p}),
\end{aligned}\tag{3.8}$$

we have the following

Theorem 10. *If there exists locally a projective Killing tensor $v_{a_1\cdots a_p}$ of degree p satisfying $v_{a_1\cdots a_p}(Q) = C_{a_1\cdots a_p}$ for any point Q of $M^m (m \geqq 3)$ and any skew symmetric constant $C_{a_1\cdots a_p}$, then M^m is a space of projectively flat.*

Proof. It follows from (3. 4) and (3. 8) by the same computation as in the proof of Theorem 9 that the equation (3. 6) holds. By the assumption, we then have (3. 7). Q. E. D.

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