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A NON-IMMERSION RESULT FOR LENS SPACES $L^n(2^m)$

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1. Introduction. The lens space $L^n(2^m)$ is the quotient of the sphere S^{2n+1} by the free action of the cyclic group $Z/2^m$ given by:

$$\zeta^k z = (\zeta^k z_0, \zeta^k z_1, \dots, \zeta^k z_n),$$

where $\zeta = \exp(i\pi/2^{m-1})$ is the generator of $Z/2^m$, and $z = (z_0, z_1, \dots, z_n) \in \mathbf{C}^{n+1}$ is such that $\sum_{i=0}^n |z_i|^2 = 1$. A classical question is to determine the smallest integer k such that $L^n(2^m)$ immerses into $\mathbf{R}^{2n+1+k}$. In [3], we have seen that for m sufficiently large, k is greater or equal than $2n - 2\alpha(n)$, where $\alpha(n)$ denotes the number of 1 in the dyadic expansion of n . More precisely, we have proved the following theorem

Theorem 1.1. *For $m \geq [\log_2 n] + [n/2]$, $L^n(2^m)$ does not immerse into $\mathbf{R}^{4n-2\alpha(n)}$.*

Here $[x]$ denotes the integer part of x . Some other results have been published in the same direction, (see [1], [5], [6] and [7]). In this note, we are completing theorem 1.1 for the case $m \leq [\log_2 n] + [n/2] - 1$. Let $l(n)$ be the integer

$$l(n) = \max \left\{ 1 \leq i \leq n-1 \text{ such that } \binom{n+i+1}{n} \not\equiv 0 \pmod{4} \right\}.$$

We prove:

Theorem 1.2. *Let $m \geq 2$.*

- a) *If $n \neq 2^s + 1$ and $n \geq 2$, $L^n(2^m)$ does not immerse in $\mathbf{R}^{2n+1+2l(n)}$.*
- b) *If $n = 2^s + 1$, with $s \geq 1$, $L^n(2^m)$ does not immerse into $\mathbf{R}^{2n+2l(n)} = \mathbf{R}^{4n-4}$.*

We apply theorem 1.2 to some particular values of n , and we obtain

Corollary 1.1. *Let $m \geq 2$.*

- a) *If $n = 2^s$ with $s \geq 1$, $L^n(2^m)$ does not immerse in $\mathbf{R}^{4n-1}$.*
- b) *If $n = 2^s + 2^t$, with $s > t \geq 1$, $L^n(2^m)$ does not immerse in $\mathbf{R}^{4n-3}$.*

This improves for these two cases the results obtained by theorem 1.1. Recalling that for $n = 2^s$, the space $L^n(2^m)$ immerses in $\mathbf{R}^{4n}$, we note that our result is the best possible for this case.

2. Preliminaries. In this section we establish some cohomology properties of the spaces $B(n, k)$ defined in [4] (see also [2]). This properties will be used to prove theorem 1.2. We begin with a result about spherical fibrations and recall that for any sphere bundle $S^k \xrightarrow{i} E \xrightarrow{p} B$ there is long exact sequence of $H^*(B; \mathbf{Z})$ -modules called the Gysin sequence (see [8] p.143 or [9] p.356)

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H^q(B; \mathbf{Z}) & \xrightarrow{p^*} & H^q(E; \mathbf{Z}) & \xrightarrow{\phi} & H^{q-k}(B; \mathbf{Z}) \xrightarrow{\cup e} \\ & & & & & & H^{q+1}(B; \mathbf{Z}) \longrightarrow \cdots \end{array}$$

where e is the Euler-class of the fibration. In particular we have:

Lemma 2.1. *If in the above spherical fibration, B is connected and the Euler-class e is zero, then*

$$H^*(E; \mathbf{Z}) \cong H^*(B; \mathbf{Z}) \oplus a \cup H^*(B; \mathbf{Z})$$

as an $H^*(B; \mathbf{Z})$ -module, where a is an element of $H^k(E; \mathbf{Z})$ such that $\phi(a)$ is a generator of $H^0(B; \mathbf{Z}) \cong \mathbf{Z}$.

The proof of this lemma is straightforward.

We now turn to the space $B(n, k)$ which by definition is the pull-back space of the diagram

$$\begin{array}{ccc} & & BSO(k) \\ & & \downarrow \\ BU(n) & \longrightarrow & BSO(2n) \end{array}$$

Inductively we can identify the space $B(n, k)$ with the pull-back of the diagram

$$\begin{array}{ccc} & & BSO(k) \\ & & \downarrow \\ B(n, k+1) & \longrightarrow & BSO(k+1) \end{array}$$

Let be $V_{2n, 2n-2j}$ the Stiefel manifold $SO(2n)/SO(2j)$, and let

$$(2.1) \quad V_{2n, 2n-2j} \xrightarrow{i_1} B(n, 2j) \xrightarrow{p} BU(n)$$

be the fibration induced from

$$V_{2n,2n-2j} \xrightarrow{i_2} BSO(2j) \longrightarrow BSO(2n)$$

by the canonical map $BU(n) \xrightarrow{r_n} BSO(2n)$.

