On a Subring of an Integral Domain Obtained by Intersecting a Field

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ON A SUBRING OF AN INTEGRAL DOMAIN OBTAINED BY INTERSECTING A FIELD

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Introduction. Let $S$ be an integral domain and let $K$ be a subfield of the quotient field of $S$. We are interested in the ring-extension $S/S \cap K$ or the subring $S \cap K$ itself. We call $S \cap K$ a subring with reduced quotient field. It is known that the subring $S \cap K$ inherits some properties from $S$; for example: if $S$ is integrally closed, so is $S \cap K$; if $S$ is local (not necessarily Noetherian), so is $S \cap K$; if $S$ is a DVR, then $S \cap K$ is either a DVR or a field; if $S$ is a Krull domain, so is $S \cap K$ (see [6],[8]). In these examples, theory of valuations plays an important role.

Our objective of this paper is to show the ring $S \cap K$ maintains several properties of $S$ under certain conditions.

In the section 1, we study the property of Noetherianness. We show mainly the following result:

(1) Let $S$ is a Noetherian normal domain of characteristic zero with quotient field $L$ and let $K$ be a subfield of $L$ such that $S$ is integral over $S \cap K$. Then $S \cap K$ is a Noetherian domain.

In the section 2, we show some basic properties of $S \cap K$ for later use. We consider some conditions for a subring $R$ of $S$ to be of type $S \cap K$ for some subfield $K$ of the quotient field of $S$. For instance,

(2) The extension $S/S \cap K$ is characterized by behavior of divisorial ideals of $S \cap K$ (Theorem 2.4).

In the section 3, we treat (2,3)-closedness, root-closedness and quasi-normality of a subring $S \cap K$.

In the section 4, we show: Let $S$ be a Noetherian almost factorial domain of characteristic zero. If $S$ is integral over $S \cap K$, then $S \cap K$ is a Noetherian almost factorial domain. (Theorem 4.2).

In the section 5, we have the following:

(3) Let $(S, M)$ be a local factorial domain. If $S$ is LCM-stable over $S \cap K$, then $S \cap K$ is factorial (Theorem 5.3).

When $S$ is not local, the faithful flatness of $S$ over $S \cap K$ does not always ensure the similar result in (3) (Remark 2).

In the section 6, we study the factoriality of $S \cap F$ for a non-local domain $S$. The obstruction of descent of factoriality is anyway that a
certain principal ideal of $S$ is not necessarily generated by elements in $S \cap K$.

In the section 7, we treat Dedekind domains.

In this paper, we mean by a ring a commutative ring with identity and by an integral domain (or a domain) a ring which has no non-trivial zero-divisors, and for an integral domain $S$, $K(S)$ denotes the quotient field of $S$ unless otherwise specified. Our unexplained technical terms are standard and are seen in [10] and [13].

1. A subring of a Noetherian domain. An integral domain is called to be integrally closed (or normal) if it is integrally closed in its quotient field. This section treats the following problem, which means a descent of Noetherianness of ring-extensions:

**Problem.** Let $S$ be a Noetherian (normal) domain with quotient field $L$ and let $K$ be a subfield of $L$. Is the ring $S \cap K$ Noetherian if $S$ is integral over $S \cap K$?

This problem is a certain converse to the well known result:

If $R$ is a Noetherian normal domain with quotient field $K$ and $L$ a finite separable extension of $K$, then the integral closure $S$ of $R$ in $L$ is Noetherian (See [10, (31.B)]).

Concerning the descent problem as above, we have known the following results among other things: Let $S \supseteq R$ be a ring-extension with a Noetherian domain $S$.

(i) (Faithfully flat descent) If $S$ is faithfully flat over $R$, then $R$ is Noetherian.

(ii) (Eakin-Nagata) If $S$ is finitely generated as an $R$-module, then $R$ is Noetherian.

The result (i) is well-known (See [10]) and the result (ii) is seen in [5] and [10], a new proof of which has been given by M. Nagata [14] recently.

Our objective of this section is to settle the problem in the case that $S$ is integral over $S \cap K$ with $\text{char}(K) = 0$ and the case that $L$ is not necessarily algebraic over $S \cap K$ under certain conditions.

Let $A$ be an integral domain with quotient field $L$. An element $\alpha$ in $L$ is called almost integral over $A$ if there exists a non-zero element $c$ in $A$ such that $c\alpha^i \in A$ for all $i \in \mathbb{N}$. It is easy to see that the set $A^\mathbb{N}$ of all
almost integral elements over $A$ forms a ring between $A$ and $L$, which is called the complete integral closure of $A$. We say that $A$ is completely integrally closed if $A^c = A$. When $A$ is Noetherian, $A$ being completely integrally closed is equivalent to $A$ being integrally closed. It is known that a Krull domain is completely integrally closed, and if $A$ is a Krull domain $A \cap K$ is also a Krull domain for a field $K$. Note that a Noetherian normal domain is a Krull domain (See [6] for details).

We require the following lemma.

**Lemma 1.1.** Let $S$ be an integral domain and let $K$ be a field. Assume that $S$ is algebraic over $S \cap K$. Let $(\ )^e$ denote the complete integral closure of $(\ )$ in its quotient field. Then $S^c \cap K = (S \cap K)^e$.

**Proof.** Since $S \cap K \subseteq S^d \cap K$, we have $(S \cap K)^e \subseteq S^d \cap K$. Take $\beta \in S^d \cap K$. There exists a non-zero element $s \in S$ such that $s\beta^i \in S$ for all $i \in N$ and hence $sS[\beta] \subseteq S$. Since $\beta \in S^d \cap K$, the quotient fields of $S[\beta]$ and $S$ coincide. Since $s$ is algebraic over $S \cap K$, there exists an algebraic dependence:

$$a_0s^n + a_1s^{n-1} + \cdots + a_n = 0,$$

where $a_i \in S \cap K$ with $a_n \neq 0$. Then $a_nS[\beta] \subseteq S$. Hence $a_n\beta^i \in S \cap K$ for all $i \in N$. Thus $\beta$ is almost integral over $S \cap K$, that is, $\beta \in (S \cap K)^e$. Therefore $S^c \cap K = (S \cap K)^e$.

**Corollary 1.1.1.** Let $S$ be a Krull domain and $K$ be a field contained in $K(S)$. Let $L$ be a finite Galois extension of $K$ containing $S$ and let $S'$ be the integral closure of $S$ in $L$. Then $S' \cap K = S \cap K$.

