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Yoshimi Kitamura*

*Tokyo Gakugei University

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A NOTE ON QUOTIENT RINGS OVER A QUASI-FROBENIUS EXTENSION

YOSHIMI KITAMURA

Let A/B be a ring extension. When A/B is a Quasi-Frobenius (QF) (resp. Frobenius) extension, it seems to be natural to ask where the maximal right quotient ring $Q_{\max}(A)$ of A in the sense of Utumi-Lambek is a QF (resp. Frobenius) extension of the maximal right quotient ring $Q_{\max}(B)$ of B or not. In the previous paper [4], we showed that the answer is affirmative if A/B is a Frobenius extension such that

(*)₀ A is finitely generated as a B -module by elements which centralize B .

In the present paper, we shall further investigate the above question under a suitable hypothesis. Our main result of this paper is the following:

If A/B is a QF (resp. Frobenius) extension satisfying ()₀ or the condition:*

(**)_B *the class of right B -modules with zero duals is closed under taking submodules*

then $Q_{\max}(A)$ is a QF (resp. Frobenius) extension of $Q_{\max}(B)$ such that $Q_{\max}(A) \cong Q_{\max}(B) \otimes {}_B A \cong \otimes {}_B Q_{\max}(B)$ canonically. Here, by the dual of a right B -module X , we mean $\text{Hom}(X_B, B_B)$.

It is interesting to observe that when A/B is QF, A satisfies (**)_A if and only if B does (**)_B (Proposition 1. 4).

Throughout this paper, A will denote a ring with identity element and B a subring of A with the common identity element. The notation and terminology are same as [4] unless otherwise specified.

1. Let R be a ring with identity element. A non-empty set $\mathfrak{F}$ of right ideals of R is called an idempotent filter of R if the following conditions are satisfied:

- (i) If $I \in \mathfrak{F}$ and $I \subset J$, then $J \in \mathfrak{F}$.
- (ii) If $I, J \in \mathfrak{F}$, then $I \cap J \in \mathfrak{F}$.
- (iii) If $I \in \mathfrak{F}$, $a \in R$, then $a^{-1}I = \{x \in R \mid ax \in I\} \in \mathfrak{F}$.
- (iv) If $I \in \mathfrak{F}$, $a^{-1}J \in \mathfrak{F}$ for every $a \in I$, then $J \in \mathfrak{F}$.

Let $\mathfrak{F}$ be an idempotent filter of R . Then $\mathfrak{F}$ is obviously a directed set by inverse inclusion and $Q_{\mathfrak{F}}(R) = \varinjlim_{I \in \mathfrak{F}} \text{Hom}_R(I, R/T_{\mathfrak{F}}(R))$ has a ring structure in a natural way, where $T_{\mathfrak{F}}(R) = \{a \in R \mid aI = 0 \text{ for}$

some $I \in \mathfrak{F}$, called the right quotient ring of R with respect to $\mathfrak{F}$, and moreover there is a canonical ring homomorphism $R \rightarrow Q_{\mathfrak{F}}(R)$ whose kernel is $T_{\mathfrak{F}}(R)$. Thus the last homomorphism is monic if and only if the following condition is enjoyed:

(v) The left annihilator of every member of $\mathfrak{F}$ is zero.

The largest set of right ideals of R satisfying (i)–(v) is the filter consisting of all dense right ideals of R , where a right ideal I of R is said to be dense if for every $x, y \in R$, $y \neq 0$, there exists some $a \in R$ such that $xa \in I$ and $ya \neq 0$, or equivalently, if $\text{Hom}(R/I_R, E(R_R)_R) = 0$, $E(R_R)$ the injective hull of R_R . The right quotient ring with respect to this filter is called the maximal right quotient ring of R and denoted by $Q_{\max}(R)$ (see for details, [6] and [8]).

Now we shall turn to a ring extension A/B .

Proposition 1.1. *Let $\mathfrak{F}$ be a non-empty set of right ideals of B and $\tilde{\mathfrak{F}}$ the set of right ideals of A which contain some member in $\mathfrak{F}$. If $\mathfrak{F}$ satisfies (i), (ii) and (iv), then $\tilde{\mathfrak{F}}$ satisfies (i), (ii) and (iv). Moreover if (*) A is generated as a B -module by elements which centralize B , then $\tilde{\mathfrak{F}}$ satisfies (iii) whenever so does $\mathfrak{F}$.*

