

Mathematical Journal of Okayama University

Volume 18, Issue 2

1975

Article 8

JUNE 1976

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Jiro Kado*

*Okayama University

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NOTE ON EXCHANGE PROPERTY

Dedicated to Professor Kiiti Morita on his 60th birthday

JIRO KADO

Every module concerned in the present note will be a unital right module over a ring R with 1. Let η be a cardinal number. Then, a module M is said to have the η -exchange property, if for any direct sum $A = \bigoplus_{B \in \mathfrak{B}} B$ such that $\eta \geq |\mathfrak{B}|$ (the cardinal of the set $\mathfrak{B}$) and $A = M \oplus M'$ there exist submodules $B' \subseteq B$ such that $A = M \oplus (\bigoplus_{B \in \mathfrak{B}} B')$. We say that M has the *finite exchange property* if M has the η -exchange property for any finite η . These properties were defined and investigated by Crawley and Jónsson [1], and also studied by Harada [3] in connection with Krull-Schmidt-Azumaya theorem. Finally, M is said to have the η -exchange property w. r. t. *completely indecomposable modules*, if for any direct sum A of completely indecomposable modules $B \in \mathfrak{B}$ such that $\eta \geq |\mathfrak{B}|$ and $A = M \oplus M'$ there exist some $B' \subseteq B$ such that $A = M \oplus (\bigoplus_{B \in \mathfrak{B}} B')$.

In our subsequent study, $\mathfrak{N}$ will represent always a set of completely indecomposable modules. We consider a sequence $\{N_i\}_1^\infty$ of modules N_i in $\mathfrak{N}$ and non-isomorphisms $f_i: N_i \longrightarrow N_{i+1}$ ($i=1, 2, \dots$). The sequence $\{f_i\}_1^\infty$ will be called a *sequence of non-isomorphisms in $\mathfrak{N}$* . Such a sequence is said to be *proper* if $N_i \not\cong N_j$ for $i \neq j$. If for any $m \in N_1$ there exists a natural number n (depending on m) such that $f_n \cdots f_1(m) = 0$, then the sequence $\{f_i\}_1^\infty$ is said to be *locally T -nilpotent*. If every sequence (resp. proper sequence) of non-isomorphisms in $\mathfrak{N}$ is locally T -nilpotent then $\mathfrak{N}$ is defined to be *locally T -nilpotent* (resp. *locally semi- T -nilpotent*). Let $\mathfrak{N} = \bigcup_{\rho \in P} \mathfrak{N}(\rho)$ be the partition of $\mathfrak{N}$ into the isomorphism classes. Moreover, let $\mathfrak{N}' = \bigcup_{\rho \in P'} \mathfrak{N}(\rho)$ and $\mathfrak{N}'' = \bigcup_{\rho \in P''} \mathfrak{N}(\rho)$ where $P' = \{\rho \in P \mid |\mathfrak{N}(\rho)| < \aleph_0\}$ and $P'' = \{\rho \in P \mid |\mathfrak{N}(\rho)| \geq \aleph_0\}$. In their paper [4] Harada and Ishii proved that if $\mathfrak{N}'$ is locally T -nilpotent then $\bigoplus_{N \in \mathfrak{N}'} N$ has the $\aleph_0$ -exchange property. The purpose of this note is to prove the following theorems.

Theorem 1. *If $\mathfrak{N}'$ is locally semi- T -nilpotent then $\bigoplus_{N \in \mathfrak{N}'} N$ has the $\aleph_0$ -exchange property.*

Theorem 2. *If $\mathfrak{N}$ is locally semi- T -nilpotent and every N in $\mathfrak{N}$ is finitely generated, then $\bigoplus_{N \in \mathfrak{N}} N$ has the $\aleph_0$ -exchange property.*

In advance of proving our theorems, we claim that the statement of

[4, Lemma 8] is obscure and should be read as follows :

Lemma 1. *Assume that a module M has the finite exchange property. If $M \oplus M' = \bigoplus_{i=1}^{\infty} A_i$ and $K_i = \bigoplus_{i=1}^{\infty} A_i$, then there exist respective direct summands A'_i and K'_{i+1} of A_i and K_{i+1} such that $M \oplus M' = M \oplus (\bigoplus_{i=1}^n A'_i) \oplus K'_{n+1}$ for any n .*