**Proof.** Put $R = S \cap K$. Take $\beta \in S' \cap K$. Then $\beta$ is integral over $R$. So $R[\beta]$ is a finite $R$-module (cf. [13, (10.1)]). Write $R[\beta] = \sum_{i=1}^{s} d_i R (d_i = b_i/c_i$ with $b_i, c_i \in R)$, where we note that $R[\beta] \subseteq K$. Put $c = \prod_{i=1}^{s} c_i$. Then $c \in R: R[\beta]$, and hence $c\beta^j \in R$ for all $j \in N$. Thus $\beta \in R^d = (S \cap K)^d$ and so $S' \cap K \subseteq (S \cap K)^d \cap K$. Since $S'$ is a Krull domain, $R = S \cap K \subseteq (S \cap K)^d \cap K = S^c \cap K = S \cap K = R$ by Lemma 1.1, that is, $S' \cap K = S \cap K = R$.

We prove the following theorem by using, so-called the Galois-descent.

**Theorem 1.2.** Let $S$ be a normal domain of characteristic zero with quotient field $L$ and let $K$ be a subfield of $L$ such that $S$ is integral over $S \cap K$. If $S$ is Noetherian, then so is $S \cap K$. 

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Proof. Let $R = S \cap K$. Let $I$ be an ideal of $R$. Then $IS = (a_1, \ldots, a_t)S$ for some $a_i \in I$. Let $J$ be the ideal of $R$ generated by $a_1, \ldots, a_t$. Take $b \in I$. Then $b = \sum_{i=1}^{t} a_i \alpha_i$ ($\alpha_i \in S$). Put $S' = S \cap K(\alpha_1, \ldots, \alpha_t)$. Then $R \subseteq S' \subseteq S$ and $S'$ is integrally closed in $K(\alpha_1, \ldots, \alpha_t)$. Note that $b \in JS'$. Noting that $\text{char}(K) = 0$, there exists a field $L'$ such that

(a) $L' \supseteq K(\alpha_1, \ldots, \alpha_t) \supseteq K$,

(b) $L'$ is a finite Galois extension of $K$.

Let $G$ denote the Galois group $G(L'/K)$ with $n = \# G$. Let $S''$ denote the integral closure of $R$ in $L'$. Then $S''$ is a Galois extension of $R$. Note that $S''^{\sigma} = S''$ for each $\sigma \in G$. Since $S$ is integral over $R$, we have $S' \subseteq S \cap L' \subseteq S''$ and $S'^{\sigma} \subseteq S''^{\sigma} = S''$ for each $\sigma \in G$. Hence $\alpha_i^{\sigma} \in S''$ for any $\sigma \in G$. By [6, (1.3)] $S''$ is a Krull domain because $L'$ is a finite extension of $K$. We see that $nb = \sum_{\sigma \in G} b^{\sigma} = \sum_{\sigma \in G} \sum_{i=1}^{t} (a_i \alpha_i)^{\sigma} = \sum_{i=1}^{t} \sum_{\sigma \in G} a_i^{\sigma} \alpha_i^{\sigma} = \sum_{i=1}^{t} a_i (\sum_{\sigma \in G} \alpha_i^{\sigma})$. Since $\sum_{\sigma \in G} \alpha_i^{\sigma}$ is invariant under every element in $G$. Hence $\sum_{\sigma \in G} \alpha_i^{\sigma} \in K \cap S'' = K \cap S$ by Corollary 1.1.1. Hence $nb \in \sum_{i=1}^{t} a_i R$. Since $\text{char}(K) = 0$, we have $b \in J$. The implication $I \supseteq J$ is trivial, and hence $I = J = (a_1, \ldots, a_t)R$, a finitely generated ideal of $R$. Therefore $R = S \cap K$ is Noetherian.

Corollary 1.2.1. Let $R$ be an integrally closed domain with quotient field $K$ of characteristic zero and let $L$ be a field extension of $K$. If the integral closure of $R$ in $L$ is a Noetherian ring, then $R$ is Noetherian.

Proof. This follows from Theorem 1.2.

Let $S$ be an integral domain with quotient field $L$. We say that $S$ is N-1 if the integral closure of $S$ in its quotient field $L$ is a finite $S$-module; and that $S$ is N-2 if, for any finite extension $T$ of $L$, the integral closure of $S$ in $T$ is a finite $S$-module. It is known that N-1 is equivalent to N-2 when $S$ is a Noetherian integral domain of characteristic zero ([10, p.232]). A ring $A$ is called a Nagata ring if it is Noetherian and if $A/P$ is N-2 for every $P \in \text{Spec}(A)$.

Corollary 1.2.2. Let $R$ be an N-1 domain with quotient field $K$ of characteristic zero and let $L$ be an algebraic field extension of $K$. Let $S$ denote the integral closure of $R$ in $L$. If $S$ is a Noetherian domain, then so is $R$.

Proof. Since $S$ is a Noetherian normal domain, $S \cap K$ is Noetherian by Theorem 1.2. Since the quotient field of $S$ is algebraic over $K$, we have
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$S \cap K = S^2 \cap K = (S \cap K)^2$ by Lemma 1.1. Hence $S \cap K$ is the integral closure of $R$ in $K$ because $S \cap K$ is Noetherian. Since $R$ is a N-1 domain, $S \cap K$ is a finite $R$-module. So by Eakin-Nagata's Theorem, we conclude that $R$ is Noetherian.

A ring $A$ is called locally Noetherian if $A_P$ is a Noetherian ring for each prime ideal $P$ of $A$.

**Remark 1.** (1) The following is known in [7, (12.7)]: Let $R$ be an integral closed integral domain with quotient field $K$ and let $S$ be an integral domain containing $R$ such that $S$ is integral over $R$. Then for each prime ideal $M$ of $S$, $S_M \cap K = R_{M \cap R}$.

(2) Let $S$ be an integral domain and let $K$ be a subfield of the quotient field $K(S)$ of $S$ such that $K(S)$ is finite algebraic over $K$. Assume that $S$ is integral over $S \cap K$ and that $S$ is locally Noetherian. Then for each prime ideal $p$ of $S \cap K$, $S_p$ is Noetherian, where $S_p$ denotes $(S \cap K \setminus p)^{-1}S$. Indeed, there are only finitely many prime ideals $P_1, \ldots, P_n$ of $S$ lying over $p$ by [10, p.296]. Let $T = S \setminus \bigcup_{i=1}^{n} P_i$, a multiplicatively closed subset of $S$. Then $S_p = T^{-1}S$ by [7, (11.10)]. Let $I$ be an ideal of $S_p$. Then for each $1 \leq i \leq n$, $I_{P_i} = (a_{i1}, \ldots, a_{ir_i})S_{P_i}$ for some $a_{ij} \in I$. Put $J = \sum a_{ij}S_p$. Then $I_{P_i} = J_{P_i}$ for each $1 \leq i \leq n$. Thus $I = J$, which means that $S_p$ is Noetherian.

