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KO -GROUP OF $\mathrm{PSp}(2^{4n})$

Dedicated to Professor Teiichi Kobayashi on his 60th birthday

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Let $Sp(n)$ be the symplectic group of degree n and $\mathrm{PSp}(n)$ be the projective group associated with $Sp(n)$, that is, $\mathrm{PSp}(n) = Sp(n)/C$ where C denotes the center of $Sp(n)$ which is generated by the scalar matrix with all diagonal entries -1 .

Our purpose here is to compute the real K -group $KO^*(\mathrm{PSp}(2^{4n}))$. As for the complex K -group, $K^*(\mathrm{PSp}(\ell))$ has been determined in [7,9] for any $\ell \geq 1$. But we begin with the calculation of $K^*(\mathrm{PSp}(2^{4n}))$ by our method for convenience of calculation. The way getting these groups is quite parallel to that of [12]. As it turns out that there is a $\mathbb{Z}/2$ -map from S^{8n+3} to $Sp(2^{4n})$ where the generator of $\mathbb{Z}/2$ acts on S^{8n+3} as antipodal involution and on $Sp(2^{4n})$ as the generator of C respectively, the multiplicative structures of the K -groups of $\mathrm{PSp}(2^{4n})$ can be reduced to those of the K -groups of S^{8n+3} and $Sp(2^{4n})$ just as in the case of $SO(8\ell)$ [12] by making use of this $\mathbb{Z}/2$ -map and applying a device to the equivariant K -theories associated with $\mathbb{Z}/2$.

This paper is arranged as follows. Section 1 consists of preparations for the subsequent sections. Sections 2 and 3 deal with the computation of $K^*(\mathrm{PSp}(2^{4n}))$ and $KO^*(\mathrm{PSp}(2^{4n}))$ respectively.

1. Let Γ denote the multiplicative group generated by -1 and H denote the canonical non-trivial 1-dimensional real representation of Γ .

We write nH for the direct sum of n copies of H . And by $B(pH \oplus \mathbf{R}^q)$ and $S(pH \oplus \mathbf{R}^q)$ we denote the unit ball and the unit sphere in $pH \oplus \mathbf{R}^q$ centered at the origin o , and let $\Sigma^{p,q} = B(pH \oplus \mathbf{R}^q)/S(pH \oplus \mathbf{R}^q)$ with the collapsed $S(pH \oplus \mathbf{R}^q)$ as base point. Here $\mathbf{R}$ denotes the field of real numbers. Also, for later use we fix the notations $\mathbf{C}$ and $\mathbf{H}$ for the fields of complex numbers and quaternions as usual.

Let $\Delta^+ : Spin(8n+4) \rightarrow U(2^{4n+1})$ be one of the half-spin representations of $Spin(8n+4)$. It is known [10], §13 that Δ^+ is the restriction of a quaternionic representation of $Spin(8n+4)$, denoted by

$$\bar{\Delta}^+ : Spin(8n+4) \longrightarrow Sp(2^{4n})$$

below. Assume that the generator of Γ acts on $Spin(8n+4)$ and $Sp(2^{4n})$ as the elements -1 and $-I$ of these groups respectively where I is the unit matrix, and thus consider these two groups as Γ -spaces. Then $\bar{\Delta}^+$ becomes a Γ -map obviously. Moreover we know [6] that $Spin(8n+4)$ contains $S^{8n+4,0}$ as an invariant subspace. This follows from the fact that $Spin(8n+4)$ is a subgroup of the Clifford algebra C_{8n+3} multiplicatively generated by the elements of the unit sphere S^{8n+3} ([10], §11). Therefore we have the following result similar to [6], (1.14).

(1.1) There exists a Γ -map $\iota: S^{8n+4,0} \rightarrow Sp(2^{4n})$, so that we have a homeomorphism

$$(S^{8n+4,0} \times Sp(2^{4n}))/\Gamma \approx P^{8n+3} \times Sp(2^{4n}).$$

In fact, this homeomorphism is induced by the assignment $(x, g) \mapsto (\pi(x), \iota(x)^{-1}g)$ for $x \in S^{8n+4,0}$ and $g \in Sp(2^{4n})$, where $P^{8n+3} = S^{8n+4,0}/\Gamma$, the real projective space of dimension $8n+3$, and π is the canonical projection from $S^{8n+4,0}$ to P^{8n+3} .