Proof. For the sake of completeness, we shall give here the proof. Assume that we have found A'_i and K'_{i+1} ($i=1, 2, \dots, n$) such that $M \oplus M' = M \oplus (\bigoplus_{i=1}^m A'_i) \oplus K'_{m+1}$ for all $m \leq n$. We may set $K_{n+1} = K'_{n+1} \oplus K''_{n+1}$ with $K''_{n+1} \subseteq M \oplus (\bigoplus_{i=1}^n A'_i)$. Then, $M \oplus (\bigoplus_{i=1}^n A'_i) = K'_{n+1} \oplus L$ with some L . Let $A_i = A'_i \oplus A''_i$. Since $M \approx (\bigoplus_{i=1}^n A''_i) \oplus K''_{n+1}$, K''_{n+1} has the finite exchange property by [1, Lemma 3.10]. Then, by $K_{n+1} = A_{n+1} \oplus K_{n+2}$ there exist respective direct summands A'_{n+1} and K'_{n+2} of A_{n+1} and K_{n+2} such that $K_{n+1} = K'_{n+1} \oplus A'_{n+1} \oplus K'_{n+2}$. Thus, $M \oplus M' = M \oplus (\bigoplus_{i=1}^n A'_i) \oplus K'_{n+1} = K'_{n+1} \oplus L \oplus K'_{n+1} = K'_{n+1} \oplus A'_{n+1} \oplus K'_{n+2} \oplus L = M \oplus (\bigoplus_{i=1}^{n+1} A'_i) \oplus K'_{n+2}$.

Proof of Theorem 1. Let $M = \bigoplus_{N \in \mathfrak{N}} N$, $N = \bigoplus_{\rho \in P'} (\bigoplus_{N \in \mathfrak{N}(\rho)} N)$. Then M has the finite exchange property by [6, Prop. 1.7]. Assume $A = \bigoplus_{i=1}^{\infty} A_i = M \oplus M'$, and put $K_i = \bigoplus_{i=1}^{\infty} A_i$. By Lemma 1, there exist some A'_i and K'_{i+1} such that $A_i = A'_i \oplus A''_i$, $K_{i+1} = K'_{i+1} \oplus K''_{i+1}$ and $A = M \oplus (\bigoplus_{i=1}^n A'_i) \oplus K'_{n+1}$ for any n . Since $(\bigoplus_{i=1}^n A''_i) \oplus K''_{n+1} \approx M$, by Kanbra theorem and Krull-Schmidt-Azumaya theorem we have

$$(*) \quad A''_i = \bigoplus_{\rho \in P'} (\bigoplus_{B \in \mathfrak{B}(i, \rho)} B)$$

where every $B \in \mathfrak{B}(i, \rho)$ is isomorphic to $N \in \mathfrak{N}(\rho)$ and $\sum_i |\mathfrak{B}(i, \rho)| \leq |\mathfrak{N}(\rho)| < \aleph_0$. Suppose $M^* = M \oplus (\bigoplus_{i=1}^{\infty} A'_i) \neq A$. Then, noting that $A = (\bigoplus_{i=1}^{\infty} A'_i) \oplus (\bigoplus_{i=1}^{\infty} A''_i)$, we can find i_1 and ρ_1 such that $\mathfrak{B}(i_1, \rho_1)$ contains some B_1 which is not contained in M^* . Now, let $a_1 \in B_1 \setminus M^*$, and consider the projection $p: A \rightarrow K'_{m+1}$ with respect to the decomposition $A = M \oplus (\bigoplus_{i=1}^m A'_i) \oplus K'_{m+1}$ where $m = i_1$. Then $x_1 = p(a_1) \notin M^*$. Writing x_1 in $K_{m+1} = (\bigoplus_{i=m+1}^{\infty} A'_i) \oplus (\bigoplus_{i=m+1}^{\infty} A''_i)$, we have $x_1 = \sum_i x'_i + \sum_i x''_i$. There exists then some $i_2 > i_1$ such that $x''_{i_2} \notin M^*$. From (*) we can find some $\mathfrak{B}(i_2, \rho_2)$ which contains B_2 such that $q(x_1) \in B_2 \setminus M^*$, where $q: A \rightarrow B_2$ is the projection with respect to the decomposition $A = (\bigoplus_{i=1}^{\infty} A'_i) \oplus (\bigoplus_{\rho \in P'} (\bigoplus_{B \in \mathfrak{B}(i, \rho)} B))$. Putting $g_1 = qp|_{B_1}: B_1 \rightarrow B_2$, we have $a_2 = g_1(a_1) \in B_2 \setminus M^*$. Repeating the same argument to a_2 and so on, we can find $g_k: B_k \rightarrow B_{k+1}$ ($B_k \in \mathfrak{B}(i_k, \rho_k)$, $i_k < i_{k+1}$) such that $g_k \cdots g_1(a_1) \notin M^*$. Since $\sum_i |\mathfrak{B}(i, \rho_i)| < \aleph_0$, there exists a natural number k_1 such that $f_1 = g_{k_1} \cdots g_1: B_1 \rightarrow B_{k_1+1}$ is a non-isomorphism. Repeating the same procedure, we can

