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On the Ko-theory of Lie Groups and Symmetric Spaces

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ON THE KO-THEORY OF LIE GROUPS AND SYMMETRIC SPACES

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0. Introduction. Let G be a compact, 1-connected simple Lie group of rank 2. Then G is one of the following: $SU(3)$, $Sp(2)$ and G_2. In this note we shall describe explicitly the KO-theory of G, together with the action of the Adams operations ψ^k on it, and also describe the KO-theory of symmetric spaces $SU(2n+1)/SO(2n+1)$ and $SU(2n)/Sp(n)$ for $n \geq 1$. In particular, for the first topic, the following fact should be noted. For a compact, connected, semisimple and simply-connected Lie group G, Seymour [16, Theorem 5.6] described theoretically the ($\boldsymbol{Z}/(8)$-graded) ring $KO^*(G)$; its additive structure was determined completely and its multiplicative structure was almost done. However, it seems that papers containing an explicit description of $KO^*(G)$ are [13], [14] and [15].

This paper is arranged as follows. In section 1 we compute the Adams operations in $K^*(G)$. Section 2 consists of preparations for subsequent sections and involves a review of Seymour's work. The ($\boldsymbol{Z}$-graded) ring $KO^*(G)$ will be described in section 3, and the rings $KO^*(SU(2n+1)/SO(2n+1))$ and $KO^*(SU(2n)/Sp(n))$ in section 4.

We shall deal with the $\boldsymbol{Z}$-graded objects, simultaneously with the associated $\boldsymbol{Z}/(2)$- or $\boldsymbol{Z}/(8)$-objects.

1. The Adams operations in $K^*(G)$. Since the Chern character of G was described explicitly in [17], the Adams operations in $K^*(G)$ should be computed. This is what we put into practice in this section.

We begin by recalling some facts on complex K-theory needed in the sequel. For details, see [2], [3] and [9]. We will use the following notation: $\boldsymbol{R}$ is the field of real numbers; $\boldsymbol{C}$ is the field of complex numbers; $\boldsymbol{H}$ is the algebra of quaternions; $\boldsymbol{K}$ is the algebra of Cayley numbers. Let X be a space with nondegenerate base point. The Adams operations $\psi^k \colon K(X) \to K(X)$, $k \in \boldsymbol{Z}$, are homomorphisms of rings. They are closely related with the Chern character $ch \colon K(X) \to H^*(X; \boldsymbol{Q})$. That is, if

$$ch(x) = \sum_{q \geq 0} ch_q(x), \quad ch_q(x) \in H^{2q}(X; \boldsymbol{Q}),$$

for $x \in K(X)$, then

$$ch(\psi^k(x)) = \sum_{q \geq 0} k^q ch_q(x). \tag{1.1}$$

This ch extends to a multiplicative natural transformation of $\boldsymbol{Z}/(2)$-graded cohomology theories ch: $K^*(\ ) \to H^{**}(\ ;\boldsymbol{Q})$. The coefficient ring of reduced $\boldsymbol{Z}$-graded K-theory is

$$\widetilde{K}^*(S^0) = \boldsymbol{Z}[g, g^{-1}]/(gg^{-1} - 1),$$

where $g \in \widetilde{K}^{-2}(S^0)$ is the Bott generator. The action of ψ^k on $\widetilde{K}^*(S^0)$ is given by

$$\psi^k(g) = kg, \quad \text{in particular} \quad \psi^{-1}(g) = -g. \tag{1.2}$$

The K-ring $K(X)$ and the complex representation ring $R(G)$ are λ-rings (see [9, 12(1.1)]); roughly speaking, they possess the exterior power operations λ^k for $k \geq 0$. Let β: $R(G) \to \widetilde{K}^{-1}(G)$ be the homomorphism of abelian groups, introduced in [8], called the beta-construction. Notice that β does not commute with λ^k.

We now consider the case $G = SU(3)$. The space $\boldsymbol{C}^3$ becomes a $SU(3)$-$\boldsymbol{C}$-module in the usual way. We write λ_1 for the class of $\boldsymbol{C}^3$ in $R(SU(3))$, and put $\lambda_k = \lambda^k(\lambda_1) \in R(SU(3))$. Then $R(SU(3))$ equals the polynomial algebra $\boldsymbol{Z}[\lambda_1, \lambda_2]$ (see [1, Theorem 7.4] or [9, 13(3.1)]). Therefore, by the theorem of Hodgkin [8, Theorem A], $K^*(SU(3))$ equals the exterior algebra $\Lambda_{\boldsymbol{Z}}(\beta(\lambda_1), \beta(\lambda_2))$ as a $\boldsymbol{Z}/(2)$-graded Hopf algebra over $\boldsymbol{Z}$. On the other hand, $H^*(SU(3); \boldsymbol{Z}) = \Lambda_{\boldsymbol{Z}}(x_3, x_5)$, where $x_i \in H^i(SU(3); \boldsymbol{Z})$. With this notation, we may set

$$\begin{aligned} ch(\beta(\lambda_1)) &= ax_3 + bx_5, \\ ch(\beta(\lambda_2)) &= cx_3 + dx_5 \end{aligned} \tag{1.3}$$

for some $a, b, c, d \in \boldsymbol{Q}$ (as seen below, these numbers are known). Using the relations $x_3^2 = 0$, $x_5x_3 = -x_3x_5$ and $x_5^2 = 0$, we have

$$\begin{aligned} ch(\beta(\lambda_1)\beta(\lambda_2)) &= adx_3x_5 + bcx_5x_3 \\ &= (ad - bc)x_3x_5 \end{aligned} \tag{1.4}$$

in the $\boldsymbol{Z}/(2)$-graded ring $H^{**}(SU(3); \boldsymbol{Q})$. Since $\{\beta(\lambda_1), \beta(\lambda_2)\}$ is a basis for $\widetilde{K}^{-1}(SU(3)) = \widetilde{K}(\Sigma SU(3)) \cong \boldsymbol{Z} \oplus \boldsymbol{Z}$, we may set

$$\psi^k(\beta(\lambda_1)) = e\beta(\lambda_1) + f\beta(\lambda_2)$$

for some $e, f \in \boldsymbol{Z}$. Let s: $\widetilde{H}^*(G;\boldsymbol{Q}) \to \widetilde{H}^{*+1}(\Sigma G;\boldsymbol{Q})$ denote the suspension isomorphism. Then, by (1.3), ch: $\widetilde{K}(\Sigma SU(3)) \to \widetilde{H}^*(\Sigma SU(3);\boldsymbol{Q})$ satisfies

$$\begin{aligned} ch(\beta(\lambda_1)) &= as(x_3) + bs(x_5), \\ ch(\beta(\lambda_2)) &= cs(x_3) + ds(x_5). \end{aligned}$$

It follows from this and (1.1) that

$$ch(\psi^k(\beta(\lambda_1))) = ak^2 s(x_3) + bk^3 s(x_5),$$

while

$$\begin{aligned} ch(\psi^k(\beta(\lambda_1))) &= ch(e\beta(\lambda_1) + f\beta(\lambda_2)) \\ &= (ae + cf)s(x_3) + (be + df)s(x_5). \end{aligned}$$

Since $a = -1$, $b = 1/2$, $c = -1$ and $d = -1/2$ by [17, Theorem 2], we obtain $-e - f = -k^2$ and $e - f = k^3$. These reduce to $e = k^2(k+1)/2$ and $f = -k^2(k-1)/2$. We call the reader's attention to the fact that these numbers belong to $\boldsymbol{Z}$ (see also part (i) of Theorems 1 to 3).

Similarly, if we set

$$\psi^k(\beta(\lambda_2)) = e\beta(\lambda_1) + f\beta(\lambda_2)$$

for some $e, f \in \boldsymbol{Z}$, we obtain $-e - f = -k^2$ and $e - f = -k^3$, which reduce to $e = -k^2(k-1)/2$ and $f = k^2(k+1)/2$.