A Real (Γ) -vector bundle is a complex (Γ) -vector bundle together with a conjugate (equivariant) involutive automorphism and a quaternionic (Γ) -vector bundle is a complex (Γ) -vector bundle together with a conjugate (equivariant) anti-involutive automorphism. It is clear by definition that the external tensor product $E \hat{\otimes}_C F$ of two quaternionic (Γ) -vector bundles E and F admits an obvious Real structure.

Let KR and KSp denote the Real and quaternionic K -theories and let KR_Γ and KSp_Γ denote the equivariant ones associated with Γ . But $KR(X) \cong KO(X)$ and $KR_\Gamma(X) \cong KO_\Gamma(X)$ canonically if X has a trivial Real structure. Since all spaces of this note are such ones, we identify these isomorphisms throughout this paper. Then the above external tensor product $x \hat{\otimes}_C y$ defines uniquely an element $x \wedge_C y$ of either $\widetilde{KO}(X \wedge Y)$ or $\widetilde{KO}_\Gamma(X \wedge Y)$ according as $x \in \widetilde{KSp}(X)$, $y \in \widetilde{KSp}(Y)$ or $x \in \widetilde{KSp}_\Gamma(X)$, $y \in \widetilde{KSp}_\Gamma(Y)$.

Considering $S^{0,3}$ to be the unit quaternions $Sp(1)$ yields a generator of $\widetilde{KSp}(\Sigma^{0,4})$ in a canonical way. We write α for this element. Then

$$\widetilde{KSp}(\Sigma^{0,4}) = \mathbb{Z} \cdot \alpha$$

and also α satisfies

$$(1.2) \quad \alpha \otimes_C H = \eta_4, \quad \alpha \wedge_C \alpha = \eta_8 \quad \text{and} \quad s(\alpha) = \mu^2$$

where η_4 , η_8 and μ denote the canonical generators of $\widetilde{KO}(\Sigma^{0,4})$, $\widetilde{KO}(\Sigma^{0,8})$ and $\widetilde{K}(\Sigma^{0,2})$, (the last two generators are called the Bott class), and s denotes the natural complexification $KSp \rightarrow K$.

From [3,11,14] we now recall the equivariant Thom isomorphism theorems. Consider the isomorphism $S^{8n+4,0} \times H^{2^{4n}} \cong S^{8n+4,0} \times H^{2^{4n}} \otimes_R H$ of Γ -quaternionic vector bundles over $S^{8n+4,0}$ given by the assignment $(x, v) \mapsto (x, \iota(x)v)$ for $x \in S^{8n+4,0}$, $v \in H^{2^{4n}}$ where ι is as in (1.1). Then, in a canonical manner, this isomorphism yields a generator τ_H of $\widetilde{KSp}_\Gamma(\Sigma^{8n+4,0})$ such that its restriction to $o \in B((8n+4)H)$ is $2^{4n}(H - H \otimes_R H) \in KSp_\Gamma(o) (= RSp(\Gamma)$, the quaternionic representation ring of Γ).

Set

$$(1.3) \quad \begin{aligned} \tau &= s(\tau_H) \in \widetilde{K}_\Gamma(\Sigma^{8n+4,0}) \quad \text{and} \\ \omega &= \tau_H \wedge_C \alpha \in \widetilde{KO}_\Gamma(\Sigma^{8n+4,4}). \end{aligned}$$

