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NOTE ON FULLY IDEMPOTENT IDEALS AND *s*-UNITAL IDEALS

To Professor R. Takekuma on his sixteenth birthday
February 29, 1976

MOTOSHI HONGAN and HISAO TOMINAGA

Throughout the present note R will represent a ring (with or without identity). R is said to be *fully left idempotent* (or *left weakly regular*) if every left ideal of R is idempotent or equivalently if $a \in (Ra)^2$ for any $a \in R$ (cf. [4]). R is said to be *fully idempotent* if every ideal of R is idempotent or equivalently if $a \in (RaR)^2$ for any $a \in R$ (cf. [2]). Finally, R is said to be *left* (resp. *right*) *s-unital* (or *D-regular*) if $a \in Ra$ (resp. $a \in aR$) for any $a \in R$ (cf. [5]). If an ideal I of R is a fully left idempotent ring (resp. fully idempotent ring), then I is called a *fully left idempotent ideal* (resp. *fully idempotent ideal*). Similarly, I is called an *s-unital ideal* if I is a left and right s-unital ring. We shall denote by $W(R)$, $W'(R)$ and $W^*(R)$ the sum of all fully left idempotent ideals of R , the sum of all fully idempotent ideals of R and the sum of all s-unital ideals of R , respectively.

In this note we shall prove the following :

Theorem 1. (1) $W(R)$ is the unique maximal fully left idempotent ideal of R .

(2) $W(R/W(R)) = 0$.

(3) Let $(R)_n$ be the $n \times n$ matrix ring over R . Then $W((R)_n) = (W(R))_n$.

(4) If I is an ideal of R then $W(I) = W(R) \cap I$.

Theorem 2. (1) $W'(R)$ is the unique maximal fully idempotent ideal of R .

(2) $W'(R/W'(R)) = 0$.

(3) $W'((R)_n) = (W'(R))_n$.

(4) If I is an ideal of R then $W'(I) = W'(R) \cap I$.

Theorem 3. (1) $W^*(R)$ is the unique maximal s-unital ideal of R .

(2) $W^*(R/W^*(R)) = 0$.

(3) $W^*((R)_n) = (W^*(R))_n$.

Borrowing the idea from B. Brown and N. H. McCoy [1], V. Gupta

[3] has proved Theorem 1 for rings with identity. Moreover, the W -radical class and the W' -radical class are considered in [4] and [2], respectively. In case R is an integral domain, it is easy to see that R is left s -unital if and only if R contains 1. Accordingly, if R is an integral domain with 1 and I is an ideal of R with $R \supset I \supset 0$ then $W^*(I) = 0 \neq I = W^*(R) \cap I$, which shows that the W^* -radical class is not hereditary. Moreover, one may remark that if R is the subring of $(\text{GF}(2))_2$ consisting of 0, $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ then $eW(R)e = eW'(R)e = eW^*(R)e = 0 \neq eRe = W(eRe) = W'(eRe) = W^*(eRe)$.

In advance of proving our theorems we shall state three lemmas.

Lemma 1. *Let I be an ideal of R .*

- (1) *I is a fully left idempotent ideal if and only if $a \in (Ra)^2$ for any $a \in I$.*
- (2) *I is a fully idempotent ideal if and only if $a \in (RaR)^2$ for any $a \in I$.*

Proof. (1) If $a \in I$ and $a \in (Ra)^2$ then $(Ra)^2 = (Ra)^4 = (Ia)^2$. The converse is trivial.

(2) If $a \in I$ and $a \in (RaR)^2$ then $(RaR)^2 = (RaR)^6 = (IaI)^2$. The converse is trivial.

Lemma 2 (cf. [3, Lemma 2]). (1) *If $x \in (Ra)^2$ and $a - x \in (R(a - x))^2$ then $a \in (Ra)^2$.*

(2) *If $x \in (RaR)^2$ and $a - x \in (R(a - x)R)^2$ then $a \in (RaR)^2$.*

Proof. We shall prove only (1). Since $x \in (Ra)^2$, it follows $R(a - x) \subseteq Ra$. Hence, $a - x \in (Ra)^2$, which together with $x \in (Ra)^2$ implies $a \in (Ra)^2$.

The next is given in [5, Proposition 5 (1)], [2, Theorem 2.5] and [5, Corollary 1].