Using these results and the relations $\beta(\lambda_1)^2 = 0$, $\beta(\lambda_2)\beta(\lambda_1) = -\beta(\lambda_1)\beta(\lambda_2)$ and $\beta(\lambda_2)^2 = 0$, we have

$$\begin{aligned} \psi^k(\beta(\lambda_1)\beta(\lambda_2)) &= \psi^k(\beta(\lambda_1))\psi^k(\beta(\lambda_2)) \\ &= \frac{k^4(k+1)^2}{4}\beta(\lambda_1)\beta(\lambda_2) + \frac{k^4(k-1)^2}{4}\beta(\lambda_2)\beta(\lambda_1) \\ &= k^5\beta(\lambda_1)\beta(\lambda_2). \end{aligned}$$

Theorem 1. *The action of* ψ^k *on* $\widetilde{K}^*(SU(3))$ *is given by:*
(i) ([18, (2.5)]) *In* $\widetilde{K}^{-1}(SU(3)) = Z\{\beta(\lambda_1), \beta(\lambda_2)\}$,

$$\begin{aligned} \psi^k(\beta(\lambda_1)) &= \frac{k^2(k+1)}{2}\beta(\lambda_1) - \frac{k^2(k-1)}{2}\beta(\lambda_2), \\ \psi^k(\beta(\lambda_2)) &= -\frac{k^2(k-1)}{2}\beta(\lambda_1) + \frac{k^2(k+1)}{2}\beta(\lambda_2). \end{aligned}$$

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(ii) *In* $\widetilde{K}^{-2}(SU(3)) = \boldsymbol{Z}\{\beta(\lambda_1)\beta(\lambda_2)\}$,

$$\psi^k(\beta(\lambda_1)\beta(\lambda_2)) = k^5\beta(\lambda_1)\beta(\lambda_2).$$

We move on to consider the case $G = Sp(2)$. Since the space $\boldsymbol{H}^2$ is a left $Sp(2)$-$\boldsymbol{H}$-module, its complex restriction $(\boldsymbol{H}^2)_{\boldsymbol{C}}$ becomes a $Sp(2)$-$\boldsymbol{C}$-module. We write $\mu_1' = [(\boldsymbol{H}^2)_{\boldsymbol{C}}] \in R(Sp(2))$, and put $\mu_k' = \lambda^k(\mu_1')$. Then $R(Sp(2))$ equals the polynomial algebra $\boldsymbol{Z}[\mu_1', \mu_2']$ (see [1, Theorem 7.6] or [9, 13(6.1)]). Therefore, by the theorem of Hodgkin [8], $K^*(Sp(2)) = \Lambda_{\boldsymbol{Z}}(\beta(\mu_1'), \beta(\mu_2'))$. On the other hand, $H^*(Sp(2); \boldsymbol{Z}) = \Lambda_{\boldsymbol{Z}}(x_3, x_7)$, where $x_i \in H^i(Sp(2); \boldsymbol{Z})$. With this notation, [17, Theorem 3] tells us that

$$ch(\beta(\mu_1')) = x_3 - \frac{1}{6}x_7,$$
$$ch(\beta(\mu_2')) = 2x_3 + \frac{2}{3}x_7.$$

By a calculation similar to the case of $SU(3)$ we have

Theorem 2. *The action of* ψ^k *on* $\widetilde{K}^*(Sp(2))$ *is given by*:
(i) *In* $\widetilde{K}^{-1}(Sp(2)) = \boldsymbol{Z}\{\beta(\mu_1'), \beta(\mu_2')\}$,

$$\psi^k(\beta(\mu_1')) = \frac{k^2(k^2+2)}{3}\beta(\mu_1') - \frac{k^2(k^2-1)}{6}\beta(\mu_2'),$$
$$\psi^k(\beta(\mu_2')) = -\frac{4k^2(k^2-1)}{3}\beta(\mu_1') + \frac{k^2(2k^2+1)}{3}\beta(\mu_2').$$

(ii) *In* $\widetilde{K}^{-2}(Sp(2)) = \boldsymbol{Z}\{\beta(\mu_1')\beta(\mu_2')\}$,

$$\psi^k(\beta(\mu_1')\beta(\mu_2')) = k^6\beta(\mu_1')\beta(\mu_2').$$

Finally we consider the case $G = G_2$. The automorphism group of $\boldsymbol{K}$ is G_2. As seen in [19, Appendix A] or [20, p.217], the subspace $\boldsymbol{K}_0$ consisting of pure imaginary elements in $\boldsymbol{K}$ forms a G_2-$\boldsymbol{R}$-module of dimension 7. Hence its complexification $\boldsymbol{K}_0^{\boldsymbol{C}}$ becomes a G_2-$\boldsymbol{C}$-module. We write $\rho_1 = [\boldsymbol{K}_0^{\boldsymbol{C}}] \in R(G_2)$, and put $\rho_k = \lambda^k(\rho_1)$. Then $R(G_2)$ equals the polynomial algebra $\boldsymbol{Z}[\rho_1, \rho_2]$. Therefore, by the theorem of Hodgkin [8], $K^*(G_2) = \Lambda_{\boldsymbol{Z}}(\beta(\rho_1), \beta(\rho_2))$. On the other hand, $H^*(G_2; \boldsymbol{Z})/\mathrm{Tor.} = \Lambda_{\boldsymbol{Z}}(x_3, x_{11})$, where $x_i \in H^i(G_2; \boldsymbol{Z})$. With this notation, [17, Theorem 7] tells us that

$$ch(\beta(\rho_1)) = 2x_3 + \frac{1}{60}x_{11},$$
$$ch(\beta(\rho_2)) = 10x_3 - \frac{5}{12}x_{11}.$$

By a calculation similar to the case of $SU(3)$ we have

Theorem 3. *The action of* ψ^k *on* $\widetilde{K}^*(G_2)$ *is given by*:
(i) *In* $\widetilde{K}^{-1}(G_2) = \boldsymbol{Z}\{\beta(\rho_1), \beta(\rho_2)\}$,

$$\psi^k(\beta(\rho_1)) = \frac{k^2(k^4+5)}{6}\beta(\rho_1) - \frac{k^2(k^4-1)}{30}\beta(\rho_2),$$
$$\psi^k(\beta(\rho_2)) = -\frac{25k^2(k^4-1)}{6}\beta(\rho_1) + \frac{k^2(5k^4+1)}{6}\beta(\rho_2).$$

(ii) *In* $\widetilde{K}^{-2}(G_2) = \boldsymbol{Z}\{\beta(\rho_1)\beta(\rho_2)\}$,

$$\psi^k(\beta(\rho_1)\beta(\rho_2)) = k^8\beta(\rho_1)\beta(\rho_2).$$

2. The Bott's exact sequence. We will use the quaternionic representation ring functor $RSp(\ )$ and quaternionic K-theory $KSp^*(\ )$ as well as the real representation ring functor $RO(\ )$ and real K-theory $KO^*(\ )$. To do this we fix some notation. For details, see [1], [2], [4] and [9].

Let c: $RO(G) \to R(G)$ and c: $KO(X) \to K(X)$ be the complexifications; they preserve the λ-ring structures. Let r: $R(G) \to RO(G)$ and r: $K(X) \to KO(X)$ be the real restrictions; they preserve the additions only. Let t: $R(G) \to R(G)$ and t: $K(X) \to K(X)$ be the complex conjugations; they preserve the λ-ring structures. Finally, let c': $RSp(G) \to R(G)$ and c': $KSp(X) \to K(X)$ be the complex restrictions; they preserve the additions only. Then, among them, the following formulas hold:

(2.1a) $\quad rc = 2,\ cr = 1+t,\ tc = c,\ tc' = c'$ and $t^2 = 1$.
(2.1b) $\quad r(x \cdot c(z)) = r(x)z$ for all $x \in K(X)$ and $z \in KO(X)$.

The coefficient ring of reduced $\boldsymbol{Z}$-graded KO-theory is

$$\widetilde{KO}^*(S^0) = \boldsymbol{Z}[\eta, \nu, \sigma, \sigma^{-1}]/(2\eta, \eta^3, \eta\nu, \nu^2 - 4\sigma, \sigma\sigma^{-1} - 1),$$

where $\eta \in \widetilde{KO}^{-1}(S^0)$, $\nu \in \widetilde{KO}^{-4}(S^0)$ and $\sigma \in \widetilde{KO}^{-8}(S^0)$. When $\widetilde{K}^*(X)$ has been determined (our case is this), a basic tool for computing $\widetilde{KO}^*(X)$ is the Bott's exact sequence

$$\cdots \longrightarrow \widetilde{KO}^{1-q}(X) \xrightarrow{\eta} \widetilde{KO}^{-q}(X) \xrightarrow{c} \widetilde{K}^{-q}(X) \xrightarrow{\delta} \widetilde{KO}^{2-q}(X) \longrightarrow \cdots,$$

where η denotes multiplication by η and the map δ is defined by

(2.2) $$\delta(x) = r(g^{-1}x) \quad \text{for} \quad x \in \widetilde{K}^{-q}(X).$$

We have

(2.3a) $c(1) = 1,\ c(\eta) = 0,\quad c(\eta^2) = 0,\quad c(\nu) = 2g^2$ and $c(\sigma) = g^4$.

(2.3b) $r(1) = 2,\ r(g) = \eta^2,\ r(g^2) = \nu,\ r(g^3) = 0$ and $r(g^4) = 2\sigma$.

From this and (2.1b), one can calculate $r(g^i)$ for all $i \in \boldsymbol{Z}$. The action of ψ^k on $\widetilde{KO}^*(S^0)$ is given by

(2.4) $$\psi^k(\eta) = k\eta,\ \psi^k(\nu) = k^2\nu \quad \text{and} \quad \psi^k(\sigma) = k^4\sigma.$$

The following two lemmas can be proved by using the Bott's exact sequence and are included in [16, Theorem 4.2]. So we omit the details of their proofs. By $\widetilde{K}^*(S^0)\{x, \cdots\}$ we denote the free $\widetilde{K}^*(S^0)$-module generated by elements $x, \cdots$.