Then their restrictions to o and $\Sigma^{0,4}$ are $2^{4n+1}(1-L) \in K_\Gamma(o) = R(\Gamma)$ and $2^{4n}(1-H)\eta_4 \in \widetilde{KO}_\Gamma(\Sigma^{0,4}) = RO(\Gamma) \cdot \eta_4$ respectively where $L = C \otimes_R H$, and multiplications by τ and ω give isomorphisms $\widetilde{K}_\Gamma^*(X) \cong \widetilde{K}_\Gamma^*(\Sigma^{8n+4,0} \wedge X)$ and $\widetilde{KO}_\Gamma^*(X) \cong \widetilde{KO}_\Gamma^*(\Sigma^{8n+4,4} \wedge X)$ for any Γ -space X with base-point respectively. Here $R(\Gamma)$ and $RO(\Gamma)$ are the complex and real representation rings of Γ and $R \cdot g$ denotes an R -module generated by a single element g for a ring R .

By h we denote the K - or KO -functor. For $X = +$ (a point), $Sp(2^{4n})$ we consider the exact sequence of the pair $(B((8n+4)H) \times X, S((8n+4)H) \times X)$ in h_Γ -theory. In general if Γ acts on X freely then there is a natural isomorphism $h_\Gamma^*(X) \cong h^*(X/\Gamma)$. Combining this with (1.1) and (1.3) gives rise to the following exact sequences.

$$(1.4a) \quad \cdots \xrightarrow{\delta} h_\Gamma^*(+) \xrightarrow{J} h_\Gamma^*(+) \xrightarrow{I} h^*(P^{8n+3}) \xrightarrow{\delta} \cdots,$$

$$(1.4b) \quad \cdots \xrightarrow{\delta} h^*(PG) \xrightarrow{J} h^*(PG) \xrightarrow{I} h^*(P^{8n+3} \times G) \xrightarrow{\delta} \cdots$$

where $G = Sp(2^{4n})$ and there holds the equality $\delta(xI(y)) = \delta(x)y$ in either case.

We write G for $Sp(2^{4n})$ for simplicity in the subsequent sections.

2. By the same symbol $\bar{\sigma}$ we denote the reduced bundles of the canonical line bundles $(S^{8n+4,0} \times H)/\Gamma \rightarrow P^{8n+3}$ and $(G \times H)/\Gamma \rightarrow PG$. And we write $\sigma = c(\bar{\sigma})$ where c denotes the complexification $KO \rightarrow K$. Since

$H^2 = 1$ in $RO(\Gamma)$ there hold obviously

$$\bar{\sigma}^2 + 2\bar{\sigma} = 0 \quad \text{and} \quad \sigma^2 + 2\sigma = 0.$$

Let $\bar{\nu} = p^*(\eta_8^{n+1}) \in \widetilde{KO}^{-5}(P^{8n+3})$ and $\nu = p^*(\mu^{4n+2}) \in \widetilde{K}^{-1}(S^{8n+3})$ where p is the map $P^{8n+3} \rightarrow S^{8n+3}$ obtained by collapsing the outside of a top dimensional cell in P^{8n+3} to a point. Then the equalities

$$c(\bar{\nu}) = \mu^2 \nu \quad \text{and} \quad r(\nu) = \eta_4 \bar{\nu}$$

follow from the relations $c(\eta_4) = 2\mu^2$ and $\eta_4^2 = 4$.

We consider the complex and real K -theories the $\mathbb{Z}/2$ - and $\mathbb{Z}/8$ -graded cohomology theories with the coefficient rings $K^*(+) = \mathbb{Z}[\mu]/(\mu^2 - 1)$ and $KO^*(+) = \mathbb{Z}[\eta_1, \eta_4, \eta_8]/(2\eta_1, \eta_1^3, \eta_1\eta_4, \eta_4^2 - 4, \eta_8 - 1)$ respectively where $\eta_1 \in KO^{-1}(+)$ and the others are as in Section 1. But the complex K -theory is viewed as $\mathbb{Z}/8$ -graded, so that $K^*(+) = \mathbb{Z}[\mu]/(\mu^4 - 1)$, when we discuss the relation between these two kinds of K -theories.