**Corollary 1.2.3.** Let $S$ be a locally Noetherian, normal domain of characteristic zero and let $K$ be a subfield of the quotient field $K(S)$ of $S$ such that $K(S)$ is finite algebraic over $K$. Assume that $S$ is integral over $S \cap K$. Then $S \cap K$ is locally Noetherian.

**Proof.** Note first that for each prime ideal $P$ of $S \cap K$, there exists a prime ideal $M$ of $S$ such that $M \cap K = P$ because $S$ is integral over $S \cap K$. Hence Remark 1(2) and Theorem 1.2 yield our conclusion.

**Example.** Let $k$ be a field ($\text{char} k \neq 1$) and let $t_i$ ($i \in N$) and $X$, $Y$ be indeterminates. Put $S = k(t_1, t_2, \ldots)[X,Y]$, which is a Noetherian domain, and for $i \in N$, put $d_i = t_{2i-1}X + t_{2i-1}Y$. Let $K = k(d_1, d_2, \ldots)$. Then $S \cap K = k[d_1, d_2, \ldots] := R$, which is not Noetherian. Note that $S/R$ is not integral.

**Proposition 1.3** (cf. [8, p.73, Ex.4]). Let $(S, M)$ be a local domain and $K$ a subfield of the quotient field $K(S)$ of $S$. Then $S \cap K$ is a local domain with the maximal ideal $M \cap K$.
Proof. Suppose that there exists a maximal ideal \( m \) which properly contains \( M \cap K \). Then \( mS = S \) and we have \( \sum_{i=1}^{n} a_i \beta_i = 1 \) in \( S \) with \( a_i \in m \) and \( \beta_i \in S \). Since \( S \) is a local domain with maximal ideal \( M \), there exists \( i \), say \( i = 1 \) such that \( a_1 \) is a unit in \( S \). Hence \( a_1 \alpha = 1 \) for some \( \alpha \in S \). So we have \( \alpha = 1/a_1 \in S \cap K \), which means that \( a_1 \) is a unit in \( S \cap K \). This is absurd. Therefore \( S \cap K \) is a local domain with the maximal ideal \( M \cap K \).

2. Basic properties of a subring with reduced quotient field.
In this section, we study the conditions for a subring to be a subring with reduced quotient field and show some preliminary results which will be used later. We start with the following lemma.

Lemma 2.1. Let \( S \) be an integral domain, let \( K \) be a subfield of the quotient field of \( S \) and let \( R \) be a subring of \( S \) which is contained in \( K \). Then the following statements are equivalent:
(i) \( aS \cap K = aR \) for any \( a \in K \);
(ii) \( R = S \cap K \).
If furthermore \( K \) is the quotient field of \( R \), (i) is equivalent to the following:
(iii) \( aS \cap R = aR \) for any \( a \in R \).

Proof. (ii) \( \implies \) (i). Take \( x \in aS \cap K \). Then \( x = as \) for some \( s \in S \) and hence \( x/a = s \in S \cap K = R \). Thus \( x \in aR \).
The implications (i) \( \implies \) (ii) is trivial.
Assume that \( K \) is the quotient field of \( R \). The implications (i) \( \implies \) (iii) is trivial.
(ii) \( \implies \) (iii). Take \( s \in S \cap K \). Since \( K \) is the quotient field of \( R \), \( s = b/a \) for some \( a, b \in R \). Hence \( b = as \in R \cap aS = aR \). Thus \( s \in R \).

Corollary 2.1.1. Let \( S \) be an integral domain and let \( K \) be a subfield of the quotient field of \( S \). Then for any \( a, b \in R := S \cap K \), the following hold:
(a) \( aR = bR \) if and only if \( aS = bS \),
(b) \( \sqrt{aR} = \sqrt{bR} \) if and only if \( \sqrt{aS} = \sqrt{bS} \).
Moreover for any \( \alpha, \beta \in K \),
(a') \( \alpha R = \beta R \) if and only if \( \alpha S = \beta S \).

Proof. (a) The implication \( aR = bR \implies aS = bS \) is obvious. Conversely, \( aR = aS \cap K = bS \cap K = bR \) by Lemma 2.1 (i) \( \iff \) (ii).
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(b) Assume that $\sqrt{aS} = \sqrt{bS}$. Take $x \in \sqrt{aR}$. Then $x^n \in aR \subseteq aS \subseteq bS$ for some positive integer $n$. Hence $x^m \in bS \cap K = bR$ for some positive integer $m$ by (a). Thus $x \in \sqrt{bR}$. By symmetry, we have $\sqrt{aR} = \sqrt{bR}$. Conversely, assume that $\sqrt{aR} = \sqrt{bR}$. Then $\sqrt{aRS} = \sqrt{bRS}$ and hence $\sqrt{aS} = \sqrt{bS}$.

(a') There exist $c, d \in R$ such that $ca, db \in R$. By (a), we have $cdaR = cd\beta R \iff cdaS = cd\beta S$. Hence $\alpha R = \beta R \iff \alpha S = \beta S$.

**Corollary 2.1.2.** Let $S$, $K$ and $R$ be the same as in the above corollary 2.1.1. If $S$ satisfies the ascending chain condition for principal ideals, then so does $R$.

**Proof.** Let $a_1 R \subseteq a_2 R \subseteq \ldots$ be an ascending chain of principal ideals of $R$. Then we have the ascending chain $a_1 S \subseteq a_2 S \subseteq \ldots$ of principal ideals of $S$. Since $S$ satisfies the ascending chain condition for principal ideals, there exists an integer $r$ such that for any $n > r$, $a_n S = a_n R$. Thus by Corollary 2.1.1, we have $a_r R = a_n R$ for any $n > r$, which means that $R$ has the ascending chain condition for principal ideals.

**Proposition 2.2.** Let $S$ be an integral domain, let $K$ be a subfield of the quotient field of $S$ and let $R$ be its subring $S \cap K$. Then $(aS :_S bS) \cap K = aR :_R bR$ for any $a, b \in R$. In particular, if $a, b \in R$ is an $S$-sequence, then $a, b$ is an $R$-sequence.

