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NOTE ON H-SEPARABLE EXTENSION

Dedicated to Professor Mutsuo Takahashi on his 60th birthday

TAICHI NAKAMOTO

Throughout the present note, A/B will represent a ring extension with common identity 1, and V the centralizer $V_A(B)$ of B in A . Let N be a unital right B -module, $B' = \text{End}(N_B)$, and $A' = \text{End}(N \otimes_B A_A)$. Further, we consider the ring homomorphism $\theta: B' \longrightarrow A'$ defined by $\theta(b')(n \otimes a) = b'(n) \otimes a$. For any $v \in V$, v' will represent an element of $V_{A'}(\theta(B'))$ defined by $v'(n \otimes a) = n \otimes va$.

Following K. Hirata ([1] and [2]), A/B is said to be centrally projective if A is B - B -isomorphic to a B - B -direct summand of a finite direct sum of copies of A , and A/B H -separable if $A \otimes_B A$ is A - A -isomorphic to an A - A -direct summand of a finite direct sum of copies of A . To be easily seen, A/B is centrally projective if and only if there exist some $w_k \in V$ and $f_k \in \text{Hom}({}_B A_B, {}_B B_B)$ ($k = 1, \dots, p$) such that $\sum_k f_k(a)w_k = a$ for all $a \in A$. Such a system $\{w_k; f_k\}_k$ will be called a centrally projective coordinate system for A/B . If B is a left B -direct summand of A , especially if A/B is centrally projective (cf. [1, (5.6)]), then the ring homomorphism θ is a monomorphism, and A'/B' may be regarded as a ring extension via θ . On the other hand, A/B is H -separable if and only if there exist some $v_i \in V$ and $\sum_j x_{ij} \otimes y_{ij} \in (A \otimes_B A)^4$ ($i = 1, \dots, q$) such that $\sum_{i,j} x_{ij} \otimes y_{ij} v_i = 1 \otimes 1$. Such a system $\{v_i; \sum_j x_{ij} \otimes y_{ij}\}_i$ will be called an H -system for A/B (cf. [3]).

Theorem. *If A/B is a centrally projective H -separable extension, then so is A'/B' .*

In advance of the proof of our theorem, we state a couple of lemmas.

Lemma 1. *Let $\{v_i; \sum_j x_{ij} \otimes y_{ij}\}_i$ be an H -system for A/B . Then there holds the following:*

$$1) \sum_j n f(a x_{ij}) \otimes y_{ij} = \sum_j n f(x_{ij}) \otimes y_{ij} a,$$

$$2) \sum_{i,j} n f(a_1 v_i a_2 x_{ij}) \otimes y_{ij} = n f(a_1) \otimes a_2,$$

where $f \in \text{Hom}({}_B A_B, {}_B B_B)$, $n \in N$, and $a, a_1, a_2 \in A$.

Proof. Easy by the definition of an H -system.

Lemma 2. *Let $\{w_k; f_k\}_k$ be a centrally projective coordinate system for A/B , and $f'_k: A' \longrightarrow B'$ the B' - B' -homomorphisms defined by*

$f'_k(a')(n) = (1 \otimes f_k)(a'(n \otimes 1))$. Then $\{w'_k; f'_k\}_k$ is a centrally projective coordinate system for A'/B' .

Proof. Noting that $\sum_k (1 \otimes f_k)(n \otimes a) \otimes w_k = n \otimes \sum_k f_k(a)w_k = n \otimes a$, we obtain

$$\begin{aligned} (\sum_k f'_k(a')w'_k)(n \otimes a) &= \sum_k \theta(f'_k(a'))(n \otimes w_k a) \\ &= \sum_k (1 \otimes f_k)(a'(n \otimes 1)) \otimes w_k a \\ &= a'(n \otimes 1)a \\ &= a'(n \otimes a), \end{aligned}$$

which means that $\{w'_k; f'_k\}_k$ is a centrally projective coordinate system for A'/B' .