Lemma 4. *Suppose that, as a $\boldsymbol{Z}$-graded module with t-action, $\widetilde{K}^*(X)$ has a direct summand $T\langle x\rangle$ which is a free $\widetilde{K}^*(S^0)$-module generated by two elements $x, x' \in \widetilde{K}^n(X)$ such that $t(x) = x'$ (and so $t(x') = x$), where $x' \neq \pm x$. Then $\widetilde{KO}^*(X)$ contains the image $r(\widetilde{K}^*(S^0)\{x\})$ of $\widetilde{K}^*(S^0)\{x\}$ under r as a direct summand. It is described by:*

$$\begin{aligned}
&\widetilde{KO}^n(X) \supset \boldsymbol{Z}\{r(x)\}, && \widetilde{KO}^{n-1}(X) \supset 0,\\
&\widetilde{KO}^{n-2}(X) \supset \boldsymbol{Z}\{r(gx)\}, && \widetilde{KO}^{n-3}(X) \supset 0,\\
&\widetilde{KO}^{n-4}(X) \supset \boldsymbol{Z}\{r(g^2x)\}, && \widetilde{KO}^{n-5}(X) \supset 0,\\
&\widetilde{KO}^{n-6}(X) \supset \boldsymbol{Z}\{r(g^3x)\}, && \widetilde{KO}^{n-7}(X) \supset 0
\end{aligned}$$

where $r(g^ix') = (-1)^i r(g^ix)$ for $i \in \boldsymbol{Z}$.

Proof. We prove the last relation only.

$$\begin{aligned}
0 &= \delta c(r(g^{i+1}x)) && \text{by exactness of Bott's exact sequence}\\
&= \delta((1+t)(g^ix)) && \text{since } cr = 1 + t \text{ by (2.1a)}\\
&= \delta(g^{i+1}x + (-1)^{i+1}g^{i+1}x') && \text{since } t(g) = -g \text{ by (1.2)}\\
&= r(g^ix) + (-1)^{i+1}r(g^ix') && \text{by (2.2).}
\end{aligned}$$

Lemma 5. *Suppose that, as a Z-graded module with t-action, $\widetilde{K}^*(X)$ has a direct summand $N\langle y\rangle$ which is a free $\widetilde{K}^*(S^0)$-module generated by an element $y \in \widetilde{K}^n(X)$ such that $y = c(z)$ for some $z \in \widetilde{KO}^n(X)$. Then $\widetilde{KO}^*(X)$ contains the free $KO^*(S^0)$-module $\widetilde{KO}^*(S^0)\{z\}$ generated by z as a direct summand. It is described by:*

$$\begin{array}{ll}
\widetilde{KO}^n(X) \supset Z\{z\}, & \widetilde{KO}^{n-1}(X) \supset Z/(2)\{\eta z\}, \\
\widetilde{KO}^{n-2}(X) \supset Z/(2)\{\eta^2 z\}, & \widetilde{KO}^{n-3}(X) \supset 0, \\
\widetilde{KO}^{n-4}(X) \supset Z\{\nu z\}, & \widetilde{KO}^{n-5}(X) \supset 0, \\
\widetilde{KO}^{n-6}(X) \supset 0, & \widetilde{KO}^{n-7}(X) \supset 0.
\end{array}$$

Instead of Lemma 5, we show the following which we will often use.

Lemma 6. *Suppose that $\widetilde{K}^*(X)$ is the free $\widetilde{K}^*(S^0)$-module generated by m elements $b_1, b_2, \ldots, b_m$, $b_i \in \widetilde{K}^{n_i}(X)$, satisfying $t(b_i) = b_i$. Suppose further that there exist elements $a_1, a_2, \ldots, a_m$, $a_i \in \widetilde{KO}^{n_i}(X)$, satisfying $c(a_i) = b_i$. Then $\widetilde{KO}^*(X)$ is the free $\widetilde{KO}^*(S^0)$-module generated by $a_1, a_2, \ldots, a_m$.*

Proof. (Note that, since $tc = c$ by (2.1a), $c(a_i) = b_i$ implies $t(b_i) = b_i$.) We use the machinery of exact couples (see [10]). An exact couple is an exact triangle of graded abelian groups.

$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

Then $d = jk \colon E \to E$ satisfies $d^2 = 0$, and there is another exaxt couple (the derived couple)

$$\begin{tikzcd} D' \arrow[rr, "i'"] & & D \arrow[dl, "j'"] \\ & E' \arrow[ul, "k'"] & \end{tikzcd}$$

where $D' = i(D)$, $E' = \mathrm{Ker}\, d/\mathrm{Im}\, d$, i' is induced by i, k' is induced by k, and $j'(i(a)) = [j(a)]$ for $a \in D$.

The Bott's exact sequence yields an exact couple by setting $D = \widetilde{KO}^*(X)$, $E = \widetilde{K}^*(X)$, $i = \eta \colon \widetilde{KO}^*(X) \to \widetilde{KO}^{*-1}(X)$, $j = c \colon \widetilde{KO}^*(X) \to \widetilde{K}^*(X)$, and $k = \delta \colon \widetilde{K}^*(X) \to \widetilde{KO}^{*+2}(X)$. (With this notation, since

$\eta^3 = 0$, it follows that $D''' = 0$ and $E''' = 0$.) For any $b \in \widetilde{K}^*(X)$

$$\begin{aligned} d(b) &= c\delta(b) && \text{by the definition of } d \\ &= cr(g^{-1}b) && \text{by (2.2)} \\ &= (1+t)(g^{-1}b) && \text{since } cr = 1+t. \end{aligned}$$

By the first hypothesis,

$$E = \boldsymbol{Z}\{g^k b_i \mid i = 1, 2, \dots, m; k \in \boldsymbol{Z}\}. \tag{2.5}$$

We have

$$\begin{aligned} d(g^k b_i) &= g^{k-1} b_i + (-1)^{k-1} g^{k-1} t(b_i) \\ &\qquad \text{since } t(g) = -g \text{ and } t \text{ is a ring homomorphism} \\ &= (1 + (-1)^{k-1}) g^{k-1} b_i \quad \text{since } t(b_i) = b_i. \end{aligned}$$

Therefore

$$\operatorname{Ker} d = \boldsymbol{Z}\{g^{2k} b_i \mid i = 1, 2, \dots, m; k \in \boldsymbol{Z}\}$$

and

$$\operatorname{Im} d = \boldsymbol{Z}\{2g^{2k} b_i \mid i = 1, 2, \dots, m; k \in \boldsymbol{Z}\}.$$

Hence

$$E' = \boldsymbol{Z}/(2)\{[g^{2k} b_i] \mid i = 1, 2, \dots, m; k \in \boldsymbol{Z}\}. \tag{2.6}$$

Using the second hypothesis, we prove that the multiples of a_i by η, η^2, ν and σ are not zero. First of all, $\sigma a_i \neq 0$, since σ: $\widetilde{KO}^*(X) \to \widetilde{KO}^{*-8}(X)$ is an isomorphism. Let us verify that $\nu a_i \neq 0$. We have

$$\begin{aligned} c(\nu a_i) &= c(\nu)c(a_i) && \text{since } c \text{ is a ring homomorphism} \\ &= 2g^2 b_i && \text{since } c(\nu) = 2g^2 \text{ by (2.3a) and } c(a_i) = b_i. \end{aligned}$$

Assume that $\nu a_i = 0$. Then $2g^2 b_i = 0$. This contradicts (2.5), and proves the assertion. Let us next verify that $\eta a_i \neq 0$. Consider the homomorphism δ: $E \to D$, where E is as in (2.5). For $j = 1, 2, 3, 4$ and for $k \in \boldsymbol{Z}$, we have

$$\begin{aligned} \delta(g^{4k+j} b_i) &= r(g^{4k+j-1} b_i) \\ &= r(g^{j-1} c(\sigma^k a_i)) && \text{since } c(\sigma) = g^4 \text{ by (2.3a)} \\ &= r(g^{j-1})\sigma^k a_i && \text{by (2.1b).} \end{aligned}$$

By this and (2.3b), we find that the elements which may belong to $\operatorname{Im}\delta$ are $2\sigma^k a_i$, $\eta^2\sigma^k a_i$ (this may be zero at this point of time) and $\nu\sigma^k a_i$. Assume that $\eta a_i = 0$. Then $a_i \in \operatorname{Ker}\eta = \operatorname{Im}\delta$. This contradicts the above observation, and proves the assertion. Hence $\eta\sigma^k a_i$ becomes a nonzero element of D'. Let us finally verify that $\eta^2 a_i \neq 0$. Consider the homomorphism $\delta'\colon E' \to D'$, where E' is as in (2.6). For $k \in Z$ we have

$$
\begin{aligned}
&\delta'([g^{4k}b_i]) = \delta(g^{4k}b_i) = r(g^3)\sigma^{k-1}a_i = 0 \quad \text{and} \\
&\delta'([g^{4k+2}b_i]) = \delta(g^{4k+2}b_i) = r(g)\sigma^k a_i = \eta^2\sigma^k a_i.
\end{aligned} \tag{2.7}
$$

Assume that $\eta^2 a_i = 0$. Then, on the one hand, $\eta a_i \in \operatorname{Ker}\eta' = \operatorname{Im}\delta'$ and on the other hand, the above calculation implies that $\operatorname{Im}\delta' = 0$. This is a contradiction, and proves the assertion. Hence $\eta^2\sigma^k a_i$ becomes a nonzero element of D'.