Proof. Clearly, $\tilde{\mathfrak{F}}$ satisfies (i) and (ii). Let I be a member in $\tilde{\mathfrak{F}}$ with $I \supset J$ for some $J \in \mathfrak{F}$. Let K be a right ideal of A such that $x^{-1}K \in \mathfrak{F}$ for every $x \in I$. Since $b^{-1}(K \cap B) = b^{-1}K \cap B \in \mathfrak{F}$ for every $b \in J$, we obtain $K \cap B \in \mathfrak{F}$, and so, $K \in \tilde{\mathfrak{F}}$, proving (iv). Next assume further (*) and (iii) for $\mathfrak{F}$. Then every element a of A can be written as $a = \sum b_i a_i$ with some $a_i \in V_A(B) = \{a \in A \mid ab = ba \text{ for all } b \in B\}$ and $b_i \in B$ ($i = 1, \dots, t$). Since $b_i^{-1}(I \cap B) \in \mathfrak{F}$ for all i , we have $\cap b_i^{-1}(I \cap B) \in \mathfrak{F}$, and so, $a^{-1}I \in \mathfrak{F}$ because of $a^{-1}I \supset \cap b_i^{-1}(I \cap B)$.

To contrast with the above proposition, we shall prove the next.

Proposition 1.2. *Let $\mathfrak{F}$ be a non-empty set of right ideals of A satisfying (i), (ii) and (iii), and $\mathfrak{F}'$ the set of right ideals J of B with $J \supset I'$ for some $I \in \mathfrak{F}$, where I' denotes the right ideal of B generated by the elements of the form $f(x)$, $x \in I$, $f \in \text{Hom}({}_B A, {}_B B)$. Then $\mathfrak{F}'$ satisfies (i), (ii) and (iii). Moreover, if ${}_B A$ satisfies (*) and is projective then $\mathfrak{F}'$ satisfies (iv) whenever so does $\mathfrak{F}$.*

Proof. Obviously, $\mathfrak{F}'$ satisfies (i) and (ii). Let I and b be elements in $\mathfrak{F}$ and B respectively. Since $(b^{-1}I)' \subset b^{-1}I'$, we have $b^{-1}I' \in \mathfrak{F}'$, proving (iii). Assume further that ${}_B A$ is projective and (*) is satisfied. Let $\{a_i\}_i \subset V_A(B)$ be a generating set for the left B -module A . Then, by the projectivity of ${}_B A$, there exists a subset $\{f_i\}_i \subset$

$\text{Hom}(A_B, B_B)$ such that, for each $x \in A$, $f_i(x) = 0$ for almost all i and $x = \sum f_i(x) a_i$. (Such a system $(f_i, a_i)_i$ is called a dual basis for ${}_B A$). Let J be a right ideal of B such that $y^{-1}J \in \mathfrak{F}'$ for every $y \in I'$. For every element $x = \sum f_i(x) a_i$ in $I \cap f_i(x)^{-1}J$ is contained in $x^{-1}JA$ and contains some I'_0 ($I_0 \in \mathfrak{F}$) as a member of $\mathfrak{F}'$, and so, $I_0 \subset I'_0 \cdot A \subset x^{-1}JA$. Therefore $JA \in \mathfrak{F}$ by (iv), which implies $(JA)' \subset J \in \mathfrak{F}'$ as desired.

We now consider the following conditions on a ring R which have been investigated in [2]:

- (**) $_R$ The class of right R -modules with zero duals is closed under taking submodules.
- (***) $_R$ The class of torsionless right R -modules is closed under taking extensions.

As was mentioned in [2], if the injective hull $E(R_R)_R$ of R_R is torsionless then R satisfies (**) $_R$ and (***) $_R$. We shall show in the following proposition that these conditions are inherited for QF extensions. Although, the next lemma was cited in the foot note of [2, p. 450], for the sake of completeness, we give here a proof.

Lemma 1.3. *A ring R satisfies (**) $_R$ if and only if $\text{Hom}(X_R, E(R_R)_R) = 0$ for every X_R with zero dual.*

Proof. It is enough to prove the only if part. Let X_R be a right R -module with zero dual. If $f(X) \neq 0$ for some $f \in \text{Hom}(X_R, E(R_R)_R)$, then $f(X) \cap R \neq 0$, which contradicts (**) $_R$.