**Proof.** The implication $aR :_R bR \subseteq (aS :_S bS) \cap K$ is obvious and it is clear that $(aS :_S bS) \cap K \subseteq R$. Take $x \in (aS :_S bS) \cap K$. Then $xb \in aS \cap K = aR$ by Lemma 2.1 (i) $\iff$ (ii). Hence $x \in aR :_R bR$. Next if $aS :_S bS = aS$, then $aR :_R bR = aR$ by the above argument, which means that if $a, b \in R$ is an $S$-sequence, then $a, b$ is an $R$-sequence.

Let $S$ be an integral domain with quotient field $L$. We say that $J$ is a fractional ideal of $S$ if $J$ is an $S$-submodule of $L$ such that $sJ \subseteq S$ for some non-zero element $s \in S$. Let $J$ be a fractional ideal of $S$. We denote by $J^*$ a fractional ideal $S :_L J := \{ x \in L | xJ \subseteq S \}$. We also write $S :_L J$ for $S :_L J$ if no confusion takes place. We say that a fractional ideal $J$ of $S$ is divisorial if $J'' := S :_L (S :_L J) = J$.

**Lemma 2.3.** Let $S$ be an integral domain with quotient field $K(S)$ and let $I$ be a divisorial integral ideal of $S$. Then $I = \cap_i (b_i S :_S a_i S)$ for some $a_i, b_i \in S$. 

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Proof. Let \( y = z/x \) be an element in \( K(S) \) with \( x, z \in S \). Then \( yS \cap S = zS : x : S \). Indeed, if \( \alpha \in zS : x : S \), then \( \alpha x \in zS \) and hence \( \alpha \in (z/x)S \cap S = yS \cap S \). Conversely, if \( \alpha \in yS \cap S \), then \( \alpha = y \alpha = (z/x) \alpha \) for some \( \alpha \in S \). So \( \alpha x = \alpha z \in zS \). Hence \( \alpha \in zS : x : S \). Since \( I \) is a divisorial integral ideal of \( S \), \( I \) is an intersection of principal fractional ideals, that is, \( I = \bigcap yS \cap S \), where \( I \subseteq yS \), \( y \in K(S) \) (See [6, p.12] for details). By the above argument, \( I \) is written as \( \bigcap a_i (a_i S : b_i S) \) for some \( a_i, b_i \in S \).

**Theorem 2.4.** Let \( S \) be an integral domain and let \( R \) be its subring with quotient field \( K \). Then the following statements are equivalent:

(i) \( R = S \cap K \);
(ii) \( aS \cap R = aR \) for each \( a \in R \);
(iii) \( aS \cap K = aR \) for each \( a \in K \);
(iv) \( IS \cap R = I \) for each divisorial integral ideal \( I \) of \( R \);
(v) \( IS \cap K = I \) for each divisorial fractional ideal \( I \) of \( R \).

Proof. (i) \( \iff \) (ii) \( \iff \) (iii') have been shown in Lemma 2.1.

Let \( J \) be a fractional ideal of \( R \). Then there exists a non-zero element \( d \) in \( R \) such that \( dJ \subseteq R \). It is easy to see that if \( (dJS) \cap K = dJ \) holds, then \( JS \cap K = J \) holds. Hence in (iii') and (iv'), we can assume that \( I \) is an integral ideal, i.e., \( I \subseteq R \).

(iv) \( \implies \) (iii) (resp. (iv') \( \implies \) (iii')) follows from the implications: \( I \subseteq IS \cap R \subseteq (IS)^* \cap R = I \) (resp. \( I \subseteq IS \cap K \subseteq (IS)^* \cap K = I \)).

(iv) \( \implies \) (ii) and (iv') \( \implies \) (iii') are trivial because a principal ideal is divisorial.

We must show the implication (i) \( \implies \) (iv) (resp. (i) \( \implies \) (iv')). The ideal \( I \) is written as \( \bigcap (a_i R : b_i R) \) for some \( a_i, b_i \in R \) by Lemma 2.3. Hence we have \( IS \subseteq \bigcap (a_i R : b_i R S) \subseteq \bigcap (a_i S : b_i S) \). Thus \( IS \subseteq (IS)^* \subseteq \bigcap (a_i S : b_i S) \). So we have \( I \subseteq IS \cap R \subseteq (IS)^* \cap R \subseteq \bigcap (a_i S : b_i S) \cap R = \bigcap (a_i R : b_i R) = I \) (resp. \( I \subseteq IS \cap K \subseteq (IS)^* \cap K \subseteq \bigcap (a_i S : b_i S) \cap K = \bigcap (a_i R : b_i R) = I \)) by Proposition 2.2, which means that \( (IS)^* \cap R = I \) (resp. \( (IS)^* \cap K = I \)).

**Corollary 2.4.1.** Let \( S, K \) and \( R \) be the same as in Theorem 2.4 and assume that \( R = S \cap K \). Let \( I \) and \( J \) be divisorial fractional ideal of \( R \). Then \( I = J \) if and only if \( (IS)^* = (JS)^* \).
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Proof. The implication $I = J \implies (IS)^* = (JS)^*$ is obvious. Let $I$, $J$ be divisorial fractional ideals with $(IS)^* = (JS)^*$. Then there exist non-zero elements $a, b \in R$ such that both $aI$ and $bJ$ are integral ideals of $R$, which are divisorial. Then $(abIS)^* = ab(IS)^* = ab(JS)^* = (abJS)^*$. By Theorem 2.4, we have $abI = (abIS)^* \cap R = (abJS)^* \cap R = abJ$. Thus we have $I = J$.

For a domain $D$, $\text{Inv}(D)$ denotes the set of the invertible ideals of $D$. Define $\text{Prin}(D)$ to be the set $\{aD | a \in K(D), a \neq 0\}$. It is easy to see that $\text{Prin}(D)$ is a subgroup of $\text{Inv}(D)$. Define $\text{Pic}(D) = \text{Inv}(D)/\text{Prin}(D)$, which is equipped with the commutative group structure induced from that of $\text{Inv}(D)$. We call $\text{Pic}(D)$ the Picard group of $D$, which can be regarded as the group of isomorphic classes of invertible $D$-modules. We denote the composition in $\text{Pic}(D)$ additively.

Let $S$ and $K$ be the same as in Theorem 2.4. The inclusion $S \cap K \hookrightarrow S$ induces the canonical map $\varphi: \text{Inv}(S \cap K) \rightarrow \text{Inv}(S)$ defined by sending $I \in \text{Inv}(S \cap K)$ to $IS \in \text{Inv}(S)$.

Corollary 2.4.2. Let $S$ and $K$ be the same as above. Then $\varphi: \text{Inv}(S \cap K) \rightarrow \text{Inv}(S)$ is injective.

