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NOTE ON MORPHISMS OF AFFINE ALGEBRAIC GROUPS

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Throughout the present note, F will denote a field. As for notations and terminologies used here, we follow [1].

Let (G, A) and (H, B) be the structures of affine algebraic groups over F . If $\rho: G \rightarrow H$ is a morphism (of affine algebraic groups), i. e., a group homomorphism with $B \circ \rho \subset A$, then the map $\rho^*: B \rightarrow A$ ($b \mapsto b \circ \rho$) is a morphism (of Hopf algebras), which we call the morphism associated to ρ . Conversely, every morphism $\varphi: B \rightarrow A$ induces a morphism $\varphi_*: G \rightarrow H$ in the natural fashion. To be easily seen, $(\rho^*)_* = \rho$ and $(\varphi_*)^* = \varphi$. Hence, there exists a 1-1 correspondence between the set of all morphisms $G \rightarrow H$ and the set of all morphisms $B \rightarrow A$. Furthermore, if (K, C) is the structure of another affine algebraic group, and $\sigma: H \rightarrow K$ and $\psi: C \rightarrow B$ are morphisms, then $(\sigma\rho)^* = \rho^*\sigma^*$ and $(\psi\varphi)_* = \varphi_*\psi_*$. Finally, regarding G as an affine algebraic set in $A^n(F)$ for suitable n , G is a topological space with Zariski topology. In this sense, every algebraic subgroup of G is characterized as a subgroup of G which is closed, and every morphism of affine algebraic groups is continuous. If F is algebraically closed, every morphism $\rho: G \rightarrow H$ is closed, and so $\text{Im } \rho$ is an algebraic subgroup of H .

Now the purpose of this note is to give elementary proofs of the following results which are known:

Theorem 1. *Let (G, A) and (H, B) be the structures of affine algebraic groups over F , and let $\rho: G \rightarrow H$ and $\varphi: B \rightarrow A$ be morphisms which are associated to each other.*

- (1) *φ is injective if and only if $\text{Im } \rho$ is dense in H .*
- (2) *ρ is injective if and only if $\text{Im } \varphi$ separates the elements of G (or equivalently, the left fixer of $\text{Im } \varphi$ is trivial).*

Theorem 2. *Let (G, A) and (H, B) be the structures of affine algebraic groups over an algebraically closed field F . If $\rho: G \rightarrow H$ is a bijective morphism and $[A_1]/[\text{Im } \rho^* \cap A_1]$ is a separable field extension, then ρ is an isomorphism, where A_1 is the component of A associated to the connected component of the neutral element in G and $[-]$ is the total ring of fractions of $-$.*

As a direct consequence of Theorem 2, we have the following well-known result (cf. [1]): Every bijective morphism of affine algebraic groups over an algebraically closed field of characteristic 0 is an isomorphism.

In advance of proving our theorems, we shall state a couple of lemmas.

Lemma 1. *Let (G, A) be the structure of affine algebraic group over F , and B a sub-Hopf algebra of A . Then, $\text{Alg}_F(B, F)$ is a group and the sequence*

$$1 \longrightarrow L_f(B) \longrightarrow G \xrightarrow{\text{res}} \text{Alg}_F(B, F)$$

is exact, where $L_f(G)$ is the left fixer of B . In case F is algebraically closed, the following is exact:

$$1 \longrightarrow L_f(B) \longrightarrow G \xrightarrow{\text{res}} \text{Alg}_F(B, F) \longrightarrow 1.$$

Proof. As is well-known, $\text{Alg}_F(B, F)$ is a group with respect to the convolution product (cf. [1], [3]), and $L_f(B) \subset \text{Ker}(\text{res}) = \{x \in G \mid x|_B = c_B\}$. If x is in $\text{Ker}(\text{res})$, then for any $y \in G$ and $b \in B$ we have

$$(x \cdot b)(y) = x(b \cdot y) = c_B(b \cdot y) = b(y),$$

whence it follows $x \cdot b = b$, i. e., $x \in L_f(B)$. This proves the first assertion. The latter is now obvious by [1, Theorem 6.4].