Proposition 1.4. *Suppose A/B is a QF extension. Then B satisfies (**) $_B$ (resp. (***) $_B$) if and only if so does A .*

Proof. Let $(\alpha_i : \sum_j x_{ij} \otimes y_{ij})_{1 \leq i \leq p}$ be a right QF system for A/B . First, we claim the followings:

a) *If Y_B is torsionless, then $\text{Hom}(A_B, Y_B)_A$ is torsionless and a mapping $\alpha_Y : Y \rightarrow \text{Hom}(A_B, Y_B)_A$ defined by $\alpha_Y(y) = (\lambda_y \cdot \alpha_i)_i$ for $y \in Y$ is monic, where λ_y denotes a mapping $B \rightarrow Y$ given by $\lambda_y(b) = yb$ for $b \in B$. In fact, the first assertion is clear, because $Y_B \subset \Pi B_B$ implies $\text{Hom}(A_B, Y_B)_A \subset \Pi \text{Hom}(A_B, B_B)_A \subset \Pi A_A$. To see the second one, let us assume $\lambda_y \cdot \alpha_i = 0$ for all i . Then $\alpha_i \cdot f(y) = 0$ in $\text{Hom}(A_B, B_B)$ for every $f \in \text{Hom}(Y_B, B_B)$, and so, $f(y) = 0$. As Y_B is torsion-*

1) A right QF system for A/B is defined as a system $(\alpha_i ; \sum_j x_{ij} \otimes y_{ij})_{1 \leq i \leq p}$ with $\sum_{i,j} \alpha_i(x_{ij}) y_{ij} = 1$, where every α_i is a B - B -homomorphism of A to B and every $\sum_j x_{ij} \otimes y_{ij}$ is a casimir element of $A \otimes_B A$, that is, an element which commutes with all elements of A . A left QF system for A/B is defined similarly (see [5]).

less, this means $y = 0$.

b) X_A is torsionless if and only if so is it as a B -module. Noting $X_A \subset \text{Hom}(A_B, X_B)_A$, this is a consequence of a).

c) ${}_B\text{Hom}(X_A, A_A) \sim {}_B\text{Hom}(X_B, B_B)$ for every X_A . In fact, ${}_B\text{Hom}(X_A, A_A) \sim {}_B\text{Hom}(X_A, \text{Hom}(A_B, B_B)) \cong {}_L\text{Hom}(X \otimes {}_A A_B, B_B) \cong {}_B\text{Hom}(X_B, B_B)$.

Now assume the validity of $(**)_{A_1}$ and let Y be a right B -module with zero dual and Y' an arbitrary submodule of Y . Since $\text{Hom}(A_B, Y'_B)_A \subset \text{Hom}(A_B, Y_B)_A$ and $\text{Hom}(\text{Hom}(A_B, Y_B)_A, A_A) \sim \text{Hom}(Y \otimes {}_B A_A, A_A) \cong \text{Hom}(Y_B, A_B) \mid \text{Hom}(Y_B, B_B)$ (as abelian groups), the validity of $(**)_{A_1}$ yields $\text{Hom}(Y'_B, A_B) = 0$, and so, $\text{Hom}(Y'_B, B_B) = 0$ as desired. The converse is clear by c). Next assume the validity of $(***)_{A_1}$. Let $0 \rightarrow Y' \rightarrow Y \rightarrow Y'' \rightarrow 0$ be a short exact sequence of tight B -modules with Y', Y'' torsionless. We have the following commutative diagram with rows exact:

$$\begin{array}{ccccccc} 0 & \longrightarrow & Y' & \longrightarrow & Y & \longrightarrow & Y'' \longrightarrow 0 \\ & & \alpha_{Y'} \downarrow & & \alpha_Y \downarrow & & \alpha_{Y''} \downarrow \\ 0 & \rightarrow & \text{Hom}(A_B, Y'_B)^p & \rightarrow & \text{Hom}(A_B, Y_B)^p & \rightarrow & \text{Hom}(A_B, Y''_B)^p \rightarrow 0 \end{array}$$

Therefore $\text{Hom}(A_B, Y_B)$ is torsionless as an A and hence as a B -module by a) and $(***)_{A_1}$. Recalling here that $\alpha_{Y'}$ and $\alpha_{Y''}$ are both monic by a), we can see α_Y to be monic and Y_B to be torsionless. Conversely, A satisfies $(***)_{A_1}$ whenever B does $(***)_{B_1}$ from b).

Proposition 1.5. Suppose that A/B is QF and that either $(*)$ or $(**)_B$ is satisfied. If I (resp. J) is a dense right ideal of A (resp. B), then so is $I \cap B$ (resp. JA) in B (resp. A).

Proof. First, we note the following fact that the canonical injection $B/I \cap B \rightarrow A/I$ induces a surjection $\text{Hom}(A/I_B, E(B_B)_B) \rightarrow \text{Hom}(B/I \cap B_B, E(B_B)_B)$. Let us now assume $(*)$. Then we have $E(A_A)_A \sim \text{Hom}(A_B, E(B_B)_B)_A$ by the proof of [4, Theorem]. Thus we have $\text{Hom}(A/I_B, E(B_B)_B) \cong \text{Hom}(A/I_A, \text{Hom}(A_B, E(B_B)_B)_A) \sim \text{Hom}(A/I_A, E(A_A)_A) = 0$ as abelian groups. Therefore, recalling the above mention, $I \cap B$ is dense in B . Further, noting $\text{Hom}(A/JA_A, E(A_A)_A) \sim \text{Hom}(B/J \otimes {}_B A_A, \text{Hom}(A_B, E(B_B)_B)_A) \sim \text{Hom}(B/J_B, E(B_B)_B) = 0$, it follows that JA is dense in A .

