Mathematical Journal of Okayama University

Volume 18, *Issue* 1 1975 *Article* 1

DECEMBER 1975

On Hardy's exponential series related to the divisor problem

Takeshi kano*

*Okayama University

ON HARDY'S EXPONENTIAL SERIES RELATED TO THE DIVISOR PROBLEM

Dedicated to Professor Tikao Tatuzawa on his 60th birthday

TAKESHI KANO

1. Introduction

It was G. H. Hardy [5] who investigated the divisor problem of Dirichlet in a systematic way by complex analysis depending on a variant of a theorem of M. Riesz concerning Dirichlet series. Defining

(1) $$D(x) = \sum_{n \leq x} d(n) = x \log x + (2\gamma - 1)x + \Delta(x),$$ [1]

he proved that

$$\Delta(x) \neq o(x^{\frac{1}{4}}) \qquad (x \to \infty)$$

and derived the identity

(2) $$\Delta(x) = \frac{1}{4} + \sqrt{x} \sum_{n=1}^{\infty} \frac{d(n)}{\sqrt{n}} \{H_1(4\pi\sqrt{xn}) - Y_1(4\pi\sqrt{xn})\},$$

which was fiirst found by G. F. Voronoï [13] where $Y_1(x)$ is the Bessel function of order 1 of the second kind and

$$H_1(x) = \frac{2}{\pi}\int_1^{\infty} \frac{t}{\sqrt{t^2-1}} e^{-xt}\, dt$$

is one of Hankel's cylinder functions.

Hardy's argument of the proof is based on the following result obtained by the complex function theory : If we put

(3) $$S_N = \sum_{n=1}^{N} \frac{d(n)}{\sqrt{n}} e^{-it\sqrt{n}}, \qquad (t > 0)$$

then we have as $N \to \infty$

(4) $$S_N = o(N^{\varepsilon}),$$

or

(5) $$S_N = \frac{2(1+i)\, d(q)}{q^{\frac{1}{4}}} N^{\frac{1}{4}} + o(N^{\varepsilon}),$$

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1) γ is Euler's constant. The longstanding hitherto unproved conjecture is $\Delta(x) = O(x^{\frac{1}{4}+\varepsilon})$ for any $\varepsilon > 0$.

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for any $\varepsilon > 0$ accordingly as $t \neq 4\pi\sqrt{q}$ or $t = 4\pi\sqrt{q}$ $(q \in N)$, where the constants implied by the o's may depend on the values of t. Thus it follows that

$$\sum_{n=1}^{\infty} \frac{d(n)}{n^{\delta}} e^{-it\sqrt{n}} \tag{6}$$

is convergent for $\delta > \frac{1}{2}$ or $\delta > \frac{3}{4}$, accordingly.

Recently the writer found a simple and elementary way[2] of improving (4) to

$$S_N = O(\log N) \qquad (N \to \infty), \tag{7}$$

which enabled him to prove elementarily that the series

$$\sum_{n=2}^{\infty} \frac{d(n)}{\sqrt{n}\,(\log n)^{\alpha}} \cos(4\pi\sqrt{nx}) \tag{8}$$

is convergent for all $x \notin N$ if $\alpha > 1$ and divergent if $\alpha \leq 1$. The result has been generalized and refined by S. Uchiyama [11, 12] who in particular improved the remainder term of (5). K. Chandrasekharan and R. Narashimhan [1, 2] adopted a method from the equiconvergence theorem of Fourier series due to A. Zygmund to prove the convergence and Riesz summability of the series

$$\sum_{n=1}^{\infty} \frac{d(n)}{n^{\alpha}} J_{\beta}(4\pi\sqrt{nx}) \tag{9}$$

and

$$\sum_{n=1}^{\infty} \frac{r(n)}{n^{\alpha}} J_{\beta}(2\pi\sqrt{nx}) \text{ [3]}. \tag{10}$$

However, it seems impossible to prove, along their lines, the result for (8) or even the divergence of (9) and (10) when $\alpha = \frac{1}{4}$. On the other hand, our argument is elementary and successful for convergence problem of (8), while we can say nothing about Riesz summability of (9) and (10) when $\alpha = \frac{1}{4}$, so in this paper we shall show another simple way to answer the questions.

