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## Some Riemannian manifolds admitting a concircular scalar field

Masami Fujii\*

\*Tsuyama Technical college

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## SOME RIEMANNIAN MANIFOLDS ADMITTING A CONCIRCULAR SCALAR FIELD

MASAMI FUJII

### Introduction

Recently, R. S. Kulkarni [2] and others have dealt with Riemannian manifolds admitting a concircular scalar field in a theory of curvature-preserving mappings or in connection with the so-called Nomizu's conjecture. Properties of concircular scalar fields, of special ones or in Einstein manifolds etc, were studied by Y. Tashiro [1] or other authors. However, we have less knowledge of concircular scalar fields in manifolds of constant scalar curvature, even in case of low-dimensional manifolds.

In this paper, we shall discuss properties of Riemannian manifolds admitting a concircular scalar field by using adapted coordinate systems for concircular scalar fields as tools. We refer to [1] as to notations and terminologies.

In 1, formulas with respect to an adapted coordinate system will be stated as preliminaries. In 2, we shall determine the structure of a 4-dimensional Riemannian manifold of constant scalar curvature admitting a concircular scalar field. We shall prove, in 3, that a 4-dimensional Einstein manifold admitting a concircular scalar field is of constant curvature, and slightly generalize Kulkarni's theorem concerning conformal map. It will be proved, in 4, that a manifold having the vanishing tensor  $H_{\sigma\omega\nu\mu\lambda}$ , defined by E. Cartan and admitting a concircular scalar field is of constant curvature, and, in 5, that a manifold satisfying  $\overset{*}{H}_{\sigma\omega\mu\lambda} = H_{\sigma\omega\nu\mu\lambda}{}^\nu = 0$  is an Einstein manifold, unless the gradient vector field of the concircular scalar field is concurrent.

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### 1. Preliminaries

We shall assume, throughout this paper, that a Riemannian manifold  $M$  is connected, differentiable and of dimension  $n$ , and the metric tensor  $g_{\mu\lambda}$  of  $M$  is positive definite. Two kinds of indices run on the ranges

$$\begin{aligned} X, \lambda, \mu, \nu, \omega, \sigma, \tau &= 1, 2, 3, \dots, n \\ h, i, j, k &= 2, 3, \dots, n \end{aligned}$$

respectively. We denote the Christoffel symbol, the curvature tensor, the Ricci tensor and the scalar curvature of  $M$  by  $\{\epsilon_{\mu\lambda}^{\kappa}\}$ ,  $K_{\nu\mu\lambda}^{\kappa}$ ,  $K_{\mu\lambda}$  and  $\kappa$  respectively, where the scalar curvature defined by  $\kappa = \frac{1}{n(n-1)}K_{\mu\lambda}g^{\mu\lambda}$ .

The scalar field  $\rho$  is said to be *concircular* if it satisfies the equation

$$(1.1) \quad \nabla_{\mu}\nabla_{\lambda}\rho = \phi g_{\mu\lambda},$$

$\nabla$  indicating covariant differentiation and  $\phi$  being a scalar field, and to be *special concircular* if it satisfies the equation

$$(1.2) \quad \nabla_{\mu}\nabla_{\lambda}\rho = (-k\rho + b)g_{\mu\lambda}$$

$k$  and  $b$  being constant.

Along any geodesic with arc-length  $u$ , the equation (1.1) becomes to the ordinary equation

$$(1.3) \quad \frac{d^2\rho}{du^2} = \phi.$$

We put  $\rho_{\lambda} = \delta_{\lambda}\rho$ . A point  $P$  is said to be *ordinary* or *stationary* according as  $\rho_{\lambda}(P) \neq 0$  or  $\rho_{\lambda}(P) = 0$ . Stationary points of a concircular scalar field  $\rho$  are isolated and there exist at most two in  $M$ , see [1]. In a neighborhood  $U$  of an ordinary point of  $\rho$ , we can choose an adapted coordinate system  $(u^{\kappa})$  having the following properties: the first coordinate  $u^1$  is the arc-length  $u$  of  $\rho$ -curves, trajectories of  $\rho^{\kappa}$ , the coordinate hypersurfaces  $u^1 = \text{constant}$  are  $\rho$ -hypersurfaces defined by  $\rho = \text{constant}$ , the field  $\rho$  is a function of  $u^1 = u$  only, and the metric form  $ds^2$  of  $M$  is given in the form

$$(1.4) \quad ds^2 = du^2 + \rho'(u)^2 \overline{ds}^2,$$

where the prime indicates ordinary derivative with respect to  $u$  and  $\overline{ds}^2 = f_{ji} du^j du^i$  is a metric form of an  $(n-1)$ -dimensional manifold  $\overline{M}$ . We indicate quantities of  $\overline{M}$  by barring. With respect to an adapted coordinate system  $(u^{\kappa})$ , the metric tensor has components

$$(1.5) \quad g_{11} = 1, \quad g_{j1} = g_{1j} = 0, \quad g_{ji} = \rho'^2 f_{ji},$$

the curvature tensor  $K_{\nu\mu\lambda}^{\kappa}$  of  $M$  has components

$$(1.6) \quad \begin{aligned} K_{1j1}^h &= -K_{j11}^h = \frac{\rho'''}{\rho'} \delta_j^h, \quad K_{1ji}^h = -K_{j1i}^h = -\rho' \rho''' f_{ji}, \\ K_{kji}^h &= \overline{K}_{kji}^h - \rho'^2 (\delta_k^h f_{ji} - \delta_j^h f_{ki}), \end{aligned}$$