Next assume $(**)_B$ for B . Thus A satisfies $(**)_{A_1}$ by Proposition 1.4. Since $\text{Hom}(A/JA_A, A_A) \cong \text{Hom}(B/J_B, A_B) \mid \text{Hom}(B/J_B, B_B)$, we have $\text{Hom}(A/JA_A, A_A) = 0$. It follows that JA is dense in A by Lemma 1.3. Moreover we have $\text{Hom}(A/I_B, B_B) \sim \text{Hom}(A/I_A, A_A)$ by c) in the proof

of Proposition 1.4, and so, $\text{Hom}(A/I_B, E(B_B)_B) = 0$ by Lemma 1.3. Therefore, recalling the mention at the beginning of the proof, $I \cap B$ is dense in B .

Remark. Assume ${}_B A$ to be torsionless. If I is a dense right ideal of A , then I' defined in Proposition 1.2 is a dense right ideal of B . In fact, let x and y be in B , with $y \neq 0$. Since I is dense in A , there exists $a \in A$ such that $xa \in I$ and $ya \neq 0$. Our assumption yields $f(ya) \neq 0$ for some $f \in \text{Hom}({}_B A, {}_B B)$. Therefore we have $xf(a) = f(xa) \in I'$ and $yf(a) \neq 0$.

Let $\mathfrak{F}$ be now an idempotent filter of B such that $\tilde{\mathfrak{F}}$ defined in Proposition 1.1 is also an idempotent filter of A . Then we can construct the right quotient rings $Q_{\mathfrak{F}}(B)$ and $Q_{\tilde{\mathfrak{F}}}(A)$ of B and A with respect to $\mathfrak{F}$ and $\tilde{\mathfrak{F}}$, respectively. Let us set $\bar{B} = B/T_{\mathfrak{F}}(B)$, $\bar{A} = A/T_{\tilde{\mathfrak{F}}}(A)$, $\tilde{B} = Q_{\mathfrak{F}}(B)$ and $\tilde{A} = Q_{\tilde{\mathfrak{F}}}(A)$. First suppose A/B to be a QF extension with a right QF system $(\alpha_i: \sum_j x_{ij} \otimes y_{ij})_{1 \leq i \leq p}$ and with a left QF system $(\beta_k: \sum_l w_{kl} \otimes z_{kl})_{1 \leq k \leq q}$. Since $T_{\tilde{\mathfrak{F}}}(A) \cap B = T_{\mathfrak{F}}(B)$, we can and shall regard $\tilde{B}$ as a subring of $\bar{A}$. Let φ be an arbitrary right B -homomorphism of A to B . It is easy to see $\varphi(T_{\tilde{\mathfrak{F}}}(A)) \subset T_{\mathfrak{F}}(B)$. Therefore, φ induces a right $\bar{B}$ -homomorphism $\bar{\varphi}: \bar{A} \rightarrow \bar{B}$ in a natural way. Furthermore, we have $T_{\mathfrak{F}}(B)A = AT_{\mathfrak{F}}(B) = T_{\tilde{\mathfrak{F}}}(A)$, since for $a \in T_{\tilde{\mathfrak{F}}}(A)$, $a = \sum_{i,j} \alpha_i(ax_{ij})y_{ij} = \sum_{k,l} w_{kl}\beta_k(z_{kl}a)$ with $\alpha_i(ax_{ij}), \beta_k(z_{kl}a) \in T_{\mathfrak{F}}(B)$. Next, let g be a right B -homomorphism of $J(J \in \mathfrak{F})$ to B . Since ${}_B A$ is finitely generated (f. g.) projective, we have an unique extension $\hat{g}: JA \rightarrow \bar{A}$ of g so that the diagram

$$\begin{array}{ccc} J \otimes {}_B A & \xrightarrow{g \otimes \text{Id}_A} & B/T_{\mathfrak{F}}(B) \otimes {}_B A \\ \parallel & \hat{g} & \parallel \\ JA & \xrightarrow{\quad} & A/T_{\mathfrak{F}}(B)A = \bar{A} \end{array}$$

is commutative, where the vertical mappings are the natural isomorphisms. Thus we can obtain a ring homomorphism $\rho: \tilde{B} \rightarrow \tilde{A}$ defined by

$$\rho([g]) = [\hat{g}]$$

for $[g] \in \tilde{B}$. It is obviously monic. Therefore, we can and shall regard $\tilde{B}$ as a subring of $\tilde{A}$ by ρ . Moreover we shall denote by $\tilde{a}$ the image of $a \in A$ under the canonical mapping $A \rightarrow \tilde{A}$, and by f' the restriction of f to J for a representative $f: JA \rightarrow \bar{A}$ of $[f] \in \tilde{A} (J \in \mathfrak{F})$.