Here we calculate $K^*(P^{8n+3})$ and $KO^*(P^{8n+3})$ whose additive structures are given in [2,5]. Consider the exact sequence of (1.4a). First note that $h_F^*(+) \cong h^*(+)[t]/(t^2 - 1)$ because of $\Gamma \cong \mathbb{Z}/2$ where $t = L$ or H according as $h = K$ or KO . From inspecting the definitions of the maps it follows that

$$(2.1) \quad \begin{aligned} \delta(\nu) &= 1 + L, \quad J(1) = 2^{4n+1}(1 - L) \quad \text{and} \quad I(L) = \sigma + 1 \quad \text{for } h = K, \\ \delta(\bar{\nu}) &= 1 + H, \quad J(1) = 2^{4n}\eta_4(1 - H) \quad \text{and} \quad I(H) = \bar{\sigma} + 1 \quad \text{for } h = KO. \end{aligned}$$

Moreover we have a unique element ζ of $KO^{-6}(P^{8n+3})$ satisfying $\delta(\zeta) = \eta_1$.

Using this and the equality $\delta(xI(y)) = \delta(x)y$ we obtain by the exactness of (1.4a) the following.

With the notation as above

$$(2.2a) \quad \widetilde{K}(P^{8n+3}) = \mathbb{Z}/2^{4n+1} \cdot \sigma, \quad \widetilde{K}^{-1}(P^{8n+3}) = \mathbb{Z} \cdot \nu$$

where the ring structure is given by

$$\sigma^2 + 2\sigma = 0, \quad \nu^2 = 0,$$

$$\begin{aligned}
(2.2b) \quad & \widetilde{KO}(P^{8n+3}) = \mathbb{Z}/2^{4n+2} \cdot \bar{\sigma}, \\
& \widetilde{KO}^{-1}(P^{8n+3}) = \mathbb{Z}/2 \cdot \eta_1 \bar{\sigma} \oplus \mathbb{Z} \cdot \eta_4 \bar{\nu}, \\
& \widetilde{KO}^{-2}(P^{8n+3}) = \mathbb{Z}/2 \cdot \eta_1^2 \bar{\sigma}, \\
& \widetilde{KO}^{-3}(P^{8n+3}) = 0, \\
& \widetilde{KO}^{-4}(P^{8n+3}) = \mathbb{Z}/2^{4n} \cdot \eta_4 \bar{\sigma}, \\
& \widetilde{KO}^{-5}(P^{8n+3}) = \mathbb{Z} \cdot \bar{\nu}, \\
& \widetilde{KO}^{-6}(P^{8n+3}) = \mathbb{Z}/2 \cdot \eta_1 \bar{\nu} \oplus \mathbb{Z}/2 \cdot \zeta, \\
& \widetilde{KO}^{-7}(P^{8n+3}) = \mathbb{Z}/2 \cdot \eta_1^2 \bar{\nu} \oplus \mathbb{Z}/2 \cdot \eta_1 \zeta
\end{aligned}$$

where the ring structure is given by

$$\begin{aligned}
\bar{\sigma}^2 + 2\bar{\sigma} &= 0, & \bar{\nu}^2 &= 0, & \zeta^2 &= 0, & \eta_4 \zeta &= 0, \\
\bar{\sigma} \zeta &= \eta_1 \bar{\nu}, & \eta_1^2 \zeta &= 2^{4n+1} \bar{\sigma}.
\end{aligned}$$

Now we are ready for computing the K -groups of PG .