Since $\operatorname{Im} c' = \operatorname{Ker}\delta'$, we have an exact sequence

$$0 \longrightarrow D'/\operatorname{Ker} c' \xrightarrow{c'} E' \xrightarrow{\delta'} \operatorname{Im}\delta' \longrightarrow 0.$$

Since $\operatorname{Ker} c' = \operatorname{Im}\eta'$ and $\operatorname{Im}\delta' = \operatorname{Ker}\eta'$, it can be rewritten in the form

$$0 \longrightarrow \operatorname{Coker}\eta' \xrightarrow{c'} E' \xrightarrow{\delta'} \operatorname{Ker}\eta' \longrightarrow 0,$$

where E' is as in (2.6),

$$c'(\eta\sigma^k a_i) = [c(\sigma^k a_i)] = [c(\sigma)^k c(a_i)] = [g^{4k}b_i]$$

and (2.7) holds. So we conclude that

$$\operatorname{Coker}\eta' = Z/(2)\{\eta\sigma^k a_i \mid i = 1, 2, \ldots, m; k \in Z\}$$

and

$$\operatorname{Ker}\eta' = Z/(2)\{\eta^2\sigma^k a_i \mid i = 1, 2, \ldots, m; k \in Z\}.$$

Consider the exact sequence

$$0 \longrightarrow \operatorname{Ker}\eta' \longrightarrow D' \xrightarrow{\eta'} D' \longrightarrow \operatorname{Coker}\eta' \longrightarrow 0.$$

Then, since $2\eta = 0$ and $\eta^3 = 0$, it follows that

$$D' = Z/(2)\{\eta\sigma^k a_i, \eta^2\sigma^k a_i \mid i = 1, 2, \ldots, m; k \in Z\}. \tag{2.8}$$

There is a short exact sequence

$$0 \longrightarrow \mathrm{Ker}\, c \longrightarrow D \xrightarrow{c} \mathrm{Im}\, c \longrightarrow 0.$$

Since $\mathrm{Ker}\, c = \mathrm{Im}\,\eta = D'$ and $\mathrm{Im}\, c = \mathrm{Ker}\,\delta$, it can be rewritten in the form

$$0 \longrightarrow D' \longrightarrow D \xrightarrow{c} \mathrm{Ker}\,\delta \longrightarrow 0.$$

From a description of the behavior of δ given in the preceding paragraph, we see that

$$\mathrm{Ker}\,\delta = \boldsymbol{Z}\{g^{4k}b_i, 2g^{4k+2}b_i \mid i = 1, 2, \ldots, m; k \in \boldsymbol{Z}\}.$$

Therefore $D \cong D' \oplus \mathrm{Ker}\,\delta$, and since $c(\sigma^k a_i) = g^{4k}b_i$ and $c(\nu\sigma^k a_i) = 2g^{4k}b_i$, it follows from (2.8) that

$$\begin{aligned} D = \boldsymbol{Z}/(2)&\{\eta\sigma^k a_i, \eta^2\sigma^k a_i \mid i = 1, 2, \ldots, m; k \in \boldsymbol{Z}\} \\ &\oplus \boldsymbol{Z}\{\sigma^k a_i, \nu\sigma^k a_i \mid i = 1, 2, \ldots, m; k \in \boldsymbol{Z}\}. \end{aligned}$$

Thus the proof is completed.

Let G be a compact connected Lie group. Recall from [1, Proposition 3.27] that c: $RO(G) \to R(G)$ and c': $RSp(G) \to R(G)$ are injective. A representation μ of G is said to be *self-conjugate* if $t(\mu) = \mu$. Similarly, μ is said to be *real* if it lies in the image of c: $RO(G) \to R(G)$, and μ is said to be *quaternionic* if it lies in the image of c': $RSp(G) \to R(G)$. According to [1, Proposition 3.56], if μ is irreducible and self-conjugate, it is either real or quaternionic, but not both. The following is a collection of results from [1, Chapter 7] and [20, Chapter 5]:

Proposition 7. *For $G = SU(3)$, $Sp(2)$ and G_2, the action of t on $R(G)$ is given by*:

(1) *In* $R(SU(3)) = \boldsymbol{Z}[\lambda_1, \lambda_2]$,

$$t(\lambda_1) = \lambda_2 \quad \textit{and} \quad t(\lambda_2) = \lambda_1.$$

(2) *In* $R(Sp(2)) = \boldsymbol{Z}[\mu_1', \mu_2']$,

$$t(\mu_i') = \mu_i' \quad \textit{for} \quad i = 1, 2$$

where μ_1' is quaternionic and μ_2' is real.

(3) *In* $R(G_2) = \boldsymbol{Z}[\rho_1, \rho_2]$,

$$t(\rho_i) = \rho_i \quad for \quad i = 1, 2$$

where both ρ_1 *and* ρ_2 *are real.*

Since β commutes with t, from Proposition 7 we know the action of t on $\widetilde{K}^*(G)$, which can be deduced from Theorems 1 to 3 by taking $k = -1$, since $t = \psi^{-1}$ (see [2]). These are essential data for computing $KO^*(G)$.

There is a folkloric result which tells us how to represent the complexification map c: $\widetilde{KO}^*(X) \to \widetilde{K}^*(X)$ as a (weak) map of Ω-spectra $c = \{c_i; i \in \boldsymbol{Z}\}$: $\boldsymbol{KO} \to \boldsymbol{K}$ and can be read off from [5] or [7]. To state it, we need some notation. A monomorphism i_F: $H \to G$ of (not necessarily compact) Lie groups induces the following maps

$$G \xrightarrow{\pi_F} G/H \xrightarrow{j_F} BH \xrightarrow{\rho_F} BG.$$

We denote by $i_{\boldsymbol{C}}$ the natural inclusions $SO(n) \to SU(n)$, $O(n) \to U(n)$ and their stable versions, by $i_{\boldsymbol{H}}$ the natural inclusions $SU(n) \to Sp(n)$, $U(n) \to Sp(n)$, etc., by $i_{\boldsymbol{R}}$ the standard monomorphism $U(n) \to O(2n)$ etc., and by $i_{\boldsymbol{C}'}$ the standard monomorphisms $Sp(n) \to SU(2n)$, $Sp(n) \to U(2n)$, etc., which arise from the correspondence

$$\boldsymbol{H} \ni \alpha + \boldsymbol{j}\beta \longmapsto \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \in M(2, \boldsymbol{C})$$

where $\alpha, \beta \in \boldsymbol{C}$ and $\bar{\alpha}$ is the complex conjugate of α (see [11]).

Proposition 8. *For each* $i \in \boldsymbol{Z}$, c_i *is given as follows.*

(i) *If* $i \equiv 0 \pmod 8$, $c_i = \rho_{\boldsymbol{C}}$: $BO \times \boldsymbol{Z} \to BU \times \boldsymbol{Z}$.

(ii) *If* $i \equiv 1 \pmod 8$, *then* c_i: $U/O \to U$ *is defined by*

$$c_i(xH) = x\sigma(x)^{-1} \quad for \quad xH \in G/H \tag{2.9}$$

where $(G, H) = (U, O)$ *and* $\sigma = \sigma_\infty$: $U \to U$ *is the limit of maps* σ_n: $U(n) \to U(n)$ *defined by*

$$\sigma_n(A) = \bar{A} \tag{2.10}$$

where $\bar{A}$ *is the complex conjugate of* A.

(iii) *If* $i \equiv 2 \pmod 8$, $c_i = j_{\boldsymbol{H}}$: $Sp/U \to BU \times \boldsymbol{Z}$.

(iv) *If* $i \equiv 3 \pmod 8$, $c_i = i_{C'}: Sp \to U$.

(v) *If* $i \equiv 4 \pmod 8$, $c_i = \rho_{C'}: BSp \times \boldsymbol{Z} \to BU \times \boldsymbol{Z}$.

(vi) *If* $i \equiv 5 \pmod 8$, *then* $c_i: U/Sp \to U$ *is defined as in* (2.9), *where* $(G, H) = (U, Sp)$ *and* $\sigma = \sigma'_\infty: U \to U$ *is the limit of maps* $\sigma'_n: U(2n) \to U(2n)$ *defined by*

$$\sigma'_n(A) = J_n \bar{A} J_n{}^{-1} \tag{2.11}$$

where if I_n *denotes the unit matrix of degree* n,

$$J_n = \begin{pmatrix} O & -I_n \\ I_n & O \end{pmatrix}.$$

(vii) *If* $i \equiv 6 \pmod 8$, $c_i = j_{\boldsymbol{R}}: O/U \to BU \times \boldsymbol{Z}$.

(viii) *If* $i \equiv 7 \pmod 8$, $c_i = i_{\boldsymbol{C}}: O \to U$.

In the notation of Lemmas 4 and 5, Seymour [16, Theorem 5.6] showed that $\widetilde{K}^*(G)$ is a direct sum of $T\langle x\rangle$'s and $N\langle y\rangle$'s, and then $\widetilde{KO}^*(G)$ is a direct sum of $r(\widetilde{K}^*(S^0)\{x\})$'s and $\widetilde{KO}^*(S^0)\{z\}$'s correspondingly.