Then we claim the followings.

a) Every right B -homomorphism φ of A to B induces a right $\tilde{B}$ -homomorphism $\tilde{\varphi}$ of $\tilde{A}$ to $\tilde{B}$ defined by

$$\tilde{\varphi}([f]) = [\tilde{\varphi} \circ f']$$

for $[f] \in \tilde{A}$. In particular, if $(\varphi_s, a_s)_{1 \leq s \leq s_0}$ ($\varphi_s \in \text{Hom}(A_B, B_B)$, $a_s \in A$) is a dual basis for A_B , then $(\tilde{\varphi}_s, \tilde{a}_s)_{1 \leq s \leq s_0}$ is also a dual basis for $\tilde{A}_{\tilde{B}}$.

b) For $\psi_r \in \text{Hom}({}_B A_B, {}_B B_B)$, $\tilde{\psi}_r$ which can be defined by a) regarding ψ_r as a right B -homomorphism is contained in $\text{Hom}({}_{\tilde{B}} \tilde{A}_{\tilde{B}}, {}_{\tilde{B}} \tilde{B}_{\tilde{B}})$.

c) If $\sum_i x_i \otimes y_i \in A \otimes_B A$ is a casimir element, then $\sum \tilde{x}_i \otimes \tilde{y}_i \in \tilde{A} \otimes_{\tilde{B}} \tilde{A}$ is also a casimir element.

$$\text{d) } \sum_{i,j} \tilde{a}_i(\tilde{x}_{ij}) \tilde{y}_{ij} = \tilde{1} \text{ and } \sum_{k,l} \tilde{w}_{kl} \tilde{\rho}_k(\tilde{z}_{kl}) = \tilde{1}.$$

The above assertions a), b) and d) can be easily verified. Since $\tilde{A}_{\tilde{B}}$ is f.g. projective by a), a mapping $\tilde{A} \otimes_{\tilde{B}} \tilde{A} \ni u \otimes v \mapsto (\gamma \mapsto \gamma(u)v) \in \text{Hom}({}_{\tilde{B}} \text{Hom}(\tilde{A}_{\tilde{B}}, \tilde{B}_{\tilde{B}}), {}_{\tilde{B}} \tilde{A})$ is bijective. Hence, to see c), it is enough to show the following equation

$$\sum_i \tilde{\varphi}_s(\tilde{x}_i) \tilde{y}_i \cdot [f] = \sum_i \tilde{\varphi}_s([f] \cdot \tilde{x}_i) \tilde{y}_i \text{ for } [f] \in \tilde{A},$$

where $(\tilde{\varphi}_s, \tilde{a}_s)_s$ is a dual basis for $\tilde{A}_{\tilde{B}}$ induced by a dual basis $(\varphi_s, a_s)_s$ for A_B . So, let $f: JA \rightarrow \tilde{A}$ ($J \in \mathfrak{F}$) be a representative of $[f] \in \tilde{A}$. Then we have

$$\begin{aligned} \tilde{y}_i \cdot [f] &= [JA \ni bx \mapsto \tilde{y}_i f(b)x \in \tilde{A}], \\ \tilde{\varphi}_s(\tilde{x}_i) &= [A \ni x \mapsto \tilde{\varphi}_s(\tilde{x}_i)x \in \tilde{A}], \end{aligned}$$

where, for $a \in A$, $\tilde{a}$ denotes the image of $a \in A$ under the canonical mapping $A \rightarrow \tilde{A}$ and $[X \ni x \mapsto y \in Y]$ denotes a class to which a mapping $X \ni x \mapsto y \in Y$ belongs. Therefore the left hand of the above equation is equal to $[JA \ni bx \mapsto \sum_i \tilde{\varphi}_s(\tilde{x}_i) \tilde{y}_i f(b)x \in \tilde{A}]$. On the other hand, $\cap_i x_i^{-1}(JA)$ contains some member J_1 in $\mathfrak{F}$, and so, $\cap_i y_i^{-1}(J_1 A)$ contains some member J_2 in $\mathfrak{F}$. We have then

$$\begin{aligned} \tilde{y}_i &= [A \ni x \mapsto \tilde{y}_i x \in \tilde{A}], \\ \tilde{\varphi}_s([f] \cdot \tilde{x}_i) &= [J_1 A \ni bx \mapsto \tilde{\varphi}_s(f(x_i b)) \cdot x \in \tilde{A}]. \end{aligned}$$