2) It should be noted, however, that the underlying idea thereof had substantially been proposed by Ju. V. Linnik in 1952.

3) $r(n)$ denotes as usual the number of integer solutions in a, b of $a^2+b^2=n$.

2. It was proved by J. R. Wilton [14] and also by A. L. Dixon and W. L. Ferrar [4] that (10) with $\alpha=\frac{1}{4}$ oscilates finitely for all $x \notin N$ while it is summable (C, ε) for all $x \notin N$ and any $\varepsilon>0$. In this section we shall show that (9) with $\alpha=\frac{1}{4}$ oscilates infinitely for all $x \notin N$, and it will be proved in the next section that it is summable (C, ε) for all $x \notin N$ and any $\varepsilon>0$. From the asymptotic formulae of $J_\beta(x)$, we see that the summability problem of (9) with $\alpha=\frac{1}{4}$ is reduced to that of

$$\sum_{n=1}^{\infty}\frac{d(n)}{\sqrt{n}}\begin{matrix}\cos\\ \sin\end{matrix}(4\pi\sqrt{nx}) \,. \tag{11}$$

We then obtain the following theorem.

Theorem 1.

$$\begin{aligned}\sum_{n=1}^{N}\frac{d(n)}{\sqrt{n}}\cos(x\sqrt{n}) &= \frac{2}{x}\log N\cdot\sin(x\sqrt{N}) \\ &+\left(\frac{4\gamma}{x}+\frac{a}{2}x\right)\sin(x\sqrt{N})+(2\gamma-1)\cos x-\frac{4\gamma}{x}\sin x \\ &+A_N(x)\,,\end{aligned} \tag{12}$$

where $A_N(x)$ *denotes a certain function of* N *and* x *which converges, as* $N\to\infty$, *uniformly in* x *over any finite positive interval free from the points* $4\pi\sqrt{q}$ $(q\in N)$.

Our method of the proof differs from that of Wilton or of Dixon and Ferrar, and is based on the following representation [3, 13], which is easy to handle because of the absolute convergence of the series on the right-hand side and is less difficult to obtain the direct one for $\Delta(t)$ [9];

$$\begin{aligned}G(t)=\int_0^t\Delta(u)\,du &= at+b+\frac{t^{\frac{3}{4}}}{2\sqrt{2}\pi^2}\sum_{n=1}^{\infty}\frac{d(n)}{n^{\frac{5}{4}}}\sin\left(4\pi\sqrt{nt}-\frac{\pi}{4}\right) \\ &+\frac{15\,t^{\frac{1}{4}}}{2^6\sqrt{2}\pi^3}\sum_{n=1}^{\infty}\frac{d(n)}{n^{\frac{7}{4}}}\cos\left(4\pi\sqrt{nt}-\frac{\pi}{4}\right)+O\left(t^{-\frac{1}{4}}\right).\end{aligned} \tag{13}$$

Proof of Theorem 1. Let us put

$$C_N = \sum_{n=1}^{N} \frac{d(n)}{\sqrt{n}} \cos(x\sqrt{n}).$$

Then, by a known summation formula ([10] S. 371), we have

$$C_N = \frac{D(N)}{\sqrt{N}} \cos(x\sqrt{N}) - \int_1^N D(t)\, g'(t)\, dt,$$

where

$$D(t) = \sum_{n\le t} d(n), \quad g(t) = \frac{\cos(x\sqrt{t})}{\sqrt{t}}.$$

On observing that

$$g'(t) = -\frac{1}{2} t^{-\frac{3}{2}} \cos(x\sqrt{t}) - \frac{x}{2} \cdot \frac{\sin(x\sqrt{t})}{t},$$

we obtain

$$(14)\quad C_N = \frac{D(N)}{\sqrt{N}} \cos(x\sqrt{N}) + \frac{1}{2}\int_1^N \frac{D(t)}{t^{\frac{3}{2}}} \cos(x\sqrt{t})dt + \frac{x}{2}\int_1^N \frac{D(t)}{t} \sin(x\sqrt{t})dt.$$