Let u_j be the generator of $H^{2j}(V_{2n,2n-2j}; \mathbb{Z}) \cong \mathbb{Z}$ such that

$$i_2^*(e_j) = -2u_j,$$

where $e_j \in H^{2j}(BSO(2j); \mathbb{Z})$ is the universal Euler-Poincaré class. By the pull-back property, there is a map $BU(j) \xrightarrow{h} B(n, 2j)$ and a commutative diagramm

$$\begin{array}{ccccc} BU(j) & & & & \\ & \searrow h & & \searrow r_j & \\ & & B(n, 2j) & \xrightarrow{f_{2j}} & BSO(2j) \\ & \searrow g_j & \downarrow p & & \downarrow \\ & & BU(n) & \xrightarrow{r_n} & BSO(2n) \end{array}$$

where all the others maps are canonical maps.

Lemma 2.2. *For every $n \geq 1$ and $1 \leq j \leq n-1$, there is an element a_j in the abelian group $H^{2j}(B(n, 2j); \mathbb{Z})$ such that*

$$f_{2j}^*(e_j) = p^*(c_j) - 2a_j, \quad i_1^*(a_j) = u_j, \quad h^*(a_j) = 0.$$

Proof. There is an exact sequence coming from the Serre spectral sequence of the fibration (2.1)

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^{2j}(BU(n); \mathbb{Z}) & \xrightarrow{p^*} & H^{2j}(B(n, 2j); \mathbb{Z}) & \xrightarrow{i_1^*} & \\ & & & & H^{2j}(V_{2n,2n-2j}; \mathbb{Z}) & \longrightarrow & 0 \end{array}$$

since $V_{2n,2n-2j}$ is $(2j-1)$ -connected and $BU(n)$ is 1-connected without cohomology in odd degree. Let $x \in H^{2j}(B(n, 2j); \mathbb{Z})$ be such that $i^*(x) = u_j$. Since the map g_j^* is an isomorphism in degree $\leq 2j$ we can replace x by $a_j = x - p^*((g_j^*)^{-1}(h^*(x)))$ so that $h^*(a_j) = 0$. The above exact sequence splits and we have an isomorphism

$$H^{2j}(B(n, 2j); \mathbb{Z}) \cong \text{im}(p^*) \oplus \mathbb{Z}a_j \cong H^{2j}(BU(n); \mathbb{Z}) \oplus \mathbb{Z}a_j.$$

On the other hand $h^*(f_{2j}^*(e_j)) = r_j^*(e_j) = c_j$ so $f_{2j}^*(e_j) = p^*(c_j) + ma_j$. As $i_1^*((f_{2j}^*(e_j))) = i_2^*(e_j) = -2u_j$ we see that $m = -2$ and $f_{2j}^*(e_j) = p^*(c_j) - 2a_j$.

Let now

$$(2.2) \quad S^{r-1} \longrightarrow B(n, r-1) \xrightarrow{p_{r-1}} B(n, r)$$

be the spherical fibration induced from

$$S^{r-1} \longrightarrow BSO(r-1) \longrightarrow BSO(r)$$

by the map $B(n, r) \xrightarrow{f_r} BSO(r)$. We consider the Gysin sequence of (2.2) which becomes, using lemma 2.2,

$$\begin{aligned} \dots \longrightarrow H^0(B(n, 2j); \mathbb{Z}) &\xrightarrow{\cup(p^*(c_j) - 2a_j)} H^{2j}(B(n, 2j); \mathbb{Z}) \xrightarrow{p_{2j-1}^*} \\ &H^{2j}(B(n, 2j-1); \mathbb{Z}) \longrightarrow \dots \end{aligned}$$

By exactness, $p_{2j-1}^*(p^*(c_j)) = 2p_{2j-1}^*(a_j)$.

In the following, we note $p_{2j-1}^*(a_j) = b_j$, more generally, $(p_k^* \circ p_{k+1}^* \circ \dots \circ p_{2j-1}^*)(a_j) = b_j$ and for simplicity $(p_k^* \circ p_{k+1}^* \circ \dots \circ p_{2j-1}^*)(p^*(c_j)) = c_j$. So we have for every space $B(n, k)$ a family of elements b_i , $[k/2] + 1 \leq i \leq n-1$, such that $2b_i = c_i$. We can now give the additive structure of $H^*(B(n, k); \mathbb{Z})$. This result already appears in [2] and [4].