Furthermore, we recall Seymour's comment [16, Lemma 5.3] on the summand $N\langle y\rangle$. Let G be a compact, 1-connected Lie group. Suppose that an irreducible, self-conjugate representation μ of G is given. Then, by the theorem of Hodgkin [8], $\widetilde{K}^*(G)$ has a summand $\widetilde{K}^*(S^0)\{\beta(\mu)\}$ with $t(\beta(\mu)) = \beta(\mu)$. Here, two cases can occur. The first case is that μ is real and the second is that μ is quaternionic, as mentioned earlier. The beta-construction has the real and quaternionic analogues

$$\beta_{\boldsymbol{R}}: RO(G) \longrightarrow \widetilde{KO}^{-1}(G),$$
$$\beta_{\boldsymbol{H}}: RSp(G) \longrightarrow \widetilde{KSp}^{-1}(G)$$

which satisfy $c\beta_{\boldsymbol{R}} = \beta c$ and $c'\beta_{\boldsymbol{H}} = \beta c'$, respectively. In the first case, there exists a unique element $\widehat{\mu} \in RO(G)$ such that $c(\widehat{\mu}) = \mu$. Then $\beta_{\boldsymbol{R}}(\widehat{\mu}) \in \widetilde{KO}^{-1}(G)$ and $c(\beta_{\boldsymbol{R}}(\widehat{\mu})) = \beta(\mu)$. Thus $\widetilde{KO}^*(G)$ has a summand $\widetilde{KO}^*(S^0)\{\beta_{\boldsymbol{R}}(\widehat{\mu})\}$ corresponding to the summand $N\langle\beta(\mu)\rangle$ in $\widetilde{K}^*(G)$. In the second case, there exists a unique element $\widehat{\mu} \in RSp(G)$ such that $c'(\widehat{\mu}) = \mu$. Then $\beta_{\boldsymbol{H}}(\widehat{\mu}) \in \widetilde{KSp}^{-1}(G)$ and $c'(\beta_{\boldsymbol{H}}(\widehat{\mu})) = \beta(\mu)$. By Proposition 8(iv), since $\widetilde{KSp}^{-1}(X) = [X, Sp]$ (where $[\ ,\ ]$ denotes the set of homotopy classes of maps preserving base points) is identified with $\widetilde{KO}^{-5}(X)$, it is restated as $c(\beta_{\boldsymbol{H}}(\widehat{\mu})) = g^2\beta(\mu)$, where we regard $\beta_{\boldsymbol{H}}(\widehat{\mu})$ as an element of $\widetilde{KO}^{-5}(G)$. Thus $\widetilde{KO}^*(G)$ has a summand $\widetilde{KO}^*(S^0)\{\beta_{\boldsymbol{H}}(\widehat{\mu})\}$ corresponding to the summand $N\langle g^2\beta(\mu)\rangle$ in $\widetilde{K}^*(G)$.

After all, Seymour determined the ring structure of $\widetilde{KO}^*(G)$ except the following point (see [16, Appendix]): what are the squares $\beta_{\boldsymbol{R}}(\widehat{\mu})^2$ and $\beta_{\boldsymbol{H}}(\widehat{\mu})^2$ in the above situation? The Bott's exact sequence tells us only that they are in the image of η. M. Crabb [6, p.67] and H. Minami [14, Proposition 2.2] answered these questions. They showed that

$$\beta_{\boldsymbol{R}}(\widehat{\mu})^2 = \eta \cdot \lambda^2(\beta_{\boldsymbol{R}}(\widehat{\mu})) \quad \text{and} \quad \beta_{\boldsymbol{H}}(\widehat{\mu})^2 = \eta\sigma \cdot \beta_{\boldsymbol{R}}(\widehat{\lambda^2(\mu)}) \tag{2.12}$$

where in the first relation, $\mu \in R(G)$ is real, and in the second relation, $\mu \in R(G)$ is quaternionic and $\lambda^2(\mu) \in R(G)$ is real (see [1, Remark 3.63]). So the problems reduce to determine $\lambda^2(\beta_{\boldsymbol{R}}(\widehat{\mu}))$ and $\beta_{\boldsymbol{R}}(\widehat{\lambda^2(\mu)})$ in $\widetilde{KO}^{-1}(G)$.

3. The rings $KO^*(SU(3))$, $KO^*(Sp(2))$ and $KO^*(G_2)$. Using the notation of the previous section, we describe the $KO^*(pt)$-algebra structure of $KO^*(G)$, together with the action of ψ^k on it.

$\underline{G = SU(3)}$

As seen in [16, Theorem 5.6], there exists an element $\zeta_{1,2} \in \widetilde{KO}^0(SU(3))$ such that $c(\zeta_{1,2}) = g^{-1}\beta(\lambda_1)\beta(\lambda_2)$. It follows from this and Proposition 7(1) that

$$\widetilde{K}^*(SU(3)) = T\langle\beta(\lambda_1)\rangle \oplus N\langle g^{-1}\beta(\lambda_1)\beta(\lambda_2)\rangle \tag{3.1}$$

Theorem 9. *As a $\boldsymbol{Z}$-graded module,*

$$\widetilde{KO}^*(SU(3)) = r(\widetilde{K}^*(S^0)\{\beta(\lambda_1)\}) \oplus \widetilde{KO}^*(S^0)\{\zeta_{1,2}\}.$$

More precisely,

$$\begin{aligned}
\widetilde{KO}^0(SU(3)) &= \boldsymbol{Z}\{\zeta_{1,2}\},\\
\widetilde{KO}^{-1}(SU(3)) &= \boldsymbol{Z}/(2)\{\eta\zeta_{1,2}\} \oplus \boldsymbol{Z}\{r(\beta(\lambda_1))\},\\
\widetilde{KO}^{-2}(SU(3)) &= \boldsymbol{Z}/(2)\{\eta^2\zeta_{1,2}\},\\
\widetilde{KO}^{-3}(SU(3)) &= \boldsymbol{Z}\{r(g\beta(\lambda_1))\},\\
\widetilde{KO}^{-4}(SU(3)) &= \boldsymbol{Z}\{\nu\zeta_{1,2}\},\\
\widetilde{KO}^{-5}(SU(3)) &= \boldsymbol{Z}\{r(g^2\beta(\lambda_1))\},\\
\widetilde{KO}^{-6}(SU(3)) &= 0,\\
\widetilde{KO}^{-7}(SU(3)) &= \boldsymbol{Z}\{r(g^3\beta(\lambda_1))\}.
\end{aligned}$$

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Its $\widetilde{KO}^*(S^0)$-module structure is given by

$$\eta \cdot r(g^i\beta(\lambda_1)) = 0, \quad \nu \cdot r(g^i\beta(\lambda_1)) = 2r(g^{i+2}\beta(\lambda_1)) \quad \text{and}$$
$$\sigma \cdot r(g^i\beta(\lambda_1)) = r(g^{i+4}\beta(\lambda_1)).$$

Its multiplicative structure is given by

$$r(g^i\beta(\lambda_1)) \cdot r(g^j\beta(\lambda_1)) = (-1)^j r(g^{i+j+1})\zeta_{1,2},$$
$$r(g^i\beta(\lambda_1)) \cdot \zeta_{1,2} = 0 \quad \text{and} \quad \zeta_{1,2}^2 = 0.$$

The action of ψ^k on $\widetilde{KO}^*(SU(3))$ is given by

$$\psi^k(r(g^i\beta(\lambda_1))) = \begin{cases} k^{i+2}r(g^i\beta(\lambda_1)) & \text{if } i \equiv 0 \pmod 2 \\ k^{i+3}r(g^i\beta(\lambda_1)) & \text{if } i \equiv 1 \pmod 2 \end{cases}$$
$$\text{and} \quad \psi^k(\zeta_{1,2}) = k^4\zeta_{1,2}.$$

Proof. The additive structure follows from (3.1) and Lemmas 4, 5. For the $\widetilde{KO}^*(S^0)$-module structure, we have

$$\begin{aligned} \eta \cdot r(g^i\beta(\lambda_1)) &= r(g^i\beta(\lambda_1))\eta \\ &= r(g^i\beta(\lambda_1) \cdot c(\eta)) && \text{by (2.1b)} \\ &= 0 && \text{since } c(\eta) = 0 \text{ by (2.3a)} \end{aligned}$$

and the other equalities are obtained similarly.

For the multiplicative structure, we have

$$\begin{aligned} & r(g^i\beta(\lambda_1)) \cdot r(g^j\beta(\lambda_1)) \\ &= r(g^i\beta(\lambda_1) \cdot cr(g^j\beta(\lambda_1))) && \text{by (2.1b)} \\ &= r(g^i\beta(\lambda_1) \cdot (1+t)(g^j\beta(\lambda_1))) && \text{by (2.1a)} \\ &= r(g^i\beta(\lambda_1) \cdot (g^j\beta(\lambda_1) + (-1)^j g^j\beta(\lambda_2))) && \text{since } t(\beta(\lambda_1)) = \beta(\lambda_2) \\ &= r(g^{i+j}\beta(\lambda_1)^2) + (-1)^j r(g^{i+j}\beta(\lambda_1)\beta(\lambda_2)) \\ &= (-1)^j r(g^{i+j+1}g^{-1}\beta(\lambda_1)\beta(\lambda_2)) && \text{since } \beta(\lambda_1)^2 = 0 \\ &= (-1)^j r(g^{i+j+1})\zeta_{1,2} \quad \text{by (2.1b) and the definition of } \zeta_{1,2} \end{aligned}$$

and the other equalities are obtained similarly.