Lemma 3. (1) *R is fully left idempotent if and only if so is $(R)_n$.*

(2) *R is fully idempotent if and only if so is $(R)_n$.*

(3) *R is left s -unital if and only if so is $(R)_n$.*

Proof of Theorem 1. (1) By Lemma 1 (1), it suffices to show that if the principal ideals (u) and (v) are fully left idempotent then $u + v \in (R(u + v))^2$. Let $a = u + v$ and $x = \sum_i x_i a y_i a$, where $u = \sum_i x_i u y_i u$ ($x_i, y_i \in R$). Then $a - x = v - \sum_i x_i u y_i v - \sum_i x_i v y_i a \in (v)$, whence it follows $a - x \in$

$(R(a-x))^2$. Hence $a \in (Ra)^2$ by Lemma 2 (1).

(2) If $x \in (Ra)^2$ and $a-x \in W(R)$ then $a \in (Ra)^2$ by (1) and Lemma 2 (1). Now, (2) is obvious by Lemma 1 (1).

(3) Since $(W(R))_n \subseteq W((R)_n)$ by (1) and Lemma 3 (1), it remains only to prove the converse inclusion. Given $x \in R$, $E_{\lambda\mu}(x)$ will denote the element of $(R)_n$ with x in the (λ, μ) -position and zeros elsewhere. Let $A = (a_{ij})$ be an arbitrary element of $W((R)_n)$. Since $A \in ((R)_n A)^2$, there exist some $X_k = (x_{ij}^{(k)})$ and $Y_k = (y_{ij}^{(k)})$ such that $A = \sum_k X_k A Y_k$. Then by a brief computation we have

$$W((R)_n) \ni \sum_{k, \lambda, \mu} E_{1\lambda}(x_{p\lambda}^{(k)}) A E_{\mu 1}(y_{\mu q}^{(k)}) = E_{11}(a_{pq}).$$

Hence $W((R)_n) = (I)_n$, where I is the ideal of R consisting of all the elements x which appear in the $(1, 1)$ -position of some elements in $W((R)_n)$. Recalling that $E_{11}(a_{pq}) \in ((R)_n E_{11}(a_{pq}))^2$, one will easily see that $a_{pq} \in (Ra_{pq})^2$, namely, $I \subseteq W(R)$ (Lemma 1 (1)).

(4) By (1) and Lemma 1 (1), $W(R) \cap I \subseteq W(I)$. Conversely, if $a \in W(I)$ and $x \in R$ then $ax \in (Ia)^2 x \subseteq IaI \subseteq W(I)$ and similarly $xa \in W(I)$. Hence, $W(I)$ is an ideal of R and $W(I) \subseteq W(R) \cap I$ by (1).

Proof of Theorem 2. Although (1) and (2) are given in [2, Theorem 4.4], the proof is quite similar to that of Theorem 1 and may be left to readers.

Proof of Theorem 3. (1) Let I and J be s-unital ideals of R . If $a \in I$ and $b \in J$ then there exist some $f \in I$ and $g \in J$ such that $fa = a$ and $g(b-fb) = b-fb$. Obviously, $e = f + g - gf \in I + J$ and $e(a+b) = a+b$, and similarly $(a+b)e' = a+b$ for some $e' \in I + J$. Hence, $W^*(R)$ is s-unital.

(2) Let $I \supseteq W^*(R)$ be an ideal of R such that for any $a \in I$ there exist $e, e' \in I$ with $a - ea, a - ae' \in W^*(R)$. Then by (1) there exist $f, f' \in W^*(R)$ such that $f(a - ea) = a - ea$ and $(a - ae')f' = a - ae'$. Since $a = (e + f - fe)a = a(e' + f' - e'f')$ and $e + f - fe, e' + f' - e'f' \in I$, it follows $I \subseteq W^*(R)$.

(3) By (1) and Lemma 3 (3), $(W^*(R))_n \subseteq W^*((R)_n)$. If $A = (a_{ij}) \in W^*((R)_n)$ then there exist some $X = (x_{ij})$ and $Y = (y_{ij})$ such that $A = XA = AY = XAY$. It is easy to see that

$$W^*((R)_n) \ni \sum_{\lambda, \mu} E_{1\lambda}(x_{p\lambda}) A E_{\mu 1}(y_{\mu q}) = E_{11}(a_{pq}).$$

Hence $W^*((R)_n) = (I)_n$, where I is the ideal of R consisting of all the elements x which appear in the $(1, 1)$ -position of some elements in $W^*((R)_n)$. Recalling that there exist $Z, Z' \in W^*((R)_n)$ such that $E_{11}(a_{pq}) = ZE_{11}(a_{pq}) = E_{11}(a_{pq})Z'$, one will easily see $a_{pq} = za_{pq} = a_{pq}z'$ for some $z, z' \in I$, and hence $I \subseteq W^*(R)$.

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