Let ρ be the canonical, non-trivial, 2^{4n} -dimensional complex representation of G and $\lambda^i \rho$ be the i -th exterior power of ρ . Since the restriction of $\lambda^{2i} \rho$ to the center of G is trivial clearly, it factors through the canonical projection $\pi: \rightarrow PG$. So we view $\lambda^{2i} \rho$ also as a representation of PG below. Moreover, as is well known, an element of $\widetilde{K}^{-1}(PG)$ is represented as the homotopy class of a map from PG to the infinite dimensional unitary group U . Hence we see that $\lambda^{2i} \rho$ yields naturally an element $\beta(\lambda^{2i} \rho)$ of $K^{-1}(PG)$, which is called the β -construction of $\lambda^{2i} \rho$ [8]. Because $\dim_{\mathbb{C}} \lambda^{2i+1} \rho = \binom{2^{4n+1}}{2i+1}$ and $2^{4n+1} \parallel \binom{2^{4n+1}}{2i+1}$, $d_{2i+1} = \binom{2^{4n+1}}{2i+1} / 2^{4n+1}$ is odd. Let $\ell \rho$ denote the direct sum of ℓ copies of ρ . The map $PG \rightarrow U\left(\binom{2^{4n+1}}{2i+1}\right)$ given by the assignment $\pi(g) \mapsto (d_{2i+1} \rho)(g) \lambda^{2i+1} \rho(g)$ defines a similar element $\beta(d_{2i+1} \rho + \lambda^{2i+1} \rho)$ of $K^{-1}(PG)$.

We describe explicitly the image of $\beta(\rho) \in K^{-1}(G)$ by the transfer map $\pi_*: K^{-1}(G) \rightarrow K_F^{-1}(G) = K^{-1}(PG)$. Let us view $E = G \times (\mathbb{C}^{2^{4n+1}} \oplus \mathbb{C}^{2^{4n+1}})$ as a product Γ -vector bundle over G provided with the Γ -action given by $(g, u, v) \mapsto (-g, v, u)$ for $g \in G$, $u, v \in \mathbb{C}^{2^{4n+1}}$. Then the assignment $(g, u, v) \mapsto (g, \rho(g)u, -\rho(g)v)$ gives an equivariant bundle automorphism of E . In a canonical way this gives rise to an element of $K_F^{-1}(G)$ which is just $\pi_*(\beta(\rho))$ and is written $\beta(\rho, \Gamma)$ below.

Then we have

Theorem 2.3 ([7,9]). *With the notation as above*

$$K^*(PSp(2^{4n})) = \mathbb{Z}[\sigma]/(2^{4n+1}\sigma, \sigma^2 + 2\sigma) \\ \otimes A(\beta(d_{2i-1}\rho + \lambda^{2i-1}\rho), \beta(\lambda^{2j}\rho), \beta(\rho, \Gamma) \\ (2 \leq i \leq 2^{4n-1}, 1 \leq j \leq 2^{4n-1}))/I$$

as a ring where I is the ideal generated by

$$\sigma\beta(\rho, \Gamma).$$

Proof. We observe the exact sequence of (1.4b). According to [8]

$$K^*(G) = A(\beta(\rho), \beta(\lambda^2\rho), \dots, \beta(\lambda^{2^{4n}}\rho)).$$

Since $K^*(G)$ is torsion-free we have the Künneth isomorphism

$$K^*(P^{8n+3} \times G) \cong K^*(P^{8n+3}) \otimes K^*(G).$$

Then we get similarly to (2.1) the following.

$$(2.4) \quad \delta(\nu \times 1) = \sigma + 2, \quad J(1) = -2^{4n+1}\sigma \quad \text{and} \quad I(\sigma) = \sigma + 1.$$

Now $2^{4n+1}\sigma = 0$ follows because of $\rho(-1) = -I$. Hence (1.4b) becomes a short exact sequence

$$0 \longrightarrow K^*(PG) \xrightarrow{I} K^*(P^{8n+3} \times G) \longrightarrow \delta K^*(PG) \longrightarrow 0$$

provided with $\delta(xI(y)) = \delta(x)y$. Further by inspecting definition we have

$$(2.5) \quad \begin{aligned} I(\beta(\lambda^{2i}\rho)) &= 1 \times \beta(\lambda^{2i}\rho), \\ I(\beta(d_{2i-1}\rho + \lambda^{2i-1}\rho)) \\ &= (\sigma + 1) \times d_{2i-1}\beta(\rho) + 1 \times \beta(\lambda^{2i-1}\rho) + d_{2i-1}\nu \times 1, \\ I(\beta(\rho, \Gamma)) &= (\sigma + 2) \times \beta(\rho) + \nu \times 1, \\ \delta(1 \times \beta(\rho)) &= -1. \end{aligned}$$

Let R denote the ring on the right-hand side of the equality of the theorem. Using the last formula of (2.4) and the first three formulas of (2.5), the injectivity of I shows that R is a subring of $K^*(PG)$.