Theorem 2.1. $H^*(B(n, k); \mathbb{Z})$ is a free $\mathbb{Z}$ -module determined by the isomorphism

$$H^*(B(n, k); \mathbb{Z}) \cong \begin{cases} \mathbb{Z}[c_1, \dots, c_t] \otimes \Delta(a_t, b_{t+1}, \dots, b_{n-1}) & \text{if } k = 2t \\ \mathbb{Z}[c_1, \dots, c_t] \otimes \Delta(b_{t+1}, \dots, b_{n-1}) & \text{if } k = 2t + 1 \end{cases}$$

where $\Delta(x_1, \dots, x_m)$ is the free abelian group generated by the elements

$$x_{i_1} x_{i_2} \dots x_{i_s}, \quad 1 \leq i_1 < i_2 < \dots < i_s \leq m.$$

Proof of theorem 2.1. We proceed by induction descending over k , beginning with $k = 2n - 1$. In this case the result is valid since $B(n, 2n - 1) = BU(n - 1)$. Next we examine the case $k = 2n - 2$. Here we consider the spherical fibration (2.2) with $r = 2n - 1$. As $H^{2n-1}(BU(n-1); \mathbb{Z}) = 0$, the Euler class of this fibration is 0 and the Gysin sequence splits into short exact sequences

$$\begin{aligned} 0 \longrightarrow H^{2q}(BU(n-1); \mathbb{Z}) &\xrightarrow{p_{2n-2}^*} H^{2q}(B(n, 2n-2); \mathbb{Z}) \xrightarrow{\phi} \\ &H^{2q-2n+2}(BU(n-1); \mathbb{Z}) \longrightarrow 0. \end{aligned}$$

In lemma 2.2 we have seen that

$$H^{2n-2}(B(n, 2n-2); \mathbb{Z}) \cong \text{im}(p^*) \oplus \mathbb{Z}a_{n-1}.$$

But $\text{im}(p^*) \subset \text{im}(p_{2n-2}^*) = \ker(\phi)$, so the element $\phi(a_{n-1})$ is a generator of $H^0(BU(n-1); \mathbb{Z}) = \mathbb{Z}$. Under the map p_{2n-2}^* we have a $H^*(BU(n-1); \mathbb{Z})$ -module structure over $H^*(B(n, 2n-2); \mathbb{Z})$. With the help of lemma 2.1, we can see that this structure is given by the isomorphism

$$\begin{aligned} H^*(B(n, 2n-2); \mathbb{Z}) &\cong H^*(BU(n-1); \mathbb{Z}) \oplus a_{n-1} \cup H^*(BU(n-1); \mathbb{Z}) \\ &\cong \mathbb{Z}[c_1, \dots, c_{n-1}] \otimes \Delta(a_{n-1}) \end{aligned}$$

this achieves the proof in this case. Moreover the multiplicative structure is well-known in this case, since

$$\begin{aligned} (c_{n-1} - 2a_{n-1})^2 &= f_{2n-2}^*(e_{n-1}^2) = f_{2n-2}^*(P_{n-1}) \\ &= p^*(r_n^*(P_{n-1})) = p^*(c_{n-1}^2 - 2c_{n-2}c_n) \\ &= c_{n-1}^2 \end{aligned}$$

where P_{n-1} is the $(n-1)^{\text{th}}$ -Pontrjagin class in $H^{4n-4}(BSO(2n-2); \mathbb{Z})$ or in $H^{4n-4}(BSO(2n); \mathbb{Z})$. The relations used here are proved for example in [8] (see also proof of lemma (2.3)). So we have $a_{n-1}^2 = c_{n-1}a_{n-1}$.

Now we suppose that the result is valid for $r \leq k \leq 2n-1$. We consider the Gysin sequence of the sphere bundle (2.2):

$$\begin{aligned} \dots \longrightarrow H^q(B(n, r); \mathbb{Z}) &\xrightarrow{p_{r-1}^*} H^q(B(n, r-1); \mathbb{Z}) \xrightarrow{\phi} \\ &H^{q-r+1}(B(n, r); \mathbb{Z}) \xrightarrow{\cup e} H^{q+1}(B(n, r); \mathbb{Z}) \longrightarrow \dots \end{aligned}$$

1) If r is odd, say $r = 2j+1$, we prove exactly as above that

$$\begin{aligned} H^*(B(n, 2j); \mathbb{Z}) &\cong H^*(B(n, 2j+1); \mathbb{Z}) \oplus a_j \cup H^*(B(n, 2j+1); \mathbb{Z}) \\ &\cong H^*(B(n, 2j+1); \mathbb{Z}) \otimes \Delta(a_j) \\ &\cong \mathbb{Z}[c_1, \dots, c_j] \otimes \Delta(a_j, b_{j+1}, \dots, b_{n-1}). \end{aligned}$$

Moreover the group homomorphism

$$\begin{aligned} \psi_n : \mathbb{Z}[c_1, \dots, c_{j-1}, c_j - 2a_j] &\otimes \Delta(a_j, b_{j+1}, \dots, b_{n-1}) \\ &\longrightarrow H^*(B(n, 2j); \mathbb{Z}) \end{aligned}$$

defined by $\psi_n(x \otimes y) = x \cup y$, is an isomorphism.