Proof. Take two invertible ideals $I$ and $J$ of $S \cap K$ such that $IS = JS$. Then $I = IS \cap K = JS \cap K = J$ by Theorem 2.4, which means $\varphi$ is injective.

Question. Let $S$ and $K$ be the same as above. When is the canonical group homomorphism $\text{Pic}(S \cap K) \rightarrow \text{Pic}(S)$ injective i.e., $\text{Inv}(S \cap K) \cap \text{Prin}(S) = \text{Prin}(S \cap K)$?

Let $S$ be an integral domain and let $D(S)$ denote the collection of divisorial fractional $S$-ideals. Define $D(S) \times D(S) \rightarrow D(S)$ by $(a, b) \mapsto S:(S:ab)$. Then $D(S)$ is a commutative monoid. It is known that $D(S)$ is a group if and only if $S$ is completely integral closed [6, (3.4)]. Note here that a Krull domain is completely integral closed [6, (3.6)].

Let $R \subseteq S$ be Krull domains. We say that $S/R$ satisfies the condition (PDE) if $\text{ht}(P \cap R) \leq 1$ for each $P \in Ht_1(S)$.

It is known that if $S$ is a Krull domain, then $S \cap K$ is also a Krull domain for any field [6, (1.2)].

Proposition 2.5. Let $S$ be a Krull domain and let $K$ be a subfield of the quotient field of $S$. Then the extension $S \cap K \subseteq S$ satisfies (PDE)
and the canonical group homomorphism \( D(S \cap K) \rightarrow D(S) \) defined by \( I \mapsto (IS)^{**} \) is injective.

**Proof.** The second statement follows from Corollary 2.4.1. Since \( S \) is a Krull domain, \( S = \bigcap_i V_i \), where \( V_i \) is a DVR on the quotient field of \( S \) which contains \( S \). Let \( m_i \) denote the maximal ideal of \( V_i \). Then \( S \cap K = \bigcap_i (V_i \cap K) \), where \( V_i \cap K \) is either a DVR with maximal ideal \( m_i \cap K \) or a field. Take \( P \in Ht_1(S) \). Then there exists a DVR \( V_i \) such that \( m_i \cap S = P \). Hence \( P \cap K = m_i \cap S \cap K = m_i \cap K \) is (0) or in \( Ht_1(S \cap K) \).

3. (2,3)-closed, root-closed and quasinormal. Let \( D \) be an integral domain with quotient field \( K(D) \) and let \( L \) be a field containing \( K(D) \). We say that \( D \) is \((2,3)\)-closed in \( L \) if every element \( \alpha \in L \) such that \( \alpha^2, \alpha^3 \in D \) is an element of \( D \), and we say \"(2,3)-closed\" when \( L = K(D) \). We say that \( D \) is root-closed in \( L \) if every element \( \alpha \in L \) such that \( \alpha^n \in D \) for some \( n \in \mathbb{N} \) is an element of \( D \). We say that \( D \) is quasinormal if the canonical homomorphism: \( \text{Pic}(D) \rightarrow \text{Pic}(D[X, X^{-1}]) \) is an isomorphism, where \( X \) denotes an indeterminate over \( D \).

**Theorem 3.1.** Let \( S \) be an integral domain and let \( L \) be a field containing the quotient field \( K(S) \) of \( S \). Let \( K \) be a field. If \( S \) is \((2,3)\)-closed in \( L \), then \( S \cap K \) is \((2,3)\)-closed in \( L \cap K \).

**Proof.** Take \( \alpha \in L \cap K \) with \( \alpha^2, \alpha^3 \in S \cap K \). Then \( \alpha^2, \alpha^3 \in S \) implies \( \alpha \in S \) because \( S \) is \((2,3)\)-closed in \( L \). Hence \( \alpha \in S \cap K \), which means that \( S \cap K \) is \((2,3)\)-closed in \( L \cap K \).

In [4], the following is proved:

**Lemma 3.2.** Let \( D \) be an integral domain and let \( X \) be an indeterminate over \( D \). Then the following conditions are equivalent:

(i) \( D \) is \((2,3)\)-closed,

(ii) the canonical homomorphism \( \text{Pic}(D) \rightarrow \text{Pic}(D[X]) \) is an isomorphism.

**Corollary 3.2.1.** Let \( S, K \) be the same as in Theorem 3.1 and let \( S[X] \) be a polynomial ring. If \( \text{Pic}(S) \rightarrow \text{Pic}(S[X]) \) is an isomorphism, then \( \text{Pic}(S \cap K) \rightarrow \text{Pic}((S \cap K)[X]) \) is an isomorphism.

**Proof.** This follows from Theorem 3.1 and Lemma 3.2.
**Theorem 3.3.** Let $S$, $L$ and $K$ be the same as in Theorem 3.1. If $S$ is root-closed in $L$, then $S \cap K$ is root-closed in $L \cap K$.

**Proof.** Take $\alpha \in L \cap K$ with $\alpha^n \in S \cap K$ for some $n \in \mathbb{N}$. Then $\alpha^n \in S$ implies $\alpha \in S$ because $S$ is root-closed in $L$. Hence $\alpha \in S \cap K$, which means that $S \cap K$ is root-closed in $L$.

Let $D$ be integral domain and let $I$ be an invertible ideal of $D$. We denote by $[I]$ the equivalence class containing $I$ in $\text{Pic}(D)$.

**Theorem 3.4.** Let $S$ be an integral domain, let $X$ be indeterminate and let $K$ be a field. Assume that the canonical homomorphism $\text{Pic}((S \cap K)[X, X^{-1}]) \to \text{Pic}(S[X, X^{-1}])$ is injective. If $S$ is quasinormal, then so is $S \cap K$.

**Proof.** Put $R := S \cap K$. Take $I \in \text{Inv}(R[X, X^{-1}])$. Consider the commutative diagram:

$$
\begin{array}{ccc}
\text{Pic}(R) & \xrightarrow{i_1} & \text{Pic}(S) \\
\varphi_{/K} \downarrow \psi_{/K} & & \varphi \downarrow \psi \\
\text{Pic}(R[X, X^{-1}]) & \xrightarrow{i_2} & \text{Pic}(S[X, X^{-1}])
\end{array}
$$

where $\varphi$ and $\varphi_{/K}$ are the canonical maps and $\psi$ and $\psi_{/K}$ are the ones induced from the maps sending $X$ to 1. It is clear that $\psi_{/K} \circ \varphi_{/K} = 1$ and $\psi \circ \varphi = 1$. So $\varphi$ and $\varphi_{/K}$ are injective. By definition, $\psi_{/K}([I]) = [I']$ for some $I' \in \text{Inv}(R)$. Since $\varphi \cdot i_1([I']) = \varphi([I'S]) = [I'S[X, X^{-1}]]$, we have $[I'S[X, X^{-1}]] \in \text{Im}i_2$. By the diagram above, we have $i_2([I]) = \varphi \cdot i_2([I']) = \varphi \cdot i_1([I']) = i_2 \circ \varphi_{/K}([I'])$. Since $i_2$ is injective, we have that $[I] = \varphi_{/K}([I'])$. Thus $\varphi_{/K}$ is bijective.