Thus it follows from formula (1) that

$$\begin{aligned} C_N &= \left(\sqrt{N}\log N + (2\gamma - 1)\sqrt{N} + \frac{\Delta(N)}{\sqrt{N}}\right)\cos(x\sqrt{N}) \\ &+ \frac{1}{2}\int_1^N \frac{\log t}{\sqrt{t}} \cos(x\sqrt{t})dt + \frac{1}{2}(2\gamma - 1)\int_1^N \frac{\cos(x\sqrt{t})}{\sqrt{t}} dt \\ (15)\qquad &+ \frac{1}{2}\int_1^N \frac{\Delta(t)}{t^{\frac{3}{2}}} \cos(x\sqrt{t})\, dt + \frac{x}{2}\int_1^N (\log t)\sin(x\sqrt{t})\, dt \\ &+ \frac{x}{2}(2\gamma - 1)\int_1^N \sin(x\sqrt{t})\, dt + \frac{x}{2}\int_1^N \frac{\Delta(t)}{t} \sin(x\sqrt{t})\, dt \\ &= \left(\sqrt{N}\log N + (2\gamma - 1)\sqrt{N} + \frac{\Delta(N)}{\sqrt{N}}\right)\cos(x\sqrt{N}) + \sum_{j=1}^{6} I_j, \end{aligned}$$

say. We shall henceforth consider each of $I_1, I_2, \cdots\cdots, I_6$.

I_1 and I_2: A simple calculation shows that

$$(16)\quad \begin{aligned} I_1 &= \frac{1}{x}\sin(x\sqrt{N})\log N - \frac{2}{x}\int_1^{\sqrt{N}} \frac{\sin(xu)}{u} du, \\ I_2 &= \frac{2\gamma - 1}{x}\{\sin(x\sqrt{N}) - \sin x\}. \end{aligned}$$

I_4 and I_5: Also simply we have

$$(17)\quad \begin{aligned} I_4 = &- \sqrt{N}\log N \cdot \cos(x\sqrt{N}) + \frac{1}{x}\sin(x\sqrt{N}) \cdot \log N \\ &+ \frac{2}{x}\{\sin(x\sqrt{N}) - \sin x\} - \frac{2}{x}\int_1^{\sqrt{N}} \frac{\sin(xu)}{u}\,du, \end{aligned}$$

$$I_5 = (2\gamma - 1)\left\{-\sqrt{N}\cos(x\sqrt{N}) + \cos x + \frac{\sin(x\sqrt{N})}{x} - \frac{\sin x}{x}\right\}.$$

I_3: Since we know that for some $\delta > 0$

$$(18)\quad \Delta(t) = O(t^{\frac{1}{2}-\delta})^{4)} \qquad (t \to \infty),$$

we have

$$\int_1^N \frac{|\Delta(t)|}{t^{\frac{3}{2}}} |\cos(x\sqrt{t})|\,dt = O\left(\int_1^N \frac{dt}{t^{1+\delta}}\right) = O(1).$$

So, I_3 converges absolutely and uniformly in x as $N \to \infty$.
It remains only to consider I_6. By partial integration we have

$$(19)\quad \begin{aligned} I_6 = \frac{x}{2}\int_1^N G'(t)\frac{\sin(x\sqrt{t})}{t}\,dt &= \frac{x}{2}\left[\frac{G(N)}{N}\sin(x\sqrt{N}) - G(1)\sin x\right] \\ &+ \frac{x}{2}\int_1^N \frac{G(t)}{t^2}\sin(x\sqrt{t})\,dt - \left(\frac{x}{2}\right)^2\int_1^N \frac{G(t)}{t^{\frac{3}{2}}}\cos(x\sqrt{t})\,dt. \end{aligned}$$