We proceed by induction on n , beginning with $n = j + 1$. In this case, the morphism ψ_n becomes

$$\psi_{j+1}: \mathbf{Z}[c_1, \dots, c_{j-1}, c_j - 2a_j] \otimes \Delta(a_j) \longrightarrow H^*(B(j+1, 2j); \mathbf{Z}).$$

We have seen above that $a_j^2 = a_j c_j$ in $H^*(B(j+1, 2j); \mathbf{Z})$, so

$$\begin{aligned} c_j &= \psi_{j+1}((c_j - 2a_j) \otimes 1 + 2(1 \otimes a_j)) \\ c_j^2 &= \psi_{j+1}((c_j - 2a_j)^2 \otimes 1) \end{aligned}$$

and ψ_{j+1} is surjective. As we have a bijection between the $\mathbf{Z}$ -module basis, ψ_{j+1} is an isomorphism.

Suppose now that the result is true for $n - 1$, and let $h: B(n - 1, 2j) \rightarrow B(n, 2j)$ the map induced by the pull-back property of $B(n, 2j)$. Let

$$A = \mathbf{Z}[c_1, \dots, c_{j-1}, c_j] \otimes \Delta(a_j, b_{j+1}, \dots, b_{n-2})$$

and

$$B = \mathbf{Z}[c_1, \dots, c_{j-1}, c_j - 2a_j] \otimes \Delta(a_j, b_{j+1}, \dots, b_{n-2}).$$

By definition of h^* we have a short exact sequence

$$\ker(h^*) \longrightarrow H^*(B(n, 2j); \mathbf{Z}) \xrightarrow{h^*} H^*(B(n - 1, 2j); \mathbf{Z})$$

where $\ker(h^*) = A \cup b_{n-1}$. We also have a $\mathbf{Z}$ -modules isomorphism

$$H^*(B(n, 2j); \mathbf{Z}) \cong A \oplus A \cup b_{n-1}.$$

If $(x_q)_{q \geq 1}$ is the canonical $\mathbf{Z}$ -module basis of A , $(x_q \cup b_{n-1})_{q \geq 1}$ is a basis of $A \cup b_{n-1}$. Since h^* is a ring homomorphism, we have the commutative diagramm

$$\begin{array}{ccc} & B & \\ \psi_n|B \swarrow & & \searrow \psi_{n-1}|B \\ H^*(B(n, 2j); \mathbf{Z}) & \xrightarrow{h^*} & H^*(B(n - 1, 2j); \mathbf{Z}) \end{array}$$

By the induction hypothesis, ψ_{n-1} is an isomorphism and so $\psi_n|B$ is a monomorphism and there is a basis $(y_q)_{q \geq 1}$ of B , such that

$$\psi_n(y_q) = x_q + z_q \cup b_{n-1} \quad \text{for } q \geq 1, \text{ with } z_q \in A.$$

As $b_{n-1}^2 = 0$ in $H^*(B(n, 2j); \mathbf{Z})$, $\psi_n|_{B \cup b_{n-1}}$ is injective and $\psi_n(B \cup b_{n-1}) = A \cup b_{n-1}$.

2) If r is even, say $r = 2j$, we know by lemma 2.2 that the Euler-class of the spherical fibration (2.2) is the element $c_j - 2a_j$ and since ψ_n is injective, we can say that the multiplication by the Euler-class is injective, so the map $\phi = 0$ in the Gysin sequence of (2.2) and we have the group isomorphisms

$$\begin{aligned} H^*(B(n, 2j-1); \mathbf{Z}) &\cong H^*(B(n, 2j); \mathbf{Z}) / \langle c_j - 2a_j \rangle \\ &\cong \mathbf{Z}[c_1, \dots, c_{j-1}] \otimes \Delta(a_j, b_{j+1}, \dots, b_{n-1}). \end{aligned}$$

We can now describe the multiplicative structure of $H^*(B(n, 2j); \mathbf{Z})$ as follows.

Lemma 2.3. *For every $n \geq 1$ and $1 \leq j \leq n-1$, the element a_j in the abelian group $H^*(B(n, 2j); \mathbf{Z})$ satisfies the relation*

$$(2.3) \quad a_j^2 = a_j c_j + (-1)^j \sum_{r=j+1}^{\min(2j, n-1)} (-1)^r b_r c_{2j-r}.$$

Proof. Recall that the universal Euler-Poincaré class $e_j \in H^{2j}(BSO(2j); \mathbf{Z})$, satisfies the relation

$$e_j^2 = P_j$$

where P_j is the j^{th} universal Pontrjagin class in $H^{4j}(BSO(2j); \mathbf{Z})$, and that

$$r_n^*(P_j) = c_j^2 + (-1)^j \sum_{r=j+1}^{\min(2j, n)} (-1)^r 2c_r c_{2j-r}$$

in $H^{4j}(BU(n); \mathbf{Z})$, here P_j is the j^{th} universal Pontrjagin class in $H^{4j}(BSO(2n); \mathbf{Z})$, (see [8]). From the definition of a_j and the above relations, we see that

$$f_{2j}^*(e_j^2) = c_j^2 - 4a_j c_j + 4a_j^2$$

and

$$\begin{aligned} p^*(r_n^*(P_j)) &= c_j^2 + (-1)^j \sum_{r=j+1}^{\min(2j, n-1)} (-1)^r 2c_r c_{2j-r} \\ &= c_j^2 + (-1)^j \sum_{r=j+1}^{\min(2j, n-1)} (-1)^r 4b_r c_{2j-r}. \end{aligned}$$

Since $H^*(B(n, 2j); \mathbb{Z})$ has no torsion, the relation (2.3) is valid.