4. **A subring of an almost factorial domain.** Let $S$ be an integral domain and let $K$ be a subfield of the quotient field of $S$. An ideal $I$ of $S$ is called *radically principal* if $I^f \in S$ for some $f \in S$. A Krull domain is called *almost factorial* if its divisor class group is a torsion group.

**Lemma 4.1** ([16, Proposition 7]). Let $R$ be a Krull domain. Then $R$ is almost factorial if and only if any $P \in H\ell_1(R)$ is radically principal.

**Theorem 4.2.** Let $S$ be a Noetherian almost factorial domain of characteristic zero. Assume that $S$ is integral over $S \cap K$. Then $S \cap K$ is a Noetherian almost factorial domain.
Proof. By Theorem 1.2, $S \cap K$ is Noetherian. Since $S$ is normal, so is $S \cap K$. Since $S$ is almost factorial, any prime ideal of height one is radically principal by Lemma 4.1. Take $P \in Ht_1(S \cap K)$. Then any prime divisor of $\sqrt{PS}$ is of height one by Going-Down Theorem. So $\sqrt{PS} = \sqrt{fS}$ for some $f \in PS$. Let $P = (a_1, \ldots, a_n)(S \cap K)$. Then taking a non-negative integer $s$, we have $a_i^s = f b_i$ for some $b_i \in S$. Put $S' = S \cap K(f, b_1, \ldots, b_n)$. Then $S \cap K \subseteq S' \subseteq S$ and $S'$ is integrally closed in $K(f, b_1, \ldots, b_n)$. Note here that char$(K) = 0$. There exists a field $L'$ such that

(a) $L' \supseteq K(f, b_1, \ldots, b_n) \supseteq K$,

(b) $L'$ is a finite Galois extension of $K$.

Let $G$ denote the Galois group $G(L'/K)$ with $m = \sharp G$. Let $S''$ denote the integral closure of $S \cap K$ in $L'$. Then $S''$ is a Galois extension of $S \cap K$. Note that $S''^\sigma = S''$ for each $\sigma \in G$. Since $S$ is integral over $R$, we have $S' \subseteq S \cap L' \subseteq S''$ and $S' = S'' = S''^\sigma$ for each $\sigma \in G$. Hence $f^\sigma, b_1^\sigma, \ldots, b_n^\sigma \in S''$ for each $\sigma \in G$. By [6, (1.3)], $S''$ is a Krull domain. The elements $\Pi_{\sigma \in G} f^\sigma, \Pi_{\sigma \in G} b_i^\sigma$ $(i = 1, \ldots, n)$ are invariant under every element in $G$. Hence $\Pi_{\sigma \in G} f^\sigma, \Pi_{\sigma \in G} b_i^\sigma \in K \cap S''$ for $(i = 1, \ldots, n)$. By Corollary 1.1.1, we have $S'' \cap K = S \cap K$. Thus $\Pi_{\sigma \in G} f^\sigma, \Pi_{\sigma \in G} b_i^\sigma \in K \cap S$ for $(i = 1, \ldots, n)$. So $f^\sigma = a_i/b_i$ and $\Pi_{\sigma \in G} a_i^\sigma/\Pi_{\sigma \in G} b_i^\sigma \in S \cap K$. Put $g = \Pi_{\sigma \in G} f^\sigma$. Then $a_i^{\sigma_m} = \Pi_{\sigma \in G} f^\sigma \cdot \Pi_{\sigma \in G} b_i^\sigma$, where $\sharp G = m$. Hence for a sufficiently large integer $t$, $P^t \subseteq g(S \cap K)$. Thus we have $P = \sqrt{g(S \cap K)}$, and hence $S \cap K$ is almost factorial by Lemma 4.1.

**Theorem 4.3.** Let $S$ be an almost factorial domain. Assume that $S$ is integral over $S \cap K$. Then $S \cap K$ is an almost factorial domain.

Proof. The proof is similar to that of Theorem 4.2.

**Corollary 4.3.1.** Let $R$ be a Krull domain and let $L$ be a field extension of $K(R)$. If the integral closure $S$ of $R$ in $L$ is almost factorial, then so is $R$.

Proof. Note that $S$ is a Krull domain. Since $S \cap K(R) = R$, our conclusion follows from Theorem 4.3.

5. A subring of a locally factorial domain and LCM-stability. We mean by a local ring a ring with unique maximal ideal. It is known that an integral domain $S$ is factorial domain if and only if $S$ is a Krull domain in which each $P \in Ht_1(S)$ is principal [6, (6.1)].

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Lemma 5.1. Let \((S, M)\) be a local domain and let \(K\) be a subfield of the quotient field of \(S\). Let \(I\) be an ideal of \(S \cap K\). If \(IS\) is principal, then so is \(I\).

Proof. Let \(I\) be generated by a set \(\{a_i\}_{i \in \Delta}\). Since \(IS\) is a principal ideal of \(S\), there exists \(\alpha S = IS\). So for each \(i \in \Delta\), \(a_i = \alpha s_i\) for some \(s_i \in S\). Suppose that the set \(\{s_i| i \in \Delta\}\) generates a proper ideal of \(S\). Then \(\alpha S = IS \subseteq \alpha MS \subseteq \alpha S\), that is, \(\alpha S = \alpha MS\). Hence \(S = M\), a contradiction. So there exists a unit \(s_i\) so that \(a_i S = \alpha s_i S = \alpha S = IS\). We have \(I \subseteq IS \cap K = a_i S \cap K = a_i (S \cap K) \subseteq I\) by Lemma 2.1 (i) \(\iff\) (ii). Therefore \(I = a_i(S \cap K)\).

Corollary 5.1.1. Let \((S, M)\) and \(K\) be the same as in Lemma 5.1. Assume that for each \(P \in Ht_1(S \cap K)\), \(Ass_S(S/PS) \subseteq Ht_1(S)\). If \(S\) is a factorial domain, then so is \(S \cap K\).

