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COMMUTATIVITY THEOREMS OF OUTCALT-YAQUB TYPE

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Throughout R will represent a ring. Let A be an additive subsemi-group of R , and N the set of all nilpotent elements in R . Given a non-empty subset S of R , we set $V_R(S) = \{x \in R \mid sx = xs \text{ for all } s \in S\}$ and $V_R^-(S) = \{x \in R \mid sx = -xs \text{ for all } s \in S\}$, and the subring generated by S will be denoted by $\langle S \rangle$.

In this paper, we consider the following conditions :

- A₁) For every $x \in R$, $x - x^2x' \in A$ with some $x' \in \langle x \rangle$.
- A₂) For every $x \in R$ and every positive integer n , $x - x^n x' \in A$ with some $x' \in \langle x \rangle$.
- B) $x - y \in A$ implies $x^q = y^q$ with some prime number $q = q(x, y)$ or $xy = yx$.
- B₀) $x - y \in A$ implies $x^2 = y^2$ or $xy = yx$.
- B₁) $x - y \in A$ and $y - z \in A$ imply $x^2 = z^2$ or $xy = yx$.
- B₂) $x - y \in A$ implies $x^2 = y^2$ or both $x, y \in V_R(A)$.
- B₃) Either R is commutative or $R = V_R^-(A)$ and $a^2 = 0$ for all $a \in A$.
- C₁) For every $x \in R$, $x - x^{n+1} \in N$ with some positive integer $n = n(x)$ or $x \in V_R(A)$.
- C₂) For every $x \in R$, $x^m = x^n$ with some distinct positive integers $m = m(x)$ and $n = n(x)$ or $x \in V_R(A)$.
- D) $x - y \in A$ implies $[[x, y], y] = 0$.

Remark. (a) The conditions A₁), B₂) and C₂) have been considered in [4]. Obviously, C₂) implies C₁).

(b) Let A' be the additive subgroup generated by A . If every element of A commutes with each other, then B₂) is equivalent to the same with A' instead of A .

(c) Let f be a ring homomorphism of R onto R^* . If R satisfies one of the conditions A₁)—D), then R^* does the same with $f(A)$ instead of A .

It is the purpose of this paper to prove the following commutativity theorems.

Theorem 1. *Suppose every element of A commutes with each other. If $A_1)$, $B)$, $C_1)$ and $D)$ are satisfied, then R is commutative.*

Theorem 2. *Let A be an additive subgroup of R . If $A_1)$ and $C_1)$ are satisfied, then $B_1) - B_3)$ are equivalent.*

We shall present also a corollary to Theorem 1, which will improve [4, Theorem 1]. Moreover, by [2, Lemma 3 (1) and (2)], the corollary includes [2, Lemma 4 (3)] and Theorem 2 deduces [2, Theorem 2].

In preparation for the proofs of our theorems, we establish the following lemmas.

Lemma 1. (a) $A_1)$ and $A_2)$ are equivalent. If $A_1)$ is satisfied, then N is included in A .

(b) In general, $B_3) \Rightarrow B_2) \Rightarrow B_1) \Rightarrow B_0) \Rightarrow B)$. If every element of A commutes with each other, then $B_0) - B_2)$ are equivalent.

(c) If $B)$ is satisfied, then for each $x \in R$ and $a \in A$ there exists a prime number q such that $[x^q, a] = 0$.

(d) If R contains 1, then $B_0)$ implies $D)$.

Proof. (a) is almost evident.

(b) The first assertion can be easily seen. Now, suppose that A is commutative and $B_0)$ is satisfied. Suppose $x - y \in A$ and $x^2 \neq y^2$. Then $xy = yx$. If $ax \neq xa$ with some $a \in A$, then $(x + a)x \neq x(x + a)$ implies $(x + a)^2 = x^2$. Since $(x - y)a = a(x - y)$, it follows that $ay \neq ya$, and therefore $(x + a)y \neq y(x + a)$. But then we have $y^2 = (x + a)^2 = x^2$. This contradiction means that $x, y \in V_R(A)$.

(c) Let $x \in R$, and $a \in A$. If $ax \neq xa$, then there exists a prime number q such that $(x + a)^q = x^q$. Hence, $x^q(x + a) = (x + a)^q(x + a) = (x + a)(x + a)^q = (x + a)x^q$, which simplifies to $x^qa = ax^q$.

(d) Suppose $x - y \in A$ and $xy \neq yx$. Then $x^2 = y^2$. Since $(x + 1) - (y + 1) \in A$, we have $(x + 1)^2 = (y + 1)^2$, whence it follows $2x = 2y$. Now, from those above, we readily obtain $[[x, y], y] = xy^2 + y^2x - 2yxy = 2x^3 - 2x^3 = 0$.

Lemma 2. *Suppose $A_1)$ and $B)$ are satisfied.*

(a) *Every idempotent of R is central.*

(b) *If every element of A commutes with each other, then N is an ideal of R .*

Proof. (a) Let e be an arbitrary idempotent of R . Then, by

Lemma 1 (a) and (c), $e \in V_R(A) \subseteq V_R(N)$. As is well known, the idempotent e commuting with all nilpotent elements is central.