Furthermore, for any b in $J_2 \cap J \in \mathfrak{F}$, we have $y_i b \in J_1 A$, and so, in an equation $y_i b = \sum_{i,j} \alpha_i(y_i b x_{ij}) y_{ij}$, every $\alpha_i(y_i b x_{ij})$ is contained in J_1 . It follows that the right hand of the above equation is equal to

$$[(J_2 \cap J)A \ni bx \longmapsto \sum_i \sum_{i,j} \bar{\varphi}_s(f(x_i \alpha_i(y_i b x_{ij}))) y_{ij} x \in \bar{A}].$$

However, we have

$$\begin{aligned} \sum_i \sum_{i,j} \bar{\varphi}_s(f(x_i \alpha_i(y_i b x_{ij}))) y_{ij} &= \sum_{i,j} \bar{\varphi}_s(f(b \sum_i x_i \alpha_i(y_i x_{ij}))) y_{ij} \\ &= \sum_{i,j} \bar{\varphi}_s(f(b) \sum_i x_i \alpha_i(y_i x_{ij})) y_{ij} \\ &= \sum_i \bar{\varphi}_s(f(b) x_i) \sum_{i,j} \alpha_i(y_i x_{ij}) y_{ij} \\ &= \sum_i \bar{\varphi}_s(f(b) x_i) y_i \\ &= \sum_i \bar{\varphi}_s(\bar{x}_i) \bar{y}_i f(b), \end{aligned}$$

where the first and the last equalities are followed by the fact that $\sum x_i \otimes y_i \in A \otimes_B A$ is a casimir element. Therefore we have the desired equation. Thus we have shown that $\tilde{A}/\tilde{B}$ is a QF extension with a right QF system $(\tilde{a}_i; \sum_j \tilde{x}_{ij} \otimes \tilde{y}_{ij})_i$ and with a left QF system $(\tilde{\beta}_k; \sum_l \tilde{w}_{kl} \otimes \tilde{z}_{kl})_k$. As $\tilde{\alpha}_i(\tilde{a}) = \tilde{\alpha}_i(a)$ and $\tilde{\beta}_k(\tilde{a}) = \tilde{\beta}_k(a)$ for $a \in A$, mappings

$$\tilde{B} \otimes_B A \ni v \otimes a \longmapsto v \tilde{a} \in \tilde{A}$$

and

$$A \otimes_B \tilde{B} \ni a \otimes v \longmapsto \tilde{a} v \in \tilde{A}$$

are isomorphisms whose inverses are given respectively by

$$\tilde{A} \ni u \longmapsto \sum_{i,j} \tilde{\alpha}_i(u x_{ij}) \otimes y_{ij} \in \tilde{B} \otimes_B A$$

and

$$\tilde{A} \ni u \longmapsto \sum_{k,l} w_{kl} \otimes \tilde{\beta}_k(z_{kl} u) \in A \otimes_B \tilde{B}.$$

Similarly, we can show that if A/B is a Frobenius extension with a Frobenius system $(h; r_i, l_i)_i$, then $\tilde{A}/\tilde{B}$ is a Frobenius extension with a Frobenius system $(\tilde{h}; \tilde{r}_i, \tilde{l}_i)_i$. To summarize, we have the following proposition.

Proposition 1.6. *Let $\mathfrak{F}$ be an idempotent filter of B such that $\tilde{\mathfrak{F}}$ defined in Proposition 1.1 is also an idempotent filter of A . If A/B is a QF (resp. Frobenius) extension then $Q_{\tilde{\mathfrak{F}}}(A)$ is a QF (resp. Frobenius) extension of $Q_{\mathfrak{F}}(B)$ such that the canonical mappings*

$$Q_{\mathfrak{F}}(B) \otimes_B A \ni x \otimes a \longmapsto xa \in Q_{\tilde{\mathfrak{F}}}(A)$$

and

$$A \otimes_B Q_{\mathfrak{F}}(B) \ni a \otimes x \longmapsto ax \in Q_{\tilde{\mathfrak{F}}}(A)$$

are both isomorphisms. In particular, if $Q_{\mathfrak{F}}(B)$ is flat as a left (resp. right) B -module then $Q_{\mathfrak{F}}(A)$ is flat as a left (resp. right) A -module.

We are now ready for proving the following main theorem.