To prove the theorem it therefore suffices to verify that $\text{Im } \delta = R$ since δ is surjective. The images of generators of $K^*(P^{8n+3} \times G)$ as a

module by δ can be calculated by using (2.5) together with the equality $\delta(xI(y)) = \delta(x)y$. For example, we have

$$\begin{aligned}\delta(1 \times \beta(\lambda^{2i-1}\rho)) &= -d_{2i-1}(\sigma + 1), \\ \delta(\nu \times 1) &= -(\sigma + 2), \\ \delta(\nu \times \beta(\rho)) &= \beta(\rho, \Gamma), \\ \delta(1 \times \beta(\rho)\beta(\lambda^{2i-1}\rho)) &= -\beta(d_{2i-1}\rho + \lambda^{2i-1}\rho) - d_{2i-1}\beta(\rho, \Gamma).\end{aligned}$$

Thus by repeating such a computation inductively we get $\mathrm{Im} \delta = R$, which completes our proof.

3. In this section we compute $KO^*(PG)$. First we consider the exact sequence (1.4b) for KO -theory. The complex representation ρ of G is, of course, the complexification of the 2^{4n} -dimensional quaternionic representation, for which we write $\bar{\rho}$. Clearly $\bar{\rho}$ yields an isomorphism $G \times H^{2^{4n}} \otimes_{\mathbf{R}} H \cong G \times H^{2^{4n}}$ of Γ -quaternionic vector bundles over G . Now we have $J(1) = 2^{4n}\eta_4\bar{\sigma}$ similarly to the 2nd formula of (2.1) and also $\alpha \otimes_C H = \eta_4$ by (1.2). Hence we see that $J(1) = 0$, so that (1.4b) becomes a short exact sequence

$$(3.1) \quad 0 \longrightarrow KO^*(PG) \xrightarrow{I} KO^*(P^{8n+3} \times G) \xrightarrow{\delta} KO^*(PG) \longrightarrow 0$$

provided with $\delta(xI(y)) = \delta(x)y$.

Using this exact sequence we proceed as the same way as for $K^*(PG)$.

Let $\lambda_C^k \bar{\rho}$ be the exterior power $\bar{\rho} \wedge_C \cdots \wedge_C \bar{\rho}$ of $\bar{\rho}$ over C . Then in general $\lambda_C^k \bar{\rho}$ is quaternionic. But if k is even then it has a natural Real structure. So we consider $\lambda_C^{2i} \bar{\rho}$ to be real. By the β -construction we have

$$\beta(\lambda_C^{2i-1} \bar{\rho}) \in \widetilde{KS}p^{-1}(G) \quad \text{and} \quad \beta(\lambda_C^{2i} \bar{\rho}) \in \widetilde{KO}^{-1}(G)$$

and we set

$$\bar{\beta}(\lambda_C^{2i-1} \bar{\rho}) = \alpha \wedge_C \beta(\lambda_C^{2i-1} \bar{\rho}) \in \widetilde{KO}^{-1}(\Sigma^{0,4} \wedge G) = \widetilde{KO}^{-5}(G).$$

Then, according to [15], Theorem 5.6,

$$(3.2) \quad KO^*(G) = \Lambda_{KO^*(+)}(\bar{\beta}(\lambda_C^{2i-1} \bar{\rho}), \beta(\lambda_C^{2i} \bar{\rho}) \quad (1 \leq i \leq 2^{4n-1}))$$

as a $KO^*(+)$ -module. Further by [4], §6 and [13], Corollary 2.3 we see that its generators satisfy the relations

$$(3.3) \quad \bar{\beta}(\lambda_C^{2i-1} \bar{\rho})^2 = \eta_1 \beta(\lambda_C^{4i-2} \bar{\rho}), \quad \beta(\lambda_C^{2i} \bar{\rho})^2 = \eta_1 \beta(\lambda_C^{4i} \bar{\rho}).$$

Here we note that $\lambda_C^k \bar{\rho} = \lambda_C^{2^{4n+1}-k} \bar{\rho}$ for $1 \leq k \leq 2^{4n}$. Of course this equality holds for $\lambda_C^{2^k} \bar{\rho}$ viewed as a representation of PG .