Since we know from formula (13) that

$$G(t) = at + O(t^{\frac{3}{4}}) \qquad (t \to \infty),$$

we have

$$(20)\quad \begin{aligned} \frac{x}{2}\frac{G(N)}{N}\sin(x\sqrt{N}) &= \frac{x}{2}(a + O(N^{-\frac{1}{4}})\sin(x\sqrt{N}) \\ &= \frac{a}{2}x\sin(x\sqrt{N}) + O(xN^{-\frac{1}{4}}), \end{aligned}$$

and

4) It is known that e. g. $\Delta(t) = O(t^{\frac{1}{3}})$, however, we need no such a deeper estimate. In fact even (18) is unnecessary if we argue as in the evaluation of I_6.

$$\frac{x}{2}\int_1^N \frac{G(t)}{t^2}\sin(x\sqrt{t})\,dt = \frac{x}{2}\int_1^N\left(\frac{a}{t}+O(t^{-\frac{5}{4}})\right)\sin(x\sqrt{t})\,dt$$

$$(21)\qquad = \frac{ax}{2}\int_1^N \frac{\sin(x\sqrt{t})}{t}\,dt + O\left(x\int_1^N t^{-\frac{5}{4}}\,dt\right)$$

$$= ax\int_1^N \frac{\sin(xu)}{u}\,du + O(x),$$

the last integral being convergent, as $N\to\infty$, uniformly in x over any finite positive interval. Now we are going to consider the integral

$$I_N(x) = I_N = \int_1^N \frac{G(t)}{t^{\frac{3}{2}}}\cos(x\sqrt{t})\,dt.$$

Owing to (13) we obtain

$$I_N = a\int_1^N \frac{\cos(x\sqrt{t})}{\sqrt{t}}\,dt + b\int_1^N \frac{\cos(x\sqrt{t})}{t^{\frac{3}{2}}}\,dt$$

$$+\frac{1}{2\sqrt{2}\pi^2}\int_1^N\left\{t^{-\frac{3}{4}}\sum_{n=1}^{\infty}\frac{d(n)}{n^{\frac{5}{4}}}\sin\left(4\pi\sqrt{nt}-\frac{\pi}{4}\right)\right\}\cos(x\sqrt{t})dt$$

$$(22)$$

$$+\frac{1}{2^6\sqrt{2}\pi^3}\int_1^N\left\{t^{-\frac{5}{4}}\sum_{n=1}^{\infty}\frac{d(n)}{n^{\frac{7}{4}}}\cos\left(4\pi\sqrt{nt}-\frac{\pi}{4}\right)\right\}\cos(x\sqrt{t})\,dt$$

$$+O\left(\int_1^N t^{-\frac{7}{4}}\,dt\right).$$

Here we easily find that

$$\int_1^N \frac{\cos(x\sqrt{t})}{\sqrt{t}}\,dt = \frac{2}{x}\{\sin(x\sqrt{N})-\sin x\},$$

and that the second and the fourth integrals are both uniformly absolutely convergent as $N\to\infty$. So our final task is to examine the third integral

$$J_N = \int_1^N\left\{t^{-\frac{3}{4}}\sum_{n=1}^{\infty}\frac{d(n)}{n^{\frac{5}{4}}}\sin\left(4\pi\sqrt{nt}-\frac{\pi}{4}\right)\right\}\cos(x\sqrt{t})\,dt.$$

On integrating termwise we get[5)]

$$J_N = \sum_{n=1}^{\infty}\left\{\frac{d(n)}{n^{\frac{5}{4}}}\int_1^N t^{-\frac{3}{4}}\sin\left(4\pi\sqrt{nt}-\frac{\pi}{4}\right)\cos(x\sqrt{t})\,dt\right\}.$$

5) This is permissible from the uniform convergence since $d(n)=O(n^{\varepsilon})$ for any $\varepsilon>0$.