Using the relation (2.3) we can now give the Steenrod squares of the mod 2 reduction of the elements a_j in $H^{2j}(B(n, 2j); \mathbb{Z}/2)$.

Theorem 2.2. *For every $n \geq 1$, every $1 \leq j \leq n - 1$ and every $0 \leq k \leq j$ the following relation is valid in $H^*(B(n, 2j); \mathbb{Z}/2)$.*

$$(2.4) \quad Sq^{2k}(a_j) = \sum_{r=\max(0, k+j+1-n)}^{k-1} \binom{j-r}{k-r} b_{k+j-r} c_r + a_j c_k.$$

Proof. We proceed by an induction argument over n . We begin with the case $n = 1$ where all relations are empty. For $n = 2$, $j = 1$ and $k = 0$ or 1, so the only non trivial relation in $H^*(B(2, 2); \mathbb{Z}/2)$ is $Sq^2(a_1) = a_1^2 = a_1 c_1$ which is compatible with (2.4).

Now we suppose the result is valid for $n \geq 2$. First we observe that (2.4) is still true for $k + j \leq n - 1$ in $H^*(B(n + 1, 2j); \mathbb{Z}/2)$ since

$$H^q(B(n + 1, 2j); \mathbb{Z}/2) \cong H^q(B(n, 2j); \mathbb{Z}/2) \quad q < 2n.$$

If $k + j \geq n$, we consider the following diagram, where all the arrows are canonical.

$$\begin{array}{ccccc} B(n, 2j - 2) \times \mathbb{C}P^\infty & \xrightarrow{\quad\quad\quad} & BSO(2j - 2) \times \mathbb{C}P^\infty & & \\ \downarrow & & \downarrow & & \downarrow \\ & B(n + 1, 2j) \longrightarrow & BSO(2j) & & \\ \downarrow & & \downarrow & & \downarrow \\ BU(n) \times \mathbb{C}P^\infty & \longrightarrow & BU(n + 1) & \longrightarrow & BSO(2n + 2) \end{array}$$

In particular the next square is homotopy commutative

$$(2.5) \quad \begin{array}{ccc} B(n, 2j - 2) \times \mathbb{C}P^\infty & \longrightarrow & BSO(2j) \\ \downarrow & & \downarrow \\ BU(n + 1) & \longrightarrow & BSO(2n + 2) \end{array}$$

and replacing if necessary $B(n, 2j - 2) \times \mathbb{C}P^\infty \rightarrow BSO(2j)$ by a map homotopy equivalent, we can suppose that the diagram (2.5) is commutative since the map $BSO(2j) \rightarrow BSO(2n + 2)$ is a fibration.

So there is a map $f: B(n, 2j-2) \times CP^\infty \rightarrow B(n+1, 2j)$ such that the squares

$$\begin{array}{ccc} B(n, 2j-2) \times CP^\infty & \longrightarrow & BSO(2j-2) \times CP^\infty \\ f \downarrow & & \downarrow \\ B(n+1, 2j) & \longrightarrow & BSO(2j) \end{array}$$

and

$$\begin{array}{ccc} B(n, 2j-2) \times CP^\infty & \xrightarrow{f} & B(n+1, 2j) \\ \downarrow & & \downarrow \\ BU(n) \times CP^\infty & \longrightarrow & BU(n+1) \end{array}$$

are still commutative. We can easy see that

$$\begin{aligned} f^*(c_i) &= c_i + c_{i-1}z \quad (1 \leq i \leq j), & f^*(a_j) &= b_j + a_{j-1}z, \\ f^*(b_i) &= b_i + b_{i-1}z \quad (j+1 \leq i \leq n-1), & f^*(b_n) &= b_{n-1}z \end{aligned}$$

in $H^*(B(n, 2j-2) \times CP^\infty; \mathbb{Z}) \cong H^*(B(n, 2j-2); \mathbb{Z}) \otimes H^*(CP^\infty; \mathbb{Z})$, where z is the canonical generator of $H^2(CP^\infty; \mathbb{Z})$.