Because $KO^*(G)$ is torsion-free, there holds the Künneth isomorphism $KO^*(P^{8n+3} \times G) \cong KO^*(P^{8n+3}) \otimes_{KO^*(+)} KO^*(G)$. Therefore by using (2.2b), (3.2) and (3.3), the multiplicative structure of $KO^*(P^{8n+3} \times G)$ centered in the sequence (3.1) can be described explicitly.

In order to state our theorem we provide generators of $KO^*(PG)$. Similarly to the complex case we have

$$\begin{aligned} \beta(d_{2i-1}\bar{\rho} + \lambda_C^{2^{i-1}}\bar{\rho}), \quad \beta(\bar{\rho}, \Gamma) \in \widetilde{KSp}^{-1}(PG) \quad \text{and} \\ \beta(\lambda_C^{2^i}\bar{\rho}) \in \widetilde{KO}^{-1}(PG) \end{aligned}$$

and so we set

$$\begin{aligned} \bar{\beta}(d_{2i-1}\bar{\rho} + \lambda_C^{2^{i-1}}\bar{\rho}) &= \alpha \wedge_C \beta(d_{2i-1}\bar{\rho} + \lambda_C^{2^{i-1}}\bar{\rho}), \\ \bar{\beta}(\bar{\rho}, \Gamma) &= \alpha \wedge_C \beta(\bar{\rho}, \Gamma) \in \widetilde{KO}^{-5}(PG). \end{aligned}$$

Moreover we see that

(3.4) There exists an element $\bar{\zeta} \in KO^{-6}(PG)$ such that

$$I(\bar{\zeta}) = \eta_1 \times \bar{\beta}(\bar{\rho}) + \zeta \times 1.$$

This is shown below.

Then we obtain the following.

Theorem 3.5. *With the notation as above*

$$KO^*(PSp(2^{4n})) = \mathbb{Z}[\bar{\sigma}]/(\bar{\sigma}^2 + 2\bar{\sigma}) \otimes E \otimes \Lambda_{\mathbb{Z}/2}(\bar{\zeta})/I$$

as a ring where E is a $KO^*(+)$ -module

$$\begin{aligned} \Lambda_{KO^*(+)}(\bar{\beta}(d_{2i-1}\bar{\rho} + \lambda_C^{2^{i-1}}\bar{\rho}), \beta(\lambda_C^{2^j}\bar{\rho}), \bar{\beta}(\bar{\rho}, \Gamma)) \\ (2 \leq i \leq 2^{4n-1}, 1 \leq j \leq 2^{4n-1}) \end{aligned}$$

with the relations

$$\begin{aligned} \bar{\beta}(d_{2i-1}\bar{\rho} + \lambda_C^{2^{i-1}}\bar{\rho})^2 &= \eta_1(\beta(\lambda_C^{2^i}\bar{\rho}) + \beta(\lambda_C^{4i-2}\bar{\rho})), \\ \beta(\lambda_C^{2^j}\bar{\rho})^2 &= \eta_1\beta(\lambda_C^{4j}\bar{\rho}), \\ \bar{\beta}(\bar{\rho}, \Gamma)^2 &= 0 \end{aligned}$$

and I is the ideal generated by

$$2^{4n}\bar{\sigma}\eta_4, \quad \bar{\sigma}\bar{\beta}(\bar{\rho}, \Gamma), \quad \eta_4\bar{\zeta}, \quad \bar{\sigma}\bar{\zeta} - \eta_1\bar{\beta}(\bar{\rho}, \Gamma), \quad \eta_1^2\bar{\zeta} - 2^{4n+1}\bar{\sigma}$$

(the $\otimes$'s are omitted).