Proof of Theorem. Let $\{w_k; f_k\}_k$ be a centrally projective coordinate system for A/B , and $\{v_i; \sum_j x_{ij} \otimes y_{ij}\}_i$ an H -system for A/B . Let $f'_k \in \text{Hom}_{(B', A'_{B'}, B'B'_{B'})}$ be given as in Lemma 2. Moreover, we define $a'_{ik} \in A'$ by $a'_{ik}(n \otimes a) = \sum_j n f_k(ax_{ij}) \otimes y_{ij} = \sum_j n f_k(x_{ij}) \otimes y_{ij} a$ (cf. Lemma 1). Then, $\sum_i a'_{ik} v'_i(n \otimes a) = \sum_i a'_{ik}(n \otimes v_i a) = \sum_{i,j} n f_k(v_i a x_{ij}) \otimes y_{ij} = n f_k(1) \otimes a$, which means

$$(1) \quad \sum_i a'_{ik} v'_i = \theta(f'_k(1)).$$

Next, we claim that

$$(2) \quad \sum_k f'_i(a''w'_k) a'_{ik} a' = a'_{ii} a' a'' \quad (a', a'' \in A').$$

In fact, if $a'(n \otimes 1) = \sum_r n_r \otimes a_r$ and $a''(n_r \otimes 1) = \sum_s n_{rs} \otimes a_{rs}$, then

$$\begin{aligned} (\sum_k f'_i(a''w'_k) a'_{ik} a') (n \otimes a') &= \sum_k f'_i(a''w'_k) a'_{ik} (\sum_r n_r \otimes a_r a') \\ &= \sum_{k,r} \theta(f'_i(a''w'_k)) (n_r \otimes \sum_j f_k(a_r a x_{ij}) y_{ij}) \\ &= \sum_{k,r} f'_i(a''w'_k) (n_r) \otimes \sum_j f_k(a_r a x_{ij}) y_{ij} \\ &= \sum_{k,r,s} n_{rs} f_i(a_{rs} w_k) \otimes \sum_j f_k(a_r a x_{ij}) y_{ij} \\ &= \sum_{r,s} n_{rs} \otimes \sum_{j,k} f_i(a_{rs} w_k f_k(a_r a x_{ij})) y_{ij} \\ &= \sum_{r,s} n_{rs} \otimes \sum_j f_i(a_{rs} a_r a x_{ij}) y_{ij} \\ &= a'_{ii} (\sum_{r,s} n_{rs} \otimes a_{rs} a_r a) \\ &= a'_{ii} a'' a' (n \otimes a). \end{aligned}$$

Now, let $f' = \sum_k f'_k: {}_{B'}A'_{B'} \longrightarrow {}_{B'}(B' \oplus \cdots \oplus B')_{B'}$. Since $\{w'_k; f'_k\}_k$ is a centrally projective coordinate system for A'/B' (Lemma 2), $f' \otimes 1 = \sum_k f'_k \otimes 1: {}_{B'}A' \otimes {}_{B'}A'_{A'} \longrightarrow {}_{B'}(A' \oplus \cdots \oplus A')_{A'}$ is a monomorphism. By (2), for any $a' \in A'$ we have $\sum_k f'_i(a'w'_k) a'_{ik} = a'_{ii} a' = \sum_k f'_i(w'_k) a'_{ik} a'$, which means

$$(f'_i \otimes 1) (\sum_k a' w'_k \otimes a'_{ik}) = (f'_i \otimes 1) (\sum_k w'_k \otimes a'_{ik} a').$$

Hence, it follows $\sum_k a' w'_k \otimes a'_{ik} = \sum_k w'_k \otimes a'_{ik} a'$, namely

$$\sum_k w'_k \otimes a'_{ik} \in (A' \otimes_{B'} A')^{A'}.$$

Finally, by (1) and (2) we have $\sum_{i,k} f'_i(w'_k) a'_{ik} v'_i = \sum_i a'_{ii} v'_i = \theta(f'_i(1)) = f'_i(1) \cdot 1$. Recalling again that $f' \otimes 1$ is a monomorphism, we obtain eventually

$$\sum_{i,k} w'_k \otimes a'_{ik} v'_i = 1 \otimes 1,$$

which means that $\{v'_i; \sum_k w'_k \otimes a'_{ik}\}_i$ is an H -system for A'/B' .

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