find natural numbers $k_1 < k_2 < \dots$ such that $f_n = g_{k_n} \dots g_{k_{n-1}+1}$ are non-isomorphisms. Obviously, $\{f_n\}_1^\infty$ may be regarded as a sequence of non-isomorphisms in $\mathfrak{N}$ and $f_n \dots f_1(a_1) \neq 0$ for any n . This contradiction shows that $M \oplus (\bigoplus_{i=1}^\infty A_i) = A$, namely, M has the $\mathfrak{K}_0$ -exchange property.

Corollary 1. *If $\mathfrak{N}$ is locally semi- T -nilpotent and $|P''| < \mathfrak{K}_0$, then $M = \bigoplus_{N \in \mathfrak{N}} N$ has the $\mathfrak{K}_0$ -exchange property.*

Proof. For every $\rho \in P''$, $\mathfrak{N}(\rho)$ is locally (semi-) T -nilpotent, and so $M_\rho = \bigoplus_{N \in \mathfrak{N}(\rho)} N$ has the $\mathfrak{K}_0$ -exchange property by [4, Lemma 4]. Hence, by [1, Lemma 3.10], $\bigoplus_{\rho \in P''} M_\rho$ has the same property. Combining this with Theorem 1, we see that $M = (\bigoplus_{N \in \mathfrak{N}'} N) \oplus (\bigoplus_{\rho \in P''} M_\rho)$ has the $\mathfrak{K}_0$ -exchange property again by [1, Lemma 3.10].

Proof of Theorem 2. Let $A = \bigoplus_{i=1}^\infty A_i = M \oplus M'$. As in the proof of Theorem 1, we can find A'_i and K'_{i+1} such that $A_i = A'_i \oplus A''_i$, $\bigoplus_{i=i-1}^\infty A_i = K'_{i+1} = K'_{i+1} \oplus K''_{i+1}$ and $A = M \oplus (\bigoplus_{i=1}^\infty A'_i) \oplus K'_{n+1}$ for any n . Let $p: A \rightarrow \bigoplus_{i=1}^\infty A'_i$ be the projection with respect to the decomposition $A = (\bigoplus_{i=1}^\infty A'_i) \oplus (\bigoplus_{i=1}^\infty A''_i)$. First we claim that if $\mathfrak{F}$ is an arbitrary finite subset of $\mathfrak{N}$ then $p(\bigoplus_{N \in \mathfrak{F}} N)$ is a direct summand of $\bigoplus_{i=1}^\infty A'_i$. Since $M^* = \bigoplus_{N \in \mathfrak{F}} N$ is finitely generated, $M^* \subseteq (\bigoplus_{i=1}^m A'_i) \oplus (\bigoplus_{i=1}^m A''_i)$ for some m . Accordingly, if $q: A \rightarrow (\bigoplus_{i=1}^m A'_i) \oplus K'_{m+1}$ is the projection with respect to the decomposition $A = (\bigoplus_{i=1}^m A'_i) \oplus K'_{m+1} \oplus (\bigoplus_{i=1}^m A''_i) \oplus K''_{m+1}$ then $p(M^*) = q(M^*)$. Since $q|_M$ is a monomorphism, $q(M) = q(M^*) \oplus L = p(M^*) \oplus L$ with some L , and so $p(M^*)$ is a direct summand of $(\bigoplus_{i=1}^m A'_i)$ and hence $\bigoplus_{i=1}^\infty A'_i$. Next, as in the proof of Theorem 1, we have