Let be $G = \mathbb{Z}/2[c_1, \dots, c_{j-1}] \otimes (\mathbb{Z}/2\langle a_j \rangle \oplus \mathbb{Z}/2\langle b_{j+1} \rangle \oplus \dots \oplus \mathbb{Z}/2\langle b_n \rangle)$, where $\mathbb{Z}/2\langle x \rangle$ is the group of order two with generator x . It is clear that G is a subgroup of $H^*(B(n+1, 2j); \mathbb{Z}/2)$ and we can easy see that the restriction of f^* to G is injective. Let $h: B(n, 2j) \rightarrow B(n+1, 2j)$ be the canonical map as in the proof of theorem 2.1. For $j > 1$, $k < j$ and $k+j \geq n$,

$$\begin{aligned} h^*(Sq^{2k}(a_j)) &= Sq^{2k}(h^*(a_j)) = Sq^{2k}(a_j) \\ &= \sum_{r=k+j+1-n}^{k-1} \binom{j-r}{k-r} b_{k+j-r} c_r + a_j c_k \end{aligned}$$

by the induction hypothesis, and since $\ker(h^*) = b_n \cup H^*(B(n, 2j); \mathbb{Z}/2)$, we have

$$Sq^{2k}(a_j) = \sum_{r=k+j+1-n}^{k-1} \binom{j-r}{k-r} b_{k+j-r} c_r + a_j c_k + b_n p(c_1, \dots, c_{j+k-n})$$

where $p(c_1, \dots, c_{j+k-n}) \in \mathbb{Z}/2[c_1, \dots, c_{j-1}]$. Then, the element $Sq^{2k}(a_j)$ is in G and we can give its image under f^*

$$\begin{aligned} f^*(Sq^{2k}(a_j)) &= Sq^{2k}(f^*(a_j)) = Sq^{2k}(b_j + a_{j-1}z) \\ &= Sq^{2k}(b_j) + Sq^{2k}(a_{j-1})z + Sq^{2k-2}(a_{j-1})z^2. \end{aligned}$$

Applying once more the induction hypothesis,

$$\begin{aligned} f^*(Sq^{2k}(a_j)) &= \sum_{r=k+j+1-n}^k \binom{j-r}{k-r} b_{k+j-r} c_r \\ &\quad + \sum_{r=k+j-n}^{k-1} \binom{j-1-r}{k-r} b_{k+j-1-r} c_r z \\ &\quad + \sum_{r=\max(0, k+j-1-n)}^{k-2} \binom{j-1-r}{k-1-r} b_{k+j-2-r} c_r z^2 \\ &\quad + a_{j-1} c_k z + a_{j-1} c_{k-1} z^2 \end{aligned}$$

and since $\binom{j-1-r}{k-r} \equiv \binom{j-1-r}{k-1-r} + \binom{j-r}{k-r} \pmod{2}$,

$$\begin{aligned} f^*(Sq^{2k}(a_j)) &= \sum_{r=k+j+1-n}^k \binom{j-r}{k-r} b_{k+j-r} (c_r + c_{r-1} z) \\ &\quad + \sum_{r=k+j-n}^{k-1} \binom{j-r}{k-r} b_{k+j-1-r} c_r z \\ &\quad + \sum_{r=\max(1, k+j-n)}^{k-1} \binom{j-r}{k-r} b_{k+j-1-r} c_{r-1} z^2 \\ &\quad + a_{j-1} z (c_k + c_{k-1} z). \end{aligned}$$

If $k+j > n$ we have

$$\begin{aligned} f^*(Sq^{2k}(a_j)) &= \sum_{r=k+j+1-n}^k \binom{j-r}{k-r} b_{k+j-r} (c_r + c_{r-1} z) \\ &\quad + \sum_{r=k+j-n}^{k-1} \binom{j-r}{k-r} b_{k+j-1-r} z (c_r + c_{r-1} z) \\ &\quad + a_{j-1} z (c_k + c_{k-1} z) \\ &= \sum_{r=k+j+1-n}^{k-1} \binom{j-r}{k-r} (b_{k+j-r} + b_{k+j-1-r} z) (c_r + c_{r-1} z) \\ &\quad + (b_j + a_{j-1} z) (c_k + c_{k-1} z) \\ &\quad + \binom{n-k}{n-j} b_{n-1} z (c_{j+k-n} + c_{j+k-1-n} z) \\ &= f^* \left(\sum_{r=\max(0, k+j-n)}^{k-1} \binom{j-r}{k-r} b_{k+j-r} c_r + a_j c_k \right) \end{aligned}$$

as expected since $f^*|_G$ is injective. If $k+j = n$, we proceed exactly as above. It remains two cases, the first is for $j = 1$, but the only non trivial

Steenrod operations are $Sq^0(a_1) = a_1$ and $Sq^2(a_1) = a_1^2 = a_1 c_1 + b_2$ by lemma 2.3. The second is for $k = j$ but in this case

$$\begin{aligned} Sq^{2j}(a_j) &= a_j^2 = a_j c_j + \sum_{r=2j-n}^{j-1} b_{2j-r} c_r \\ &= \sum_{r=2j-n}^{j-1} \binom{j-r}{j-r} b_{2j-r} c_r + a_j c_j \end{aligned}$$

always by lemma 2.3.