Proof. Take \(P \in Ht_1(S \cap K)\). Since \(Ass_S(S/PS) \subseteq Ht_1(S)\), \(PS\) is a divisorial ideal of \(S\) because \(S\) is a Krull domain. Since \(S\) is factorial, \(PS\) is a principal ideal and hence \(P\) is principal by Lemma 5.1.

A ring \(A\) is called locally factorial if \(A_P\) is factorial for each prime ideal \(P\).

Theorem 5.2. Let \(S\) be a locally factorial domain and \(K\) a field. Assume that \(S\) is integral over \(S \cap K\). Then \(S \cap K\) is locally factorial.

Proof. Note first that for each prime ideal \(P\) of \(S \cap K\), there exists a prime ideal \(M\) of \(S\) such that \(M \cap K = P\) because \(S\) is integral over \(S \cap K\) and that \(K\) can be assumed to be the quotient field of \(S \cap K\). Hence our assertion follows from Lemma 5.1 and Remark 1(1) in the section one.

Remark 2. In [6, (6.11)], it is seen that when a local \(R\)-algebra \(S\) is faithfully flat over \(R\), \(R\) is a factorial domain if \(S\) is factorial. But in general, not even factoriality descends through faithfully flat extensions. That is, if \(S\) is not local, then the above conclusion does not always hold. Indeed, we have the following example (cf. [6, p.39],[8, p.74],[18, p.105]): Consider a Dedekind domain \(R\) which is not a principal ideal domain. Let \(T\) be the multiplicative subset of the polynomial ring \(R[X]\) generated by the polynomials whose coefficients generate \(R\). Then the ring \(S := T^{-1}R[X]\) is factorial (more precisely, a principal ideal domain) and it is a faithfully flat extension of \(R\). But \(R\) is not factorial. Let \(K\) denote the
quotient field of $R$. Then $S \cap K = R$. This example shows that even if $S$ is a factorial domain, $S \cap K$ is not necessarily factorial for a field $K$.

Moreover even if a Noetherian normal domain $S$ is a finite Galois extension of $S \cap K$, the factoriality of $S$ does not necessarily yield that of $S \cap K$ [6, (16.5)].

Let $S$ be a ring and let $M$ be a $S$-module. We say that $M$ is LCM-stable over $S$ if $aM \cap bM = (aS \cap bS)M$ for any $a, b \in S$ and that $M$ is Q-stable over $S$ if $aM;bM = (aS;bS)M$ for any $a, b \in S$. It is easy to see that if a $S$-module $M$ is flat, then $M$ is LCM-stable over $S$, but the converse does not always hold.

Let $R \subseteq S$ be integral domains. It is known that $S$ is LCM-stable over $R$ if and only if $S$ is Q-stable over $R$ [1, Lemma 1].

We know that a maximal proper divisorial integral ideal of a Krull domain $S$ is a prime ideal of height one with the form $S:(xS + S)$ for some $x \in K(S)$, the quotient field of $S$, which is equal to $yS;xS$ for some $y, x \in S$ [6, (3.5)]. Moreover in a Krull domain $S$, $P \in Ht_1(S)$ if and only if $P$ is a maximal divisorial prime ideal [6, (3.11)].

**Theorem 5.3.** Let $(S, M)$ be a local domain and let $K$ be a subfield of the quotient field of $S$. Assume that $S$ is LCM-stable over $S \cap K$. If $S$ is a factorial domain, then so is $S \cap K$.

**Proof.** Put $R = S \cap K$. Let $P$ be a prime ideal of $R$ of height one. Then $P = aR;RbR$ for some $a, b \in R$. Since $S$ is LCM-stable over $R$, equivalently Q-stable over $R$, $PS = (aR;RbR)S = aS;bS$, which is a divisorial integral ideal of $S$. Hence $Ass_S(S/PS) \subseteq Ht_1(S)$, which yields that $R$ is a factorial domain by Corollary 5.1.1.

The following result is known: let $(R, m)$ be a local domain with quotient field $K$ and let $S$ be an integral domain containing $R$ with $mS \neq S$. If $S$ is LCM-stable over $R$, then $S \cap K = R$ (cf. [17, (1.11)]).

**Corollary 5.3.1.** Let $(R, m)$ be a local domain and let $S$ be an integral domain containing $R$ with $mS \neq S$. Assume that $S$ is LCM-stable over $R$. If $S$ is a factorial domain, then so is $R$.

**Proof.** This follows from Theorem 5.3 and the preceding known result.

**6. A subring of a factorial domain.** Let $S$ be an integral domain and let $K$ be a subfield of the quotient field of $S$. In this section, we treat
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mainly factoriality. Recall that an integral domain $S$ is a factorial domain (or a unique factorization domain or a UFD) provided every element in $S$ is uniquely (up to multiplication by a unit) a finite product of irreducible (or prime) elements. Even if $S$ is a factorial domain, $S \cap K$ is not always factorial (see [6, (16.5)] or [3, VII, §3, Ex.11] for instance). In fact, we can see the following example in [3, VII, §3, Ex.11]:

**Example.** Let $K$ be a field, $S = K[X, Y]$ be a polynomial ring and $L = K(X^2, Y/X) \subset K(X,Y)$. Then $S$ is factorial but $S \cap L$ is not.

So our aim is to study when $S \cap K$ is factorial if $S$ is factorial.

In [6, (6.1)], we see that an integral domain $S$ is factorial if and only if $S$ has the ascending chain condition for principal ideals and a maximal proper principal ideal is a prime ideal.

**Theorem 6.1.** Let $S$ be an integral domain and let $K$ be a subfield of the quotient field of $S$. Assume that $S$ satisfies the ascending chain condition for principal ideals. Then $S \cap K$ is factorial if for each $P \in Ht_1(S)$ there exists a non-unit $a \in S \cap K$ such that $P \cap K \subseteq a(S \cap K)$.

**Proof.** Put $R = S \cap K$. By Corollary 2.1.2, $R$ has the ascending chain condition for principal ideals. Let $dR$ be a maximal proper principal ideal of $R$. Then $dS$ is contained in a prime ideal in $Ht_1(S)$. Indeed, if $dS = S$ then $dR = dS \cap K = S \cap K$ by Lemma 2.1, a contradiction. So by assumption, $dR \subset P \cap K$ and $P \cap K \subseteq aS$ for some non-unit $a$ in $R$. By the maximality, we have $dR = P \cap K = aR$ and hence $dR$ is a prime ideal. Thus $R$ is a factorial domain.