Proof. Observe (3.1). By looking at the definitions of the maps and elements we have

- (i) $I(\bar{\sigma}) = \bar{\sigma} \times 1$,
- (ii) $I(\beta(\lambda_C^{2i}\bar{\rho})) = 1 \times \beta(\lambda_C^{2i}\bar{\rho})$,
- (iii) $I(\bar{\beta}(d_{2i-1}\bar{\rho} + \lambda_C^{2i-1}\bar{\rho})) = (\bar{\sigma} + 1) \times d_{2i-1}\bar{\beta}(\bar{\rho}) + 1 \times \bar{\beta}(\lambda_C^{2i-1}\bar{\rho})$
 $\quad + d_{2i-1}\bar{\nu} \times 1$,
- (iv) $I(\bar{\beta}(\bar{\rho}, \Gamma)) = (\bar{\sigma} + 2) \times \bar{\beta}(\bar{\rho}) + \bar{\nu} \times 1$,
- (v) $I(1 \times \bar{\beta}(\bar{\rho})) = -1$,
- (vi) $\delta(\bar{\nu} \times 1) = (\bar{\sigma} + 2) \times 1$,
- (vii) $\delta(\zeta \times 1) = \eta_1$.

(3.4) is immediate from (v) and (vii). Let $\bar{R}$ denote the ring on the right-hand side of the equality of Theorem 3.5. Then using (i)–(iv) and (3.4) we see that $\bar{R} \subset KO^*(PG)$ because of the injectivity of I , and by using (v)–(vii) and the equality $\delta(xI(y)) = \delta(x)y$ in addition we can verify that $\bar{R}$ fills $KO^*(PG)$ because of the surjectivity of δ . This completes the proof of the theorem.

REFERENCES

- [1] M. F. ATIYAH: K -theory and reality, Quart. J. Math. Oxford (2) **17** (1966), 367–386.
- [2] M. F. ATIYAH: K -Theory, Benjamin Inc. 1967.
- [3] M. F. ATIYAH: Bott periodicity and the index of elliptic operators, Quart. J. Math. Oxford (2) **74** (1968), 113–140.
- [4] M. C. CRABB: $\mathbb{Z}/2$ -Homotopy Theory, London Math. Soc. Lecture Note Series **44** (1980).
- [5] M. FUJII: KO -groups of projective spaces, Osaka J. Math. **4** (1967), 141–149.
- [6] R. P. HELD und U. SUTER: Die Bestimmung der unitären K -Theorie von $SO(n)$ mit Hilfe der Atiyah-Hirzebruch-Spectralreihe, Math. Z. **122** (1971), 33–52.
- [7] R. P. HELD and U. SUTER: On the unitary K -theory of compact Lie groups with finite fundamental group, Quart. J. Math. Oxford (2) **24** (1973), 343–356.
- [8] L. HODGKIN: On the K -theory of Lie groups, Topology **6** (1967), 1–36.
- [9] L. HODGKIN: The equivariant Künneth theorem in K -theory, Lecture Notes in Math. **496** (Springer-Verlag, 1975), 1–101.
- [10] D. HUSEMOLLER: Fibre Bundles, McGraw-Hill Book Co., 1966.

- [11] P. S. LANDWEBER: On equivariant maps between spheres with involutions, *Ann. of Math.* **89** (1969), 125–137.
- [12] H. MINAMI: On the K -theory of $SO(n)$, *Osaka J. Math.* **21** (1984), 789–808.
- [13] H. MINAMI: The real K -groups of $SO(n)$ for $n \equiv 3, 4$ and $5 \pmod{8}$, *Osaka J. Math.* **25** (1988), 185–211.
- [14] G. SEGAL: Equivariant K -theory, *Publ. Math. I.H.E.S.* **34** (1968), 129–151.
- [15] R. M. SEYMOUR: The Real K -theory of Lie groups and homogeneous spaces, *Quart. J. Math. Oxford (2)* **24** (1973), 7–30.

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