Theorem 1.8. *If A/B is a QF (resp. Frobenius) extension satisfying the condition (*) or $(**)_{\mathfrak{A}}$ (or $(**)_{\mathfrak{B}}$), then $Q_{\max}(A)$ is a QF (resp. Frobenius) extension of $Q_{\max}(B)$ such that the canonical mappings*

$$Q_{\max}(B) \otimes_B A \ni x \otimes a \longrightarrow xa \in Q_{\max}(A)$$

and

$$A \otimes_B Q_{\max}(B) \ni a \otimes x \longrightarrow ax \in Q_{\max}(A)$$

are both isomorphisms. In particular, if $Q_{\max}(B)$ is flat as a left (resp. right) B -module, then $Q_{\max}(A)$ is flat as a left (resp. right) A -module.

Proof. Let $\mathfrak{F}$ be the filter of dense right ideals of B . Then $\widetilde{\mathfrak{F}}$ defined in Proposition 1.1 coincides with the filter of dense right ideals of A by Propositions 1.4 and 1.5. It follows that the theorem is a direct consequence of the above proposition.

If the injective hull of B_B is torsionless, then the condition $(**)_{\mathfrak{B}}$ is enjoyed by Lemma 1.3. Thus we have the following as a consequence of the theorem.

Corollary. *If A is a QF (resp. Frobenius) extension of B such that the injective hull of B_B is torsionless, then the same conclusion as in the theorem holds.*

As will be seen from the following examples, the conditions (*) and $(**)_{\mathfrak{B}}$ for a ring extension A/B are independent even when A/B is a Frobenius extension.

Example 1. Let $\mathbb{Z}$ be the ring of integers. A group ring $\mathbb{Z}[G]$ of a finite group G over $\mathbb{Z}$ is a Frobenius extension of $\mathbb{Z}$ with a Frobenius homomorphism $h: \mathbb{Z}[G] \ni \sum a_g \cdot g \longrightarrow a_e \in \mathbb{Z}$ (e the identity of G). Obviously, $\mathbb{Z}[G]/\mathbb{Z}$ satisfies (*). However $\mathbb{Z}$ does not satisfy $(**)_{\mathbb{Z}}$.

Example 2. Let R be a triangular matrix ring $\begin{pmatrix} K & 0 \\ K & K \end{pmatrix}$ over a field K and A the 2×2 matrix ring $(R)_2$ over the ring R . As is well known, R satisfies the condition $(**)_{\mathfrak{R}}$ (in fact, R is a QF-3 ring), and so, A does $(**)_{\mathfrak{A}}$. Let us denote by σ the inner automorphism of the ring A induced by the element $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in A , by G the group generated by σ , and by B the subring of A consisting of elements of A left fixed by

every element of G . Put $x_1 = y_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $x_2 = y_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in A$. It is then easy to see that

$$\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & c \\ b & a \end{pmatrix} \text{ for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in A,$$

$$x_1 \cdot y_1 + x_2 \cdot y_2 = 1, \quad x_1 \cdot \sigma(y_1) + x_2 \cdot \sigma(y_2) = 0.$$

Therefore, A/B is a G -Galois extension. Hence it is a Frobenius extension with a Frobenius system $(h; x_i, y_i)_{i=1,2}$, where h is the trace homomorphism of A to B defined by

$$h(x) = x + \sigma(x) \text{ for } x \in A.$$

It follows that B satisfies $(**)_B$ by Proposition 1.4. However, to be easily seen, A/B never enjoys the condition $(*)$.

Now assume that ${}_B A_B \mid {}_B B_B$, that is, there exist $f_k: {}_B A_B \longrightarrow {}_B B_B$ and $a_k \in V_A(B)$ ($k=1, \dots, t$) such that $\sum a_k \cdot f_k(a) = a$ for all $a \in A$. Since each f_k sends $V_A(B)$ into the center C of B , $V_A(B)$ is an f.g. projective, faithful C -module. Hence we have $V_A(B) \sim C$ as C -modules. Moreover mappings $V_A(B) \otimes_C B \longrightarrow A$, $x \otimes y \longmapsto xy$, and $A \longrightarrow V_A(B) \otimes_C B$, $x \longmapsto \sum a_k \otimes f_k(x)$, are the mutually inverse isomorphisms. Similarly we have $B \otimes_C V_A(B) \cong A$ by the correspondence $y \otimes x \longmapsto yx$, and so, ${}_B A_B \sim {}_B B_B$.