Let us set

$$K_N=\int_1^N t^{-\frac{3}{4}}\sin\left(4\pi\sqrt{nt}-\frac{\pi}{4}\right)\cos(x\sqrt{t})\,dt$$

$$=\frac{1}{2}\int_1^N t^{-\frac{3}{4}}\left\{\sin\left(4\pi\sqrt{nt}+xt-\frac{\pi}{4}\right)+\sin\left(4\pi\sqrt{nt}-xt-\frac{\pi}{4}\right)\right\}dt,$$

and consider

$$L_N=\frac{1}{2}\int_1^N t^{-\frac{3}{4}}\sin\left(4\pi\sqrt{nt}\pm x\sqrt{t}-\frac{\pi}{4}\right)dt.$$

Making the substitution $\sqrt{t}=u$, we have

$$L_N=\int_1^{\sqrt{N}} u^{-\frac{1}{2}}\sin\left(4\pi\sqrt{n}\,u\pm xu-\frac{\pi}{4}\right)du.$$

Now let us look into the convergence of L_N as $N\to\infty$. By the second mean value theorem,

$$\int_M^{\sqrt{N}} u^{-\frac{1}{2}}\sin\left((4\pi\sqrt{n}\pm x)u-\frac{\pi}{4}\right)du$$

$$(23)\qquad =M^{-\frac{1}{2}}\int_M^{\eta}\sin\left((4\pi\sqrt{n}\pm x)u-\frac{\pi}{4}\right)du \quad (\text{for some } \eta\in[M,\ \sqrt{N}])$$

$$=O\left(M^{-\frac{1}{2}}\frac{1}{|4\pi\sqrt{n}\pm x|}\right).$$

Here we have

$$(24)\qquad |4\pi\sqrt{n}-x|=\begin{cases} x-4\pi\sqrt{n}>4\pi\left\{\left(\frac{x}{4\pi}\right)-\sqrt{\left[\left(\frac{x}{4\pi}\right)^2\right]}\right\}>0, \\ \qquad (x>4\pi\sqrt{n}) \\ 4\pi\sqrt{n}-x>4\pi\left\{\left[\left(\frac{x}{4\pi}\right)^2\right]+1\right\}^{\frac{1}{2}}-x>0. \\ \qquad (x<4\pi\sqrt{n}) \end{cases}$$

Moreover it is clear that each of the functions

$$\sqrt{t}-\sqrt{[t]},\quad \sqrt{[t]+1}-\sqrt{t}$$

is positive and continuous in any integer-free finite interval $[a,\ b]$, and so attains positive minimum there. Hence, on account of this fact together with (23) and (24), we conclude that both

$$\lim_{N\to\infty} L_N \quad \text{and} \quad \lim_{N\to\infty} K_N$$

converge uniformly in x over any finite positive interval free from the points $4\pi\sqrt{q}$ $(q \in N)$. Therefore it follows that so do also both

$$\lim_{N\to\infty} J_N \quad \text{and} \quad \lim_{N\to\infty} I_N .$$

Thus we have arrived at the formula

$$\begin{aligned} C_N = {} & \frac{2}{x}\log N\cdot\sin(x\sqrt{N}) + \left(\frac{4\gamma}{x}+\frac{ax}{2}\right)\sin(x\sqrt{N}) \\ & + (2\gamma-1)\cos x - \frac{4\gamma}{x}\sin x + A_N(x), \end{aligned} \tag{25}$$

where

$$\begin{aligned} A_N(x) = {} & I_3 - \frac{x^2}{4} I_N(x) + \left(ax - \frac{4}{x}\right)\int_1^{\sqrt{N}} \frac{\sin(xu)}{u}\,du \\ & + O(x) + O\left(xN^{-\frac{1}{4}}\right) + o(1) \end{aligned} \tag{26}$$

converges, as $N\to\infty$, uniformly in x over any finite positive interval free from the points $4\pi\sqrt{q}$ $(q \in N)$. This proves Theorem 1.

We can give in this manner a quite similar representation like (12) to the sine series also. Hence we know that (9) with $\alpha = \frac{1}{4}$ oscilates infinitely for all $x \notin N$[6].