3. Proof of Theorem 1.2. The integral cohomology and the mod 2 cohomology of $L^n(2^m)$ are well known, they are given by the isomorphisms of abelian groups:

$$\begin{aligned} H^q(L^n(2^m); \mathbb{Z}) &\cong \begin{cases} \mathbb{Z} & \text{if } q = 0, 2n+1 \\ \mathbb{Z}/2^m & \text{if } q = 2i, 0 < i \leq n \\ 0 & \text{otherwise,} \end{cases} \\ H^q(L^n(2^m); \mathbb{Z}/2) &\cong \begin{cases} \mathbb{Z}/2 & \text{if } 0 \leq q \leq 2n+1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Let be $\pi: L^n(2^m) \rightarrow \mathbb{C}P^n$ the natural projection, μ the canonical complex line bundle over $\mathbb{C}P^n$, and let denote $z = c_1(\pi^*(\mu)) = \pi^*(c_1(\mu)) \in H^2(L^n(2^m); \mathbb{Z})$. We observe that z^i is an additive generator of $H^{2i}(L^n(2^m); \mathbb{Z})$ for every $1 \leq i \leq n$.

Let us still write z^i for the mod 2 reduction of the additive generator above, we see readily that

$$(3.1) \quad Sq^2(z^i) = i z^{i+1}$$

$$(3.2) \quad Sq^4(z^i) = \binom{i}{2} z^{i+2}.$$

Finally, let $l(n)$ denote the integer

$$l(n) = \max \left\{ 0 \leq i \leq n-1 \text{ such that } \binom{n+i+1}{n} \not\equiv 0 \pmod{4} \right\}.$$

Recall the 2-divisibility of $\binom{n+i+1}{n}$:

$$\nu_2\left(\binom{n+i+1}{n}\right) = \alpha(n) + \alpha(i+1) - \alpha(n+i+1).$$

We observe that for $\alpha(n) = 1$ and $i = n-1$, we get $\nu_2\left(\binom{n+i+1}{n}\right) = \alpha(n) = 1$ and so $l(n) = n-1$. For $\alpha(n) = 2$ we obtain, likewise, $l(n) = n-2$.

For $\alpha(n) \geq 3$, we have the next result where we relate $l(n)$ with the dyadic expansion of n .

Lemma 3.1. *If $n = 2^{s_1} + 2^{s_2} + \dots + 2^{s_k}$ with $s_1 > s_2 > \dots > s_k \geq 0$ and $k \geq 3$, $l(n) = 2^{s_1} + 2^{s_2} - 2 - 2^{s_3} - \dots - 2^{s_k}$.*

Proof. The 2-divisibility of $\binom{n+i+1}{n}$ is 0 or 1 if n and $i+1$ have at most one common term in their dyadic expansion. So $i+1$ is greatest possible, if there is one common term of highest 2-valuation, here 2^{s_1} . The rest of the expansion of $i+1$ contains all powers 2^r with $r < s_2$, except $r = s_3, \dots, s_k$.

This description of $l(n)$ gives for $\alpha(n) \geq 3$:

$$(3.3) \quad l(n) \equiv \begin{cases} 2 \pmod{4} & \text{if } n \equiv 0 \pmod{4} \\ 1 \pmod{4} & \text{if } n \equiv 1 \pmod{4} \\ 0 \pmod{4} & \text{if } n \equiv 2 \pmod{4} \\ 3 \pmod{4} & \text{if } n \equiv 3 \pmod{4} \end{cases}.$$

We come back to the immersion problem for $L^n(2^m)$. We know that the stable class of the tangent bundle of $L^n(2^m)$ is $r(n+1)\sigma$ (see [10]), where r denotes the realification. So, if $L^n(2^m)$ immerses in $\mathbf{R}^{2n+1+k}$, the stable class of the normal bundle of this immersion is $-r(n+1)\sigma$ and its classifying map

$$-r(n+1)\sigma: L^n(2^m) \longrightarrow BSO(2n+2)$$

lifts to $BU(n+1)$ and to $BSO(k)$. Therefore, this map also lifts to $B(n+1, k)$, and we obtain the commutative diagram

$$\begin{array}{ccc} & B(n+1, k) & \\ \tilde{f}_k \nearrow & \downarrow p & \\ L^n(2^m) & \xrightarrow{g} & BU(n+1) \end{array}$$

where $g: L^n(2^m) \rightarrow BU(n+1)$ denotes a lifting of $-r(n+1)\sigma$ to $BU(n+1)$ and $\tilde{f}_k$ a lifting of g in $B(n+1, k)$. We also note that

$$\begin{aligned} g^*(c_i) &= c_i(-(n+1)\sigma) = \binom{-n-1}{i} z^i \\ &= (-1)^i \binom{n+i}{n} z^i \end{aligned}$$

since for the total Chern class of $-(n+1)\sigma$ we find

$$\begin{aligned} c(-(n+1)\sigma) &= c(\sigma)^{-n-1} = (1 + c_1(\sigma))^{-n-1} = (1+z)^{-n-1} \\ &= \sum_{i \geq 0} \binom{-n-1}{i} z^i. \end{aligned}$$