Lemma 2. *Let (G, A) be the structure of an affine algebraic group over algebraically closed F , and B a sub-Hopf algebra of A which separates the elements of G . Then, $A \cap [B] = B$. Especially, if $[A] = [B]$ then $A = B$.*

Proof. $\text{Alg}_F(B, F)$ is an affine algebraic group with B as the algebra of polynomial functions, and $\text{res}: G \longrightarrow \text{Alg}_F(B, F)$ is a morphism associated to the injection $B \longrightarrow A$. As B separates the elements of G , we see that $L_f(B) = 1$, so by Lemma 1, res is a bijective closed map, whence the connected components of G and $\text{Alg}_F(B, F)$ correspond to each other. Therefore if the associated direct sum decomposition of A into isomorphic integral domains is $A = A_1 \oplus \cdots \oplus A_n$, we can write that of B as $B = B_1 \oplus \cdots \oplus B_n$, where $B_i \subset A_i$ ($i = 1, \dots, n$), thus $[B] \subset [A]$. Hence it suffices to show that $A \cap [B] \subset B$. Since res is bijective, for any $a \in A \cap [B]$, $\text{Alg}_F(B, F) \cdot a$ is covered by a finite dimensional F -

subspace of $[B]$, so that a belongs to B by [1, Lemma 6.3].

Remark. If a Hopf algebra A over an algebraically closed field F is an integral domain then, without assuming that A is the algebra of polynomial functions of an affine algebraic group, there holds $A \cap [B] = B$ for any sub-Hopf algebra B of A (cf. [2]).

Corollary 1. *Let (G, A) be the structure of an affine algebraic group over algebraically closed F , and B a sub-Hopf algebra of A which separates the elements of G . Then, every element of $[A_1]$ is purely inseparably algebraic over $[A_1 \cap B]$. Especially, if $[A_1]$ is separable over $[A_1 \cap B]$ (for instance, if F is of characteristic 0), then $A = B$.*

Proof. As was seen in the proof of Lemma 2, A and B are decomposed into the following direct sums of isomorphic integral domains: $A = A_1 \oplus \cdots \oplus A_n$, $B = B_1 \oplus \cdots \oplus B_n$, $B_i \subset A_i$ ($i = 1, \dots, n$). So, to our end, it suffices to show that if A is an integral domain then every element of A is purely inseparably algebraic over B (cf. Lemma 2). Since $\text{res}: G \rightarrow \text{Alg}_F(B, F)$ is bijective by Lemma 1, our assertion is a consequence of [1, Theorem 1.6].

Proof of Theorem 1. (1) Since $b(\rho(x)) = \varphi(b)(x)$ ($b \in B$, $x \in G$), $\text{Ker } \varphi$ coincides with the Hopf ideal of B consisting of annihilators of $\text{Im } \rho$. Hence, φ is injective if and only if $\text{Im } \rho$ is dense in H , proving (1).

(2) If x is an element of G such that $x(\varphi(b)) = c_A(\varphi(b))$ for all $b \in B$, then $\rho(x)(b) = x(\varphi(b)) = c_B(b)$. Observing that B separates the elements of H , we readily obtain $x = c_A$. The converse is also easy.

Corollary 2. *Under the situation of Theorem 1, there holds the following:*

- (1) *If ρ is surjective, then φ is injective.*
- (1') *If F is algebraically closed, the converse of (1) holds.*
- (2) *If φ is surjective, then ρ is injective.*
- (2') *If F is algebraically closed and $[A_1]/[\text{Im } \varphi \cap A_1]$ is a separable field extension, the converse of (2) holds.*

Proof. Since (1) and (2) are obvious by Theorem 1, it remains only to prove (1') and (2'). In case F is algebraically closed, ρ is a closed map, and ρ is surjective if and only if $\text{Im } \rho$ is dense in H , proving (1'). Finally, (2') is evident by Corollary 1.

Proof of Theorem 2. By Corollary 2, $\rho^* : B \longrightarrow A$ is a bijective morphism. Hence, $\rho = (\rho^*)^*$ is an isomorphism again by Corollary 2.

REFERENCES

- [1] G. HOCHSCHILD : Introduction to Affine Algebraic Groups, Holden-Day, 1971.
- [2] J. B. SULLIVAN : A proof of finite generation of invariants of a normal subgroup, Pacific J. Math. 51 (1974), 571—572.
- [3] M. E. SWEEDLER : Hopf Algebras, Benjamin, 1969.

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