For the action of ψ^k, we have

$$\begin{aligned}
&c(\psi^k r)(g^i\beta(\lambda_1))) \\
&= \psi^k(cr(g^i\beta(\lambda_1))) \\
&= \psi^k(g^i\beta(\lambda_1) + (-1)^i g^i\beta(\lambda_2)) \\
&= k^i g^i\Big(\frac{k^2(k+1)}{2}\beta(\lambda_1) - \frac{k^2(k+1)}{2}\beta(\lambda_2)\Big) \\
&\qquad + (-1)^i k^i g^i(-\frac{k^2(k-1)}{2}\beta(\lambda_1) + \frac{k^2(k+1)}{2}\beta(\lambda_2)) \\
&\qquad \text{by (1.2) and Theorem 1(i)} \\
&= \frac{k^{i+2}(k+1-(-1)^i k + (-1)^i)}{2}(g^i\beta(\lambda_1) + (-1)^i g^i\beta(\lambda_2)) \\
&= c\Big(\frac{k^{i+2}(k+1-(-1)^i k + (-1)^i)}{2} r(g^i\beta(\lambda_1))\Big).
\end{aligned}$$

By examining the behavior of c: $\widetilde{KO}^{-1-2i}(SU(3)) \to \widetilde{K}^{-1-2i}(SU(3))$, we see that this gives the first equality. The second equality follows similarly, and the proof is completed.

$\underline{G = Sp(2)}$

It follows from Proposition 7(2) that

$$\widetilde{K}^*(Sp(2)) = N\langle g^2\beta(\mu_1')\rangle \oplus N\langle\beta(\mu_2')\rangle \oplus N\langle g^2\beta(\mu_1')\beta(\mu_2')\rangle. \tag{3.2}$$

Theorem 10. *As a $KO^*(pt)$-module (but not as a ring),*

$$KO^*(Sp(2)) = KO^*(pt) \otimes \Lambda_{\mathbf{Z}}(\beta_{\mathbf{H}}(\widehat{\mu_1'}), \beta_{\mathbf{R}}(\widehat{\mu_2'})).$$

Its multiplicative structure is given by

$$\beta_{\mathbf{H}}(\widehat{\mu_1'})^2 = \eta\sigma\cdot\beta_{\mathbf{R}}(\mu_2') \quad \textit{and} \quad \beta_{\mathbf{R}}(\widehat{\mu_2'})^2 = 0.$$

The action of ψ^k on $\widetilde{KO}^*(Sp(2))$ is given by

$$\begin{aligned}
\psi^k(\beta_{\mathbf{H}}(\widehat{\mu_1'})) &= \frac{k^4(k^2+2)}{3}\beta_{\mathbf{H}}(\widehat{\mu_1'}) - \frac{k^4(k^2-1)}{12}\nu\beta_{\mathbf{R}}(\widehat{\mu_2'}), \\
\psi^k(\beta_{\mathbf{R}}(\widehat{\mu_2'})) &= -\frac{2k^2(k^2-1)}{3}\nu\sigma^{-1}\beta_{\mathbf{H}}(\widehat{\mu_1'}) + \frac{k^2(2k^2+1)}{3}\beta_{\mathbf{R}}(\widehat{\mu_2'}), \\
\psi^k(\beta_{\mathbf{H}}(\widehat{\mu_1'})\beta_{\mathbf{R}}(\widehat{\mu_2'})) &= k^8\beta_{\mathbf{H}}(\widehat{\mu_1'})\beta_{\mathbf{R}}(\widehat{\mu_2'}).
\end{aligned}$$

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Proof. It follows from (3.2) and Lemma 5 that, as a $\widetilde{KO}^*(S^0)$-module,

$$\text{(3.3)}\quad \widetilde{KO}^*(Sp(2)) = \widetilde{KO}^*(S^0)\{\beta_{\boldsymbol{H}}(\widehat{\mu_1'})\} \oplus \widetilde{KO}^*(S^0)\{\beta_{\boldsymbol{R}}(\widehat{\mu_2'})\} \oplus \widetilde{KO}^*(S^0)\{\beta_{\boldsymbol{H}}(\widehat{\mu_1'})\beta_{\boldsymbol{R}}(\widehat{\mu_2'})\}.$$

So the first statement follows.

For the multiplicative structure, the first equality is a consequence of the second relation of (2.12). It remains to prove the second equality. In view of the first relation of (2.12), we have to determine $\lambda^2(\beta_{\boldsymbol{R}}(\widehat{\mu_2'}))$. For this purpose, since c: $\widetilde{KO}^{-1}(Sp(2)) \to \widetilde{K}^{-1}(Sp(2))$ is a monomorphism of λ-rings (compare (3.3) with (3.2)), it suffices to compute $\lambda^2(\beta(\mu_2'))$. We quote from Theorem 2(i) with $k = 2$ that

$$\psi^2(\beta(\mu_2')) = -16\beta(\mu_1') + 12\beta(\mu_2').$$

Using the formula $\psi^2(x) - x^2 + 2\lambda^2(x) = 0$ for $x \in K(X)$ (see [2]) and the relation $\beta(\mu_2')^2 = 0$, we have

$$\lambda^2(\beta(\mu_2')) = 8\beta(\mu_1') - 6\beta(\mu_2').$$

By (2.12), since $2\eta = 0$, this gives the second equality.

For the action of ψ^k, we have

$$\begin{aligned}
&c\psi^k(\beta_{\boldsymbol{H}}(\widehat{\mu_1'}))\\
&= \psi^k c(\beta_{\boldsymbol{H}}(\widehat{\mu_1'}))\\
&= \psi^k(g^2\beta(\mu_1'))\\
&= \psi^k(g)^2\psi^k(\beta(\mu_1'))\\
&= (kg)^2\Big(\frac{k^2(k^2+2)}{3}\beta(\mu_1') - \frac{k^2(k^2-1)}{6}\beta(\mu_2')\Big)\\
&\qquad \text{by (1.2) and Theorem 2(i)}\\
&= \frac{k^4(k^2+2)}{3}g^2\beta(\mu_1') - \frac{k^4(k^2-1)}{6}g^2\beta(\mu_2')\\
&= c\Big(\frac{k^4(k^2+2)}{3}\beta_{\boldsymbol{H}}(\widehat{\mu_1'}) - \frac{k^4(k^2-1)}{12}\nu\beta_{\boldsymbol{R}}(\widehat{\mu_2'})\Big) \quad \text{by (2.3a).}
\end{aligned}$$

Since c: $\widetilde{KO}^{-5}(Sp(2)) \to \widetilde{K}^{-5}(Sp(2))$ is injective (compare (3.3) with (3.2)), this gives the first equality. The other equalities are obtained similarly, and the proof is completed.

$\underline{G = G_2}$

It follows from Proposition 7(3) that

$$(3.4) \qquad \widetilde{K}^*(G_2) = N\langle\beta(\rho_1)\rangle \oplus N\langle\beta(\rho_2)\rangle \oplus N\langle\beta(\rho_1)\beta(\rho_2)\rangle.$$

Theorem 11. *As a $KO^*(pt)$-module (but not as a ring),*

$$KO^*(G_2) = KO^*(pt) \otimes \Lambda_{\boldsymbol{Z}}(\beta_{\boldsymbol{R}}(\widehat{\rho_1}), \beta_{\boldsymbol{R}}(\widehat{\rho_2})).$$

Its multiplicative structure is given by

$$\beta_{\boldsymbol{R}}(\widehat{\rho_1})^2 = \eta \cdot \beta_{\boldsymbol{R}}(\widehat{\rho_1}) + \eta \cdot \beta_{\boldsymbol{R}}(\widehat{\rho_2}) = \beta_{\boldsymbol{R}}(\widehat{\rho_2})^2.$$

The action of ψ^k on $\widetilde{KO}^*(G_2)$ is given by

$$\begin{aligned}
\psi^k(\beta_{\boldsymbol{R}}(\widehat{\rho_1})) &= \frac{k^2(k^4+5)}{6}\beta_{\boldsymbol{R}}(\widehat{\rho_1}) - \frac{k^2(k^4-1)}{30}\beta_{\boldsymbol{R}}(\widehat{\rho_2}), \\
\psi^k(\beta_{\boldsymbol{R}}(\widehat{\rho_2})) &= -\frac{25k^2(k^4-1)}{6}\beta_{\boldsymbol{R}}(\widehat{\rho_1}) + \frac{k^2(5k^4+1)}{6}\beta_{\boldsymbol{R}}(\widehat{\rho_2}), \\
\psi^k(\beta_{\boldsymbol{R}}(\widehat{\rho_1})\beta_{\boldsymbol{R}}(\widehat{\rho_2})) &= k^8\beta_{\boldsymbol{R}}(\widehat{\rho_1})\beta_{\boldsymbol{R}}(\widehat{\rho_2}).
\end{aligned}$$

Proof. It follows from (3.4) and Lemma 5 that, as a $\widetilde{KO}^*(S^0)$-module,

$$\begin{aligned}
(3.5) \qquad \widetilde{KO}^*(G_2) = \widetilde{KO}^*(S^0)\{\beta_{\boldsymbol{R}}(\widehat{\rho_1})\} &\oplus \widetilde{KO}^*(S^0)\{\beta_{\boldsymbol{R}}(\widehat{\rho_2})\} \\
&\oplus \widetilde{KO}^*(S^0)\{\beta_{\boldsymbol{R}}(\widehat{\rho_1})\beta_{\boldsymbol{R}}(\widehat{\rho_2})\}.
\end{aligned}$$

So the first statement follows.