3. In this section we prove that (11) is summable (C, ε) for all $x \notin N$ and any $\varepsilon > 0$. Since the situation is quite the same for the sine series, we only consider the cosine series

$$\sum_{n=1}^{\infty} \frac{d(n)}{\sqrt{n}} \cos(4\pi\sqrt{nx}).$$

Then it will suffice, after (25) and (26), to prove that

$$\log N\cdot\sin(4\pi\sqrt{xN}) \tag{27}$$

is summable (C, ε) for all positive x and ε[7].

6) Such representations for sine and cosine series contain no cancellation terms in (9).

7) $\sin(4\pi\sqrt{xN})$ is also (C, ε) summable for all positive x and ε. Since it is easily shown to be $(C, 1)$ summable, it follows from the boundedness that it is (C, ε) summable for any $\varepsilon > 0$.

For the purpose we find it convenient to take the following Riesz typical means of integral form of order $k>0$ [8];

(28) $$I_k(\omega)=\frac{k}{\omega^k}\int_1^{\omega}(\omega-t)^{k-1}S(t)\,dt$$

with

(29) $$S(t)=\sum_{n<t}\{(\log n)\sin(4\pi\sqrt{xn})-(\log(n-1))\sin(4\pi\sqrt{x(n-1)})\}.$$

Noticing that

$$\log t\cdot\sin(4\pi\sqrt{xt})-\log[t]\cdot\sin(4\pi\sqrt{x[t]})=O\left(\sqrt{x}\,\frac{\log t}{\sqrt{t}}\right)$$

as $t\to\infty$, we have for sufficiently large ω,

$$\begin{aligned}I_k(\omega)&=\frac{k}{\omega^k}\left(\int_1^T+\int_T^{\omega}\right)(\omega-t)^{k-1}S(t)\,dt\\&=\frac{k}{\omega^k}\int_1^T(\omega-t)^{k-1}S(t)dt+\frac{k}{\omega^k}\int_T^{\omega}(\omega-t)^{k-1}\log t\cdot\sin(4\pi\sqrt{xt})\,dt\\&\quad+O\left(\frac{k}{\omega^k}\int_T^{\omega}(\omega-t)^{k-1}\frac{\sqrt{x}\log t}{\sqrt{t}}\,dt\right)\\&=\frac{k}{\omega^k}\int_T^{\omega}(\omega-t)^{k-1}\cdot\log t\cdot\sin(4\pi\sqrt{xt})\,dt\\&\quad+O\left(\frac{k}{\omega^k}\int_1^T(\omega-t)^{k-1}\log t\,dt\right)+O\left(\frac{k\sqrt{x}}{\omega^k}\int_T^{\omega}(\omega-t)^{k-1}\frac{\log t}{\sqrt{t}}\,dt\right).\end{aligned}$$

The last two O-terms are

$$O\left(\frac{k}{\omega^k}\int_1^T(\omega-t)^{k-1}\log T\,dt\right)=O\left(\frac{k}{\omega^k}T^k\log T\right)$$

and

$$O\left(\frac{k}{\omega^k}\frac{\log T}{\sqrt{T}}\int_T^{\omega}(\omega-t)^{k-1}\,dt\right)=O\left(\frac{\sqrt{x}}{\omega^k}\frac{\log T}{\sqrt{T}}(\omega-T)^k\right)$$

respectively, if $\omega\geqq 2T$ and T is sufficiently large.
So, if we here specify that $T=\sqrt{\omega}$, we have

(30) $$\begin{aligned}I_k(\omega)&=\frac{k}{\omega^k}\int_{\sqrt{\omega}}^{\omega}(\omega-t)^{k-1}\cdot\log t\cdot\sin(4\pi\sqrt{xt})\,dt\\&\quad+O\left(k\omega^{-\frac{k}{2}}\log\omega\right)+O\left(\sqrt{x}\,\omega^{-\frac{1}{4}}\log\omega\right).\end{aligned}$$

8) It is known that the Riesz typical means is epuivalent to (C,k) means when $k>0$ [6, 8].