(b) Let $x \in R$, and $a \in N$ with $a^m = 0$. By Lemma 1 (c), there exists a prime number q such that $(xa)^q a = a(xa)^q$. Then, one will easily see that $(xa)^{q^m} = \{(xa)^{q-1}x\}^m a^m = 0$, and similarly $(ax)^{q^m} = 0$. We have therefore seen that $Ra \subseteq N$ and $aR \subseteq N$. Since N is commutative as a subset of A (Lemma 1 (a)), N is evidently an ideal of R .

Lemma 3. *Let A be an additive subgroup of R . If B_1 is satisfied, then either $A \subseteq V_R(A)$ or $A \subseteq V_R^-(A)$ and $a^2 = 0$ for all $a \in A$.*

Proof. Suppose there exist $a, b \in A$ such that $ab \neq ba$. Since $a + b \equiv a \equiv 0 \pmod{A}$ and $(a+b)a \neq a(a+b)$, by B_1 we have $(a+b)^2 = a^2 = 0$, and similarly $b^2 = 0$. From these it follows $ab = -ba$. Now, by making use of Brauer's trick, we readily see that A is anti-commutative. If c is an arbitrary element of A commuting with all elements of A , then $ac = ca = -ac$. Hence, $c^2 = (a+c)^2 - 2ac - a^2 = 0$.

Proof of Theorem 1. Since R is a subdirect sum of subdirectly irreducible rings by Birkhoff's theorem, we may in view of Remark (c) assume that R is subdirectly irreducible. According to Herstein's theorem [1, Theorem 19], it is enough to prove that A is included in the center of R . Suppose on the contrary that there exist $a \in A$ and $x \in R$ such that $ax \neq xa$. Then $x \notin V_R(A)$, and hence $x \notin N$ (Lemma 1 (a)). We consider the reduced ring $\bar{R} = R/N$ (Lemma 2 (b)). By C_1 , $\bar{x} = \bar{x}^{n+1}$ for some positive integer n . Since $\bar{x}^n$ is a non-zero idempotent of $\bar{R}$ and R is subdirectly irreducible, any idempotent lifted from $\bar{x}^n$ must be 1 (see Lemma 2 (a)). Hence, x is a unit of R . We claim that $a(kx) \neq (kx)a$ for some $k > 1$. In fact, a cannot commute with both $2x$ and $3x$. Then, again by C_1 , $k\bar{x} = (k\bar{x})^{m+1}$ with some positive integer m . Here, without loss of generality, we may assume that $m=n$. Then $(k^{n+1} - k)\bar{x} = 0$, which means that the characteristic of R is non-zero. Recalling that R is subdirectly irreducible, we conclude that $p^a R = 0$ with some prime number p . By B , there exists a prime number q such that $(a+x)^q = x^q$. Then there holds $[x^q, a] = 0$ (see the proof of Lemma 1 (c)). Since $0 = [[a+x, x], x] = [[a, x], x]$ by D , an easy induction proves that $[x^h, a] = hx^{h-1}[x, a]$ for all positive integers h . In particular, $0 = [x^q, a] = qx^{q-1}[x, a]$, whence it follows $q[x, a] = 0$. Combining this with $p^a R = 0$, we conclude that $p = q$. Obviously, $\langle \bar{x} \rangle$ is a finite field of characteristic q : $\langle \bar{x} \rangle = \text{GF}(r)$, where $r = q^\beta$ and $\beta > 0$. Since $x - x^r \in N \subseteq A \subseteq V_R(A)$ (Lemma 1 (a)) and $x^r a = ax^r$, it follows at once $xa = ax$,

which is a contradiction.

Corollary 1. *Suppose every element of A commutes with each other. If A_1), B_0) and C_1) are satisfied, then R is commutative.*

Proof. By Lemma 1 (d), the proof proceeds in the same way as that of Theorem 1 did.

Proof of Theorem 2. In virtue of Lemma 1 (b), it remains only to prove that B_1) implies B_3). By Lemma 3, either $A \subseteq V_R(A)$ or $A \subseteq V_R^-(A)$ and $a^2 = 0$ for all $a \in A$. If $A \subseteq V_R(A)$ then R is commutative by Corollary 1. Henceforth, we assume that $A \not\subseteq V_R(A)$ but $A \subseteq V_R^-(A)$. Let $x \in R$ and $b \in A$ such that $xb \neq bx$. Then $(x + b)^2 = x^2$ by B_1). Hence we get $xb = -bx$. Now, by making use of Brauer's trick, we can easily see that $R = V_R(A)$ or $V_R^-(A)$. Since $A \not\subseteq V_R(A)$, R must be $V_R^-(A)$.

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Added in proof. Obviously, the condition D) is satisfied if and only if $[[x, y], y] = 0$ for all $x \in A$ and $y \in R$. The principal theorem of [3] is still valid under the weaker hypothesis that $[[x, y], y] = 0$ for all $x \in N$ and $y \in R$.