For the multiplicative structure, in view of the first relation of (2.12), we have to determine $\lambda^2(\beta_{\boldsymbol{R}}(\widehat{\rho}_i))$ for $i = 1, 2$. For this purpose, since $c\colon \widetilde{KO}^{-1}(G_2) \to \widetilde{K}^{-1}(G_2)$ is a monomorphism of λ-rings (compare (3.5) with (3.4)), it suffices to compute $\lambda^2(\beta(\rho_i))$ for $i = 1, 2$. We quote from Theorem 3(i) with $k = 2$ that

$$\begin{aligned}
\psi^2(\beta(\rho_1)) &= 14\beta(\rho_1) - 2\beta(\rho_2), \\
\psi^2(\beta(\rho_2)) &= -250\beta(\rho_1) + 54\beta(\rho_2).
\end{aligned}$$

Using the formula $\psi^2(x) - x^2 + 2\lambda^2(x) = 0$ and the relation $\beta(\rho_i)^2 = 0$, we have

$$\begin{aligned}
\lambda^2(\beta(\rho_1)) &= -7\beta(\rho_1) + \beta(\rho_2), \\
\lambda^2(\beta(\rho_2)) &= 125\beta(\rho_1) - 27\beta(\rho_2).
\end{aligned}$$

By (2.12), since $2\eta = 0$, these give the stated equalities.

The equalities describing the action of ψ^k are obtained in the same way as in the proof of Theorem 10.

4. The rings $KO^*(SU(2n+1)/SO(2n+1))$ and $KO^*(SU(2n)/Sp(n))$. Lemma 6 together with Proposition 8 can be used to compute the KO-theory of compact symmetric spaces $SU(2n+1)/SO(2n+1)$ and $SU(2n)/Sp(n)$. To begin with, the following result is in [1, Remark 3.63 and Theorems 7.3, 7.6, 7.7].

Proposition 12. *For $G = SU(n+1)$, $SO(2n+1)$ and $Sp(n)$, the action of t on $R(G)$ is given by:*

(1) *In $R(SU(n+1)) = \boldsymbol{Z}[\lambda_1, \cdots, \lambda_n]$ (where $\lambda_1 = [\boldsymbol{C}^{n+1}]$ and $\lambda_k = \lambda^k(\lambda_1)$),*

$$t(\lambda_k) = \lambda_{n+1-k} \quad \textit{for} \quad k = 1, \ldots, n.$$

(2) *In $R(SO(2n+1)) = \boldsymbol{Z}[\mu_1, \cdots, \mu_n]$ (where $\mu_1 = [(\boldsymbol{R}^{2n+1})^{\boldsymbol{C}}]$ and $\mu_k = \lambda^k(\mu_1)$),*

$$t(\mu_k) = \mu_k \quad \textit{for} \quad k = 1, \ldots, n$$

where μ_k is real.

(3) *In $R(Sp(n)) = \boldsymbol{Z}[\mu'_1, \cdots, \mu'_n]$ (where $\mu'_1 = [(\boldsymbol{H}^n)_{\boldsymbol{C}}]$ and $\mu'_k = \lambda^k(\mu'_1)$),*

$$t(\mu'_k) = \mu'_k \quad \textit{for} \quad k = 1, \ldots, n$$

where μ'_{2l-1} is quaternionic and μ'_{2l} is real.

The K-rings of $SU(2n+1)/SO(2n+1)$ and $SU(2n)/Sp(n)$ were determined by H. Minami [12]. We recall his result. Let G be a compact 1-connected Lie group. Suppose that there is an automorphism $\sigma\colon G \to G$ such that $\sigma^2 = 1_G$. Then the fixed point set $G^\sigma = \{x \in G \mid \sigma(x) = x\}$ forms a closed connected subgroup of G, and the coset space G/G^σ becomes a compact symmetric space (e.g., see [11, Chapter 3, §6]). Consider the induced homomorphism $\sigma^*\colon R(G) \to R(G)$ and let $\sigma^*(\lambda) = \lambda'$, where λ is a representation of G. Then $\dim \lambda = \dim \lambda'$ $(= n)$ and $\lambda | G^\sigma = \lambda' | G^\sigma$. So we have a map $f_\lambda\colon G/G^\sigma \to U(n)$ defined by

$$f_\lambda(xG^\sigma) = \lambda(x)\lambda'(x)^{-1} \quad \text{for} \quad xG^\sigma \in G/G^\sigma. \tag{4.1}$$

Let $\iota_n\colon U(n) \to U$ be the canonical injection. Then the composite $\iota_n f_\lambda$ gives rise to a homotopy class $\beta(\lambda - \lambda')$ in $[G/G^\sigma, U] = \widetilde{K}^{-1}(G/G^\sigma)$.

Let $\sigma = \sigma_{2n+1}\colon SU(2n+1) \to SU(2n+1)$ be the involution defined as in (2.10). Then, in the notation of Proposition 12,

$$(4.2) \qquad \sigma^*(\lambda_k) = \lambda_{2n+1-k} \quad \text{and} \quad i_C^*(\lambda_k) = \mu_k = i_C^*(\lambda_{2n+1-k}) \quad \text{for } k = 1, \ldots, n.$$

Similarly, let $\sigma = \sigma_n'\colon SU(2n) \to SU(2n)$ be the involution defined as in (2.11). Then

$$(4.3) \qquad \sigma^*(\lambda_k) = \lambda_{2n-k} \quad \text{and} \quad i_{C'}{}^*(\lambda_k) = \mu_k' = i_{C'}^*(\lambda_{2n-k}) \quad \text{for } k = 1, \ldots, n.$$

Proposition 13. (1) ([12, Proposition 8.1]) *As an algebra over $K^*(pt)$,*

$$K^*(SU(2n+1)/SO(2n+1)) = K^*(pt) \otimes \Lambda_{\mathbf{Z}}(\beta(\lambda_1 - \lambda_{2n}), \cdots, \beta(\lambda_n - \lambda_{n+1})).$$

(2) ([12, Proposition 6.1]) *As an algebra over $K^*(pt)$,*

$$K^*(SU(2n)/Sp(n)) = K^*(pt) \otimes \Lambda_{\mathbf{Z}}(\beta(\lambda_1 - \lambda_{2n-1}), \cdots, \beta(\lambda_{n-1} - \lambda_{n+1})).$$

We can now deduce our main result.

Theorem 14. (1) *As an algebra over $KO^*(pt)$,*

$$KO^*(SU(2n+1)/SO(2n+1)) = KO^*(pt) \otimes \Lambda_{\mathbf{Z}}(\lambda_{1,2n}, \cdots, \lambda_{n,n+1})$$

where $\lambda_{k,2n+1-k} \in \widetilde{KO}^1(SU(2n+1)/SO(2n+1))$ is a unique element such that

$$c(\lambda_{k,2n+1-k}) = g^{-1}\beta(\lambda_k - \lambda_{2n+1-k}).$$

(2) *As an algebra over $KO^*(pt)$,*

$$KO^*(SU(2n)/Sp(n)) = KO^*(pt) \otimes \Lambda_{\mathbf{Z}}(\lambda'_{1,2n-1}, \cdots, \lambda'_{n-1,n+1})$$

where $\lambda'_{2l-1,2n-2l+1} \in \widetilde{KO}^{-3}(SU(2n)/Sp(n))$ is a unique element such that

$$c(\lambda'_{2l-1,2n-2l+1}) = g\beta(\lambda_{2l-1} - \lambda_{2n-2l+1}),$$

and $\lambda'_{2l,2n-2l} \in \widetilde{KO}^1(SU(2n)/Sp(n))$ *is a unique element such that*

$$c(\lambda'_{2l,2n-2l}) = g^{-1}\beta(\lambda_{2l} - \lambda_{2n-2l}).$$

Proof. We first show (1). Consider $\lambda_k\colon SU(2n+1) \to U(\binom{2n+1}{k})$ for $k = 1, \ldots, n$. By (4.2), $i_C^*(\lambda_k) = \mu_k$ and by Proposition 12(2), $\mu_k \in R(SO(2n+1))$ is real, i.e., there is a (unique) $\widehat{\mu_k} \in RO(SO(2n+1))$ such that $c(\widehat{\mu_k}) = \mu_k$. Therefore, in the diagram

$$\text{(4.4)}\qquad \begin{array}{ccccccc} SO(2n+1) & \xrightarrow{i_C} & SU(2n+1) & \xrightarrow{\pi_C} & SU(2n+1)/SO(2n+1) & & \\ \downarrow{\scriptstyle \kappa_{2n+1,k}\widehat{\mu_k}} & & \downarrow{\scriptstyle \iota_{2n+1,k}\lambda_k} & & {\scriptstyle \lambda_{k,2n+1-k}}\downarrow & \searrow{\scriptstyle \beta(\lambda_k-\lambda_{2n+1-k})} & \\ O & \xrightarrow{i_C} & U & \xrightarrow{\pi_C} & U/O & \xrightarrow{c_1} & U \end{array}$$