For $i \geq [k/2] + 1$, we have $p^*(c_i) = 2b_i$ in $H^*(B(n+1, k); \mathbb{Z})$, hence

$$2\tilde{f}_k^*(b_i) = \tilde{f}_k^*(2b_i) = \tilde{f}_k^*(p^*(c_i)) = g^*(c_i)$$

and therefore, if $\binom{n+i}{i} \not\equiv 0 \pmod{2^m}$,

$$\tilde{f}_k^*(b_i) = \frac{1}{2} \binom{-n-1}{i} z^i = \pm \frac{1}{2} \binom{n+i}{n} z^i$$

Now, if $k = 2i$, and $a_i \in H^{2i}(B(n+1, 2i); \mathbb{Z})$ as in the previous section, $\tilde{f}_{2i}^*(a_i)$ is an element $\lambda_i z^i$ of $H^{2i}(L^n(2^m); \mathbb{Z}) \cong \mathbb{Z}/2^m$ where $\lambda_i \in \mathbb{Z}/2^m$.

The Steenrod squares are natural and so with the help of relations (2.4), (3.1) and (3.2), we deduce for $i \leq n-2$

$$(3.4) \quad i\lambda_i = (n+1)\lambda_i + i \frac{1}{2} \binom{n+i+1}{n} \pmod{2},$$

$$(3.5) \quad \begin{aligned} \binom{i}{2} \lambda_i &= \binom{n+2}{2} \lambda_i + (i-1)(n+1) \frac{1}{2} \binom{n+i+1}{n} \\ &\quad + \binom{i}{2} \frac{1}{2} \binom{n+i+2}{n} \pmod{2}. \end{aligned}$$

We shall note that (3.4) is still valid for $i = n-1$. In the following we shall take $i = l(n)$ and $m \geq 2$.

First we suppose $n = 2^s$ with $s \geq 1$. In this case, $i = l(n) = n-1$ and (3.4) becomes

$$\frac{1}{2} \binom{n+i+1}{n} \equiv 0 \pmod{2}$$

which is impossible.

When n is even with $\alpha(n) \geq 2$, $i = l(n)$ is even and $i \leq n-2$, so by (3.4)

$$\lambda_i \equiv 0 \pmod{2}.$$

Using (3.5) we deduce

$$\begin{aligned} 0 &\equiv \frac{1}{2} \binom{n+i+1}{n} + \binom{i}{2} \frac{1}{2} \binom{n+i+2}{n} \\ &\equiv \frac{1}{2} \binom{n+i+1}{n} \pmod{2}, \end{aligned}$$

since $i + 1 > l(n)$, which is in contradiction with the definition of $l(n)$.

When n is odd with $\alpha(n) \geq 3$, $i = l(n) < n - 3$ is odd, the relation (3.4) becomes

$$\lambda_i \equiv \frac{1}{2} \binom{n+i+1}{n} \pmod{2}.$$

Now, using (3.5) and (3.3) we obtain

$$\frac{1}{2} \binom{n+i+1}{n} \equiv \begin{cases} 0 & \pmod{2} \text{ if } n \equiv 1 \pmod{4} \\ \frac{1}{2} \binom{n+i+2}{n} \pmod{2} & \text{ if } n \equiv 3 \pmod{4}. \end{cases}$$

As before we have a contradiction since $i + 1 > l(n)$ and so we have proved part a) of theorem 1.2.

Finally, if $n = 2^s + 1$ with $s \geq 1$, and if $L^n(2^m)$ immerses in $R^{2n+1+2(n-2)-1}$, the classifying map g of $-(n+1)\sigma$ lifts to $B(n+1, 2(n-2)-1)$, and also to $B(n+1, 2(n-2))$. With the same notations, relation (3.4) becomes in this case

$$\lambda_{n-2} \equiv \frac{1}{2} \binom{2n-1}{n} \pmod{2}$$

and so

$$\lambda_{n-2} \equiv 1 \pmod{2}.$$

However, if the map g lifts to $B(n+1, 2(n-2)-1)$, we have

$$\begin{aligned} \lambda_{n-2} z^{n-2} &= \tilde{f}_{2(n-2)}^*(a_{n-2}) \\ &= \tilde{f}_{2(n-2)-1}^*(p_{2n-5}^*(a_{n-2})) \\ &= \tilde{f}_{2(n-2)-1}^*(b_{n-2}) \\ &= \frac{1}{2} \binom{2n-2}{n} z^{n-2} \\ &\equiv 0 \pmod{2} \end{aligned}$$

where p_{2n-5} denotes the canonical map $B(n+1, 2(n-2)-1) \rightarrow B(n+1, 2(n-2))$. So, we have proved part b) of theorem 1.2.

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