Proposition 1.8. *Suppose that A/B is QF (resp. Frobenius) such that B is right artinian with ${}_B A_B \mid {}_B B_B$. Let M be an f.g. right B -module which is a generator and a cogenerator. Put $N = \text{Hom}(A_B, M_B)$, $A' = \text{End}(N_A)$ and $B' = \text{End}(N_B)$. Then there holds the following:*

- a) B'/A' is QF (resp. Frobenius).
- b) B' and A' are both semi-primary QF-3 rings.
- c) $A = \text{End}({}_{A'} N)$ and $B = \text{End}({}_{B'} N)$.

Proof. As is mentioned above, ${}_B A_B \sim {}_B B_B$, and so, $N_B = \text{Hom}(A_B, M_B) \sim \text{Hom}(B_B, M_B) \cong M_B$; $N \otimes {}_B A_A \sim N_A$. Thus a) is a consequence of [5, Theorem 1.1]. Next, to be easily seen, N_A and N_B are f.g. generators and cogenerators. Hence b) is a direct consequence of [7, Theorem 3.1]. Finally c) is well known.

Remark. If B is a commutative artinian ring then such a module M does always exist. Indeed, the injective hull $E(B/J_B)$ is an f.g. cogenerator, where J is the radical of B (see [1, Proposition 10.5]). Hence $M = B(B/J_B) \oplus B$ is a required one.

As an immediate consequence of the above proposition, we have the following.

Corollary. *Every QF (resp. Frobenius) algebra over a commutative artinian ring can be obtained as an endomorphism ring of a semiprimary QF-3 ring which is a QF (resp. Frobenius) extension of a semiprimary QF-3 ring.*

2. Supplements. A right R -module whose f. g. submodules are torsionless is said to be locally torsionless. In connection with [3, Proposition 2.12], we shall show the following.

Proposition 2.1. *Suppose A/B to be QF extension, Then $E(A)_A$ is locally torsionless if and only if so is $E(B)_B$.*

Proof. In general, to be easily seen, if M_R is locally torsionless then so is every submodule of M_R^n . This remark will be used freely in the sequel. First we assume $E(A)_A$ locally torsionless. Since $E(A)_B$ is injective, we can consider as $E(B)_B \subset E(A)_B$. If Y_B is an arbitrary f. g. submodule of $E(B)_B$, then $Y \subset YA \subset E(A)_A$. Thus, by our assumption, YA_A and so, YA_B is torsionless. Hence Y_B is torsionless. Conversely, assume $E(B)_B$ locally torsionless. Since $A_A | \text{Hom}(A_B, B_B)_A \subset \text{Hom}(A_B, E(B_B)_B)_A$, we have $E(A)_A | \text{Hom}(A_B, E(B_B)_B)_A$ and so, we have only to show $\text{Hom}(A_B, E(B_B)_B)_A$ locally torsionless. To this end, take an arbitrary f. g. submodule X_A of $\text{Hom}(A_B, E(B_B)_B)_A$. Then, recalling that A_B is f. g., we have $X_A \subset \text{Hom}(A_B, Y_B)_A$ for some f. g. (torsionless) submodule Y_B of $E(B)_B$. Hence X_A is torsionless as a submodule of the torsionless module $\text{Hom}(A_B, Y_B)_A$.

Proposition 2.2. *Suppose A/B a right QF extension. Then ${}_B A$ is a generator if and only if there exists a right A -module which is a cogenerator as a B -module.*

Proof. Let $(\alpha_i; \sum_j x_{ij} \otimes y_{ij})_{1 \leq i \leq p}$ be a right QF system for A/B . Then the trace ideal $\text{Hom}({}_B A, {}_B B)(A)$ of ${}_B A$ coincides with $\sum \alpha_i(A)$. Accordingly, ${}_B A$ is a generator if and only if $\sum \alpha_i(A) = B$. First assume ${}_B A$ a generator. Then, for every right B -module Y , $\alpha_Y: Y \rightarrow \text{Hom}(A_B, Y_B)^p$ defined as in the proof of Proposition 1.4 is monic. Hence every right A -module which is a cogenerator (such a module exists always) is a cogenerator as a B -module. Conversely, assume that there exists a right A -module V which is a cogenerator as a B -module. If ${}_B A$ is not a generator, then there exists a maximal right ideal J of B

containing $\sum \alpha_i(A)$. Since V_B is a cogenerator, there exists some $f \in \text{Hom}(B/J_B, V_B)$ with $f(1+J) \neq 0$. However, $\text{Ann}_A(f(1+J)) = \{a \in A \mid f(1+J)a = 0\} \supset JA \supset \sum \alpha_i(A) \cdot A = A$, which yields a contradiction $f(1+J) = 0$.

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DEPARTMENT OF MATHEMATICS,
TOKYO GAKUGEI UNIVERSITY

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