(where $\kappa_{2n+1,k}\colon O(\binom{2n+1}{k}) \to O$ and $\iota_{2n+1,k}\colon U(\binom{2n+1}{k}) \to U$ are the canonical injections), the left square is commutative. So we have a map $\lambda_{k,2n+1-k}\colon SU(2n+1)/SO(2n+1) \to U/O$ which makes the middle square commute. Indeed, it is defined by

$$\text{(4.5)}\qquad \lambda_{k,2n+1-k}(xSO(2n+1)) = (\iota_{2n+1,k}\lambda_k)(x)O$$

for $xSO(2n+1) \in SU(2n+1)/SO(2n+1)$. Since $\sigma_{2n+1}^*(\lambda_k) = \lambda_{2n+1-k} = t(\lambda_k)$ by (4.2) and Proposition 12(1), the diagram

$$\begin{array}{ccccc} SU(2n+1) & \xrightarrow{\lambda_k} & U(\binom{2n+1}{k}) & \xrightarrow{\iota_{n,k}} & U \\ {\scriptstyle \sigma_{2n+1}}\downarrow & & \downarrow{\scriptstyle \sigma_{2n+1,k}} & & \downarrow{\scriptstyle \sigma_\infty} \\ SU(2n+1) & \xrightarrow{\lambda_k} & U(\binom{2n+1}{k}) & \xrightarrow{\iota_{n,k}} & U \end{array}$$

(where $\sigma_{2n+1,k}$ is defined as in (2.10)) is commutative. So the right triangle in (4.4) is commutative:

$$\begin{aligned} &(c_1\lambda_{k,2n+1-k})(xSO(2n+1)) \\ &\quad = (\iota_{2n+1,k}\lambda_k)(x)\sigma_\infty((\iota_{2n+1,k}\lambda_k)(x))^{-1} && \text{by (2.9) and (4.5)} \\ &\quad = (\iota_{2n+1,k}\lambda_k)(x)(\iota_{2n+1,k}\lambda_k\sigma_{2n+1})(x)^{-1} \\ &\quad = \iota_{2n+1,k}(\lambda_k(x)(\lambda_k\sigma_{2n+1})(x)^{-1}) \\ &\quad = \beta(\lambda_k - \lambda_{2n+1-k})(xSO(2n+1)) && \text{by (4.1) and (4.2).} \end{aligned}$$

By Proposition 8(ii), this implies that $c(\lambda_{k,2n+1-k}) = g^{-1}\beta(\lambda_k - \lambda_{2n+1-k})$, where we regard $\lambda_{k,2n+1-k}$ as an element of $\widetilde{KO}^1(SU(2n+1)/SO(2n+1)) =$

$[SU(2n+1)/SO(2n+1), U/O]$. By this equality and Proposition 13(1), we can apply Lemma 6 to the case $X = SU(2n+1)/SO(2n+1)$ and obtain the $KO^*(pt)$-module structure of $KO^*(SU(2n+1)/SO(2n+1))$. For the multiplicative structure, as is discussed at the end of section 2, whether ${\lambda_{k,2n+1-k}}^2$ is zero or not is a remaining question. Fortunately it says in Crabb [6, Example (6.6)] that ${\lambda_{k,2n+1-k}}^2 = 0$ since $\lambda_{k,2n+1-k}$ has degree 1 and $1 \equiv -3 \pmod 4$. Hence (1) follows.

We next show (2). Consider $\lambda_k\colon SU(2n) \to U(\binom{2n}{k})$ for $k = 1, \ldots, n-1$. By (4.3), $i^*_{C'}(\lambda_k) = \mu'_k$. From now on, our argument is divided into two cases.

Suppose that k is odd, i.e., $k = 2l - 1$ for some $l \geq 1$. Then, by Proposition 12(3), $\mu'_k \in R(Sp(n))$ is quaternionic, i.e., there is a (unique) $\widehat{\mu'_k} \in RSp(Sp(n))$ such that $c'(\widehat{\mu'_k}) = \mu'_k$. Therefore, in the diagram

$$\text{(4.6)}\qquad \begin{array}{ccccccc} Sp(n) & \xrightarrow{i_{C'}} & SU(2n) & \xrightarrow{\pi_{C'}} & SU(2n)/Sp(n) & & \\ \downarrow{\scriptstyle \xi_{2n,k}\widehat{\mu'_k}} & & \downarrow{\scriptstyle \iota_{2n,k}\lambda_k} & & \downarrow{\scriptstyle \lambda'_{k,2n-k}} & \searrow^{\beta(\lambda_k-\lambda_{2n-k})} & \\ Sp & \xrightarrow{i_{C'}} & U & \xrightarrow{\pi_{C'}} & U/Sp & \xrightarrow{c_{-3}} & U \end{array}$$

(where $\xi_{2n,k}\colon Sp(\binom{2n}{k}/2) \to Sp$ and $\iota_{2n,k}\colon U(\binom{2n}{k}) \to U$ are the canonical injections), the left square is commutative. So we have a map $\lambda'_{k,2n-k}\colon SU(2n)/Sp(n) \to U/Sp$ which makes the middle square commute. Since ${\sigma'_n}^*(\lambda_k) = \lambda_{2n-k} = t(\lambda_k)$ by (4.3) and Proposition 12(1) and since $j\alpha j^{-1} = \bar{\alpha}$ for $\alpha \in \boldsymbol{C}$, the diagram

$$\begin{array}{ccccc} SU(2n) & \xrightarrow{\lambda_k} & U(\binom{2n}{k}) & \xrightarrow{\iota_{n,k}} & U \\ \downarrow{\scriptstyle \sigma'_n} & & \downarrow{\scriptstyle \sigma'_{n,k}} & & \downarrow{\scriptstyle \sigma'_\infty} \\ SU(2n) & \xrightarrow{\lambda_k} & U(\binom{2n}{k}) & \xrightarrow{\iota_{n,k}} & U \end{array}$$

(where $\sigma'_{n,k}$ is defined as in (2.11)) is commutative and so the right triangle in (4.6) is commutative. By Proposition 8(vi), this implies that $c(\lambda'_{k,2n-k}) = g\beta(\lambda_k - \lambda_{2n-k})$, where $\lambda'_{k,2n-k} \in \widetilde{KO}^{-3}(SU(2n)/Sp(n)) = [SU(2n)/Sp(n), U/Sp]$.

Suppose that k is even, i.e., $k = 2l$ for some $l \geq 1$. Then, by Proposition 12 (3), $\mu'_k \in R(Sp(n))$ is real, i.e., there is a (unique) $\widehat{\mu'_k} \in RO(Sp(n))$

such that $c(\widehat{\mu'_k}) = \mu'_k$. Therefore, in the diagram

$$
\begin{array}{ccccccc}
Sp(n) & \xrightarrow{i_{C'}} & SU(2n) & \xrightarrow{\pi_{C'}} & SU(2n)/Sp(n) & & \\
{\scriptstyle \kappa_{2n,k}\widehat{\mu'_k}}\downarrow & & \downarrow{\scriptstyle \iota_{2n,k}\lambda_k} & & \downarrow{\scriptstyle \lambda'_{k,2n-k}} & \searrow{\scriptstyle \beta(\lambda_k-\lambda_{2n-k})} & \\
O & \xrightarrow{i_C} & U & \xrightarrow{\pi_C} & U/O & \xrightarrow{c_1} & U
\end{array}
\tag{4.7}
$$

(where $\kappa_{2n,k}$: $O(\binom{2n}{k}) \to O$ is the canonical injection), the left square is commutative. So we have a map $\lambda'_{k,2n-k}$: $SU(2n)/Sp(n) \to U/O$ which makes the middle square commute. Since $\sigma'^*_n(\lambda_k) = \lambda_{2n-k} = t(\lambda_k)$ by (4.3) and Proposition 12(1), the diagram

$$
\begin{array}{ccccc}
SU(2n) & \xrightarrow{\lambda_k} & U(\binom{2n}{k}) & \xrightarrow{\iota_{n,k}} & U \\
{\scriptstyle \sigma'_n}\downarrow & & \downarrow{\scriptstyle \sigma_{2n,k}} & & \downarrow{\scriptstyle \sigma_\infty} \\
SU(2n) & \xrightarrow{\lambda_k} & U(\binom{2n}{k}) & \xrightarrow{\iota_{n,k}} & U
\end{array}
$$

(where $\sigma_{2n,k}$ is defined as in (2.10)) is commutative and so the right triangle in (4.7) is commutative. By Proposition 8(ii), this implies that $c(\lambda'_{k,2n-k}) = g^{-1}\beta(\lambda_k - \lambda_{2n-k})$, where $\lambda'_{k,2n-k} \in \widetilde{KO}^1(SU(2n)/Sp(n))$.

By these equalities and Proposition 13(2), we can apply Lemma 6 to the case $X = SU(2n)/Sp(n)$. The rest is quite similar to the proof of (1), and (2) follows.

Remark. We have no good reasons to assert that, for example, $\lambda_{k,2n+1-k}$ lies in $\widetilde{KO}^1(SU(2n+1)/SO(2n+1))$ and does not lie in $\widetilde{KO}^{8m+1}(SU(2n+1)/SO(2n+1))$ for some $m \neq 0$. But, since the CW-complex structure of $SU(2n+1)/SO(2n+1)$ is known for small n, one can compute $\widetilde{KO}^*(SU(2n+1)/SO(2n+1))$ by using cofibre sequences. Only such observation justifies our assertion.

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