**Corollary 6.1.1.** Let $S$ be a factorial domain and let $K$ be a subfield of the quotient field of $S$. Then $S \cap K$ is factorial if for each non-unit $x \in S$ there exists a non-unit $a \in S \cap K$ such that $xS \cap K \subseteq a(S \cap K)$.

**Proof.** Since $S$ is factorial, any $P \in Ht_1(S)$ is a principal ideal. So apply Theorem 6.1 and we get our conclusion.

Recall that a ring extension $S \supseteq R$ is called to be inner if $x, y \in S$ with $xy \in R$ yields $xs, ys^{-1} \in R$ for some unit $s$ in $S$ (cf. [2]). For example, let $S$ be a polynomial ring $R[X]$. Then the extension $S \supseteq R$ is inner.

Let $S$ be an integral domain and let $K$ be a subfield of the quotient field of $S$. We say that $K$ is inner with respect to $S$ if $x, y \in S$ with
$xy \in K$ yields that $xs, ys^{-1} \in K$ for some unit $s$ in $S$. This is equivalent to the extension $S/S \cap K$ being inert in the above sense.

**Theorem 6.2.** Let $S$ be an integral domain and let $K$ be a subfield of the quotient field of $S$. Assume that $K$ is inert with respect to $S$. If $S$ is factorial, then so is $S \cap K$.

**Proof.** Put $R = S \cap K$. Then $R$ is a Krull domain because $S$ is a Krull domain. By Corollary 2.1.2, $R$ has the ascending chain condition for principal ideals. Let $dR$ be a maximal proper principal ideal. We have only to show that $dR$ is prime. Suppose that $dR$ is not prime. Then $dS$ is not prime because $dS \cap K = dR$ by Lemma 2.1. So there exists a prime ideal $bS$ in $S$ containing $dS$ properly. Thus we can write $d = bs$ for some non-unit $s \in S$. Since $bs = d \in S \cap K = R$, there exists a unit $t$ in $S$ such that $bt, st^{-1} \in R$ by assumption. Hence $dS \subseteq btS \cap st^{-1}S$ and hence $dR = dS \cap K \subseteq (btS \cap K) \cap (st^{-1}S \cap K) = btR \cap st^{-1}R$ by Lemma 2.1. Since $dR$ is not prime, $dR \neq bS \cap K = btS \cap K = btR$ by Lemma 2.1 but by the maximality, $dR = btR$, which is a prime ideal of $R$, a contradiction.

We close this section by showing the following result.

**Proposition 6.3.** Let $S$ be an integral domain and let $K$ be a subfield of the quotient field of $S$. Assume that $K$ is inert with respect to $S$ and that $U(S) = U(S \cap K)$, where $U(\ )$ denotes the group of the units. Then $S \cap K$ is algebraically closed in $S$.

**Proof.** Take $\alpha \in S$. Then there is an algebraic dependence:

$$a_0 \alpha^n + a_1 \alpha^{n-1} + \cdots + a_n = 0,$$

where $a_i \in S \cap K$. Thus $\alpha(a_0 \alpha^{n-1} + a_1 \alpha^{n-2} + \cdots + a_{n-1}) \in S \cap K$. Hence there exists a unit $t$ in $S$ such that $at \in S \cap K$. By assumption, $t$ is also a unit in $S \cap K$, we have $\alpha \in S \cap K$. This shows that $S \cap K$ is algebraically closed in $S$.

**7. Remarks on Dedekind domains.** In this section, we investigate Dedekind domains.

**Proposition 7.1.** Let $S$ be a Noetherian domain, let $K(S)$ be its quotient field and let $K$ be a subfield of $K(S)$. Let $m$ be a maximal ideal of a subring $K \cap S$ of $S$ such that $mS \neq S$. Then $\text{ht}(m) \leq \text{dim } S$. 

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Proof. Put $B = K \cap S$. Since $mS \neq S$ and $m$ is a maximal ideal of $B$, there exists a prime ideal $M$ of $S$ with $M \cap B = m$. There exists a valuation ring $(W, N)$ in $K(S)$ such that $S \subseteq W$, $N \cap S = M$ and $\dim W = \text{ht}(M)$ by [13, (11.9) and its proof]. Similarly there exists a valuation ring $(V, n)$ in $K(B)$ such that $B \subseteq V$, $n \cap B = m$ and $\dim V = \text{ht}(m)$. Let $W'$ be a subring generated by $V$ and $W$ in $K(S)$. Since $W \subseteq W' \subseteq K(S)$ and $W$ is a valuation ring, $W'$ is also a valuation ring by [13, (11.3)]. Let $N'$ be the maximal ideal of $W'$. Then $N' \cap W \subseteq N$ and $W' = W_{N' \cap W}$ by [13, (11.3)]. Note that $mW' \neq W'$. Hence $m \subseteq N' \cap B \subseteq N \cap B = m$, that is, $N' \cap B = m$. Since $V \subseteq W' \cap K$, we have $N' \cap V \subseteq n$. Since $\text{ht}(m) = \text{ht}(n)$, we have $n = N' \cap V$, which yields that $W' \cap K = V$ by [13, (11.3)]. Hence $\text{ht}(m) = \dim V = \dim W' \cap K \leq \dim W' \leq \dim W = \text{ht}(M) \leq \dim S$.

We require the following Lemma:

Lemma 7.2 ([11, (12.5)]). An integral domain $A$ is a Dedekind domain if and only if $A$ is a one-dimensional Krull domain.

We have known the following example:

Example (cf. [3, VII, §2, Ex.5(a)]). Let $k$ be a field and $L = k(X, Y)$, where $X, Y$ are indeterminates. Let $S = L[Z]$ be a polynomial ring, which is actually a PID, and let $K = k(Z, X + YZ)$. Then $S \cap K$ is not a Dedekind domain. In fact, $\dim S \cap K = 2$ and $(Z, X + YZ)S = S$ for a maximal ideal $(Z, X + YZ)$ of $S \cap K$.

Proposition 7.3. Let $S$ be a Dedekind domain and $K$ a subfield of $K(S)$. Assume that $mS \neq S$ for each $m \in \text{Spec}(S \cap K)$. Then $S \cap K$ is a Dedekind domain.

Proof. Note that a Dedekind domain is a Noetherian normal domain of dimension one. Since $mS \neq S$ for any maximal ideal $m$ of $S \cap K$, $\dim S \cap K \leq 1$ by Proposition 7.1. Hence $S \cap K$ is a Krull domain of dimension one. So by Lemma 7.2, $S \cap K$ is a Dedekind domain.

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