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On making the substitution $t=\omega u^2$, we obtain

$$J_k(\omega)=\frac{k}{\omega^k}\int_{\sqrt{\omega}}^{\omega}(\omega-t)^{k-1}\log t\cdot\sin(4\pi\sqrt{xt}\,)\,dt$$

$$=2k\int_{\omega^{-\frac14}}^{1}u(1-u^2)^{k-1}\log(\omega u^2)\cdot\sin(4\pi\sqrt{x\omega}\,u)\,du$$

$$=4k\int_{\omega^{-\frac14}}^{1}u(1-u^2)^{k-1}\log u\cdot\sin(4\pi\sqrt{x\omega}\,u)\,du$$

$$+2k\log\omega\int_{\omega^{-\frac14}}^{1}u(1-u^2)^{k-1}\sin(4\pi\sqrt{x\omega}\,u)\,du$$

$$=J_1+J_2,\quad\text{say.}$$

Then we find that

$$J_1=4k\left(\int_0^1-\int_0^{\omega^{-\frac14}}\right)u\,(1-u^2)^{k-1}\log u\cdot\sin(4\pi\sqrt{x\omega}\,u)\,du$$

$$=4k\int_0^1 u\,(1-u^2)^{k-1}\log u\cdot\sin(4\pi\sqrt{x\omega}\,u)\,du$$

(31)
$$+O\left(k\int_0^{\omega^{-\frac14}}u(1-u^2)^{k-1}\log\frac{1}{u}\,du\right)$$

$$=4k\int_0^1 u\,(1-u^2)^{k-1}\log u\cdot\sin(4\pi\sqrt{x\omega}\,u)\,du$$

$$+O\left(k\,\omega^{-\frac12}\log\omega\cdot(1-\omega^{-\frac12})^{k-1}\right)$$

$$=O\left(k\,(x\omega)^{-\frac12}\right)+O\left(k\,\omega^{-\frac12}\log\omega\right).$$

Here we used the fact that $u\log u\cdot(1-u^2)^{k-1}$ is absolutely continuous on (0, 1) provided $k>0$. Now let us estimate J_2. By the second mean value theorem we get

$$\int_{\omega^{-\frac14}}^{1}u\,(1-u^2)^{k-1}\sin(4\pi\sqrt{x\omega}\,u)\,du=\int_{\omega^{-\frac14}}^{1-\omega^{-\frac12}}+\int_{1-\omega^{-\frac12}}^{1}$$

$$=(1-\omega^{-\frac12})\{1-(1-\omega^{-\frac12})^2\}^{k-1}\int_{\eta}^{1-\omega^{-\frac12}}\sin(4\pi\sqrt{x\omega}\,u)\,dt$$

$$+O\left(\int_{1-\omega^{-\frac12}}^{1}u\,(1-u^2)^{k-1}\,du\right),\quad(\text{for some }\eta\in[\omega^{-\frac14},1-\omega^{-\frac14}])$$

because $u\,(1-u^2)^{k-1}$ is monotonely increasing for $0<u<1$ when $0<k\leq1$. Hence

(32) $$J_2 = O(\log\omega\cdot\omega^{-\frac{k-1}{2}}\cdot\omega^{-\frac{1}{2}}) + O(\log\omega\cdot\omega^{-\frac{k}{2}}) = O(\omega^{-\frac{k}{2}}\log\omega).$$

Consequently we have

$$J_k(\omega) = J_1 + J_2 = O(\omega^{-\frac{1}{2}}\log\omega) + O(\omega^{-\frac{k}{2}}\log\omega) = O(\omega^{-\frac{k}{2}}\log\omega).$$

Therefore

$$I_k(\omega) = O(\omega^{-\frac{k}{2}}\log\omega) + O(\omega^{-\frac{1}{4}}\log\omega),$$

which means that (27) is summable (C, k) to 0 for all $x > 0$ provided when $0 < k \leq 1$, hence for all $k > 0$ by the convexity theorem [6]. Thus we have proved the following theorem.

Theorem 2. *The series* (9) *with* $\alpha = \frac{1}{4}$ *is summable* (C, ε) *for all* $x \notin N$ *and any* $\varepsilon > 0$.

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DEPARTMENT OF MATHEMATICS,
OKAYAMA UNIVERSITY

(*Received March 25, 1974*)