$$(*) \quad A'_i = \bigoplus_{\rho \in P} (\bigoplus_{B \in \mathfrak{B}(i, \rho)} B)$$

where every $B \in \mathfrak{B}(i, \rho)$ is isomorphic to $N \in \mathfrak{N}(\rho)$. Since $\mathfrak{N}$ is locally semi- T -nilpotent, $p(M)$ itself is a direct summand of $\bigoplus_{i=1}^\infty A'_i$ by [3, Th. 3.2.5]. Combining this with the fact that $p(M)$ has the $\mathfrak{K}_0$ -exchange property w. r. t. completely indecomposable modules (cf. [2, Cor. 2 of Prop. 1]), it follows $\bigoplus_{i=1}^\infty A'_i = p(M) \oplus (\bigoplus_{i=1}^\infty A''_i)$ with some $A''_i \subseteq A'_i$, and so $A = (\bigoplus_{i=1}^\infty A'_i) \oplus p(M) \oplus (\bigoplus_{i=1}^\infty A''_i)$. Noting that $p|_M$ is a monomorphism, we obtain eventually $A = M \oplus (\bigoplus_{i=1}^\infty (A'_i \oplus A''_i))$.

Corollary 2. *Let S be a direct sum of semi-perfect modules. Then, S is semi-perfect if and only if S has the $\mathfrak{K}_0$ -exchange property.*

Proof. By [3, Cor. 5.1.13], S is the direct sum of indecomposable semi-perfect modules: $S = \bigoplus_{T \in \mathfrak{T}} T$. If S is semi-perfect then every T is

cyclic (cf. for instance [3, Th. 5.2.4']) and $\mathfrak{A}$ is locally semi- T -nilpotent by [3, Cor. 5.1.13]. Hence, S has the $\mathfrak{K}_0$ -exchange property by Theorem 2. Conversely, if S has the $\mathfrak{K}_0$ -exchange property then $\mathfrak{A}$ is locally semi- T -nilpotent by [8, Th. 1], and so S is semi-perfect by [3, Cor. 2.2.2].

Corollary 3 (cf. [7, Th. 8]). *Assume that R is a direct sum of indecomposable right ideals. Then a projective module S is semi-perfect if and only if S has the $\mathfrak{K}_0$ -exchange property.*

Proof. By the validity of Corollary 2, it remains only to prove the "if" part. To our end, it suffices to prove that S is a direct sum of semi-perfect modules. By Kaplansky theorem, S is a direct sum of countably generated projective modules. In what follows, we may assume therefore that S is a direct summand of a countably generated free module $F = \bigoplus_{i=1}^{\infty} F_i$, where $F_{iR} \approx R_R$. Then $F = S \oplus (\bigoplus_{i=1}^{\infty} F'_i)$, where $F_i = F'_i \oplus F''_i$. Since $S \approx \bigoplus_{i=1}^{\infty} F''_i$, each F''_i has the $\mathfrak{K}_0$ -exchange property by [1, Lemma 3.10], and then one will easily see that F''_i is a direct sum of indecomposable modules. Hence, $S = \bigoplus_{j \in \mathfrak{J}} S_j$ with indecomposable S_j . Again by [1, Lemma 3.10], each S_j has the $\mathfrak{K}_0$ -exchange property, and so S_j is completely indecomposable by [5, Prop. 1]. Therefore, S_j is semi-perfect by [3, Th. 5.2.4'].

Acknowledgement. The author is grateful to Professor M. Harada for his useful comments.

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DEPARTMENT OF MATHEMATICS,
OKAYAMA UNIVERSITY

(Received February 28, 1976)