the other components being zero, the Ricci tensor has components

$$(1.7) \quad \begin{aligned} K_{11} &= -(n-1) \frac{\rho'''}{\rho'}, \quad K_{j1} = K_{1i} = 0 \\ K_{ji} &= \bar{K}_{ji} - [(n-2)\rho'^2 + \rho'\rho''']f_{ji}, \end{aligned}$$

and the scalar curvature  $\kappa$  of  $M$  is equal to

$$(1.8) \quad \kappa = \frac{1}{n(\rho')^2} [(n-2)(\kappa - \rho'^2) - 2\rho'\rho'''],$$

where  $\bar{\kappa}$  is the scalar curvature of  $\bar{M}$  defined by  $\bar{\kappa} = \frac{1}{(n-1)(n-2)} \bar{K}_{ji} f^{ji}$ .

## 2. 4-dimensional Riemannian manifolds of constant scalar curvature

Let  $M$  be a 4-dimensional Riemannian manifold of constant scalar curvature  $\kappa$  and  $\rho$  a concircular scalar field. For  $n=4$ , the equation (1.8) is reduced to

$$(2.1) \quad 2\rho'^2 + \frac{1}{2}(\rho'^2)'' = \bar{\kappa}.$$

Since the left hand side depends on  $u$  only and  $\bar{\kappa}$  is independent of  $u$ ,  $\bar{\kappa}$  is also a constant. According the signature of the constant scalar curvature  $\kappa$ , we put

$$(2.2) \quad \kappa = \begin{cases} \text{(I)} & 0 \\ \text{(II)} & -c^2 \\ \text{(III)} & c^2 \end{cases}$$

$c$  being a positive constant. By a suitable choice of the arclength  $u$ , the general solution of (2.1) is given by one of

$$(2.3) \quad \rho'^2 = \begin{cases} \text{(I, A)} & au & (\bar{\kappa}=0) \\ \text{(I, B)} & \bar{\kappa}u^2 + a & (\bar{\kappa} \neq 0) \\ \text{(II, A}_0) & a \exp 2cu - \frac{\bar{\kappa}}{2c^2} \\ \text{(II, A}_-) & a \sinh 2cu - \frac{\bar{\kappa}}{2c^2} \\ \text{(II, B)} & a \cosh 2cu - \frac{\bar{\kappa}}{2c^2} \\ \text{(III)} & a \cos 2cu + \frac{\bar{\kappa}}{2c^2} \end{cases}$$

Therefore the manifold  $M$  has a local structure such that the metric form is given by (1.4) substituted with (2.3) for  $\rho'^2$ .

Next we suppose that  $M$  is complete. Then the arc-length of any geodesic is extendable to the infinities. Since  $\rho$ -curves are geodesic, the cases (I, A) and (II, A<sub>-</sub>) do not occur, and in the other cases, the inequalities

$$(2.4) \quad \begin{cases} \text{(I, B)} & \bar{\kappa} > 0, \quad a \geq 0 \\ \text{(II, A}_0\text{)} & a > 0, \quad \bar{\kappa} < 0 \\ \text{(II, B)} & a > 0, \quad \bar{\kappa} \leq 2ac^2 \\ \text{(III)} & \bar{\kappa} > 0, \quad \bar{\kappa} \geq 2ac^2 \end{cases}$$

should be satisfied respectively, because  $\rho'^2 \geq 0$ .

Moreover, in order that there exists no stationary point of  $\rho$  in a complete manifold  $M$ , it is necessary and sufficient that the equalities in (2.4) do not appear in all cases. Then the manifold  $M$  is topologically the direct product  $I \times \bar{M}$  of a straight line  $I$  and a 3-dimensional complete manifold  $\bar{M}$ . By transferring the factor  $\bar{\kappa}$  in the case (I, B) or  $a$  in the cases (II, A<sub>0</sub>), (II, B) and (III) into the metric tensor  $f_{ji}$  of  $\bar{M}$ , in other words, applying a homothety to  $\bar{M}$ , the metric form of  $M$  is given by

$$(2.5) \quad ds^2 = \begin{cases} \text{(I, B)} & du^2 + (u^2 + a)\bar{ds}^2 & (a > 0) \\ \text{(II, A}_0\text{)} & du^2 + \left(\exp 2cu - \frac{\bar{\kappa}}{2c^2}\right)\bar{ds}^2 & (\bar{\kappa} < 0) \\ \text{(II, B)} & du^2 + \frac{1}{2}\left(\cosh 2cu - \frac{\bar{\kappa}}{c^2}\right)\bar{ds}^2 & (\bar{\kappa} < c^2) \\ \text{(III)} & du^2 + \frac{1}{2}\left(\frac{\bar{\kappa}}{c^2} - \cos 2cu\right)\bar{ds}^2 & (\bar{\kappa} > c^2) \end{cases}$$

in the whole manifold  $M$ , respectively. On the other hand, the existence of a stationary point of  $\rho$  is possible in the cases (I, B), (II, B) and (III). Then  $M$  is of constant curvature and the scalar curvature is equal to  $\bar{\kappa}=1$  in (I, B), or  $\bar{\kappa}=c^2$  in (II, B) and (III). There is one stationary point corresponding to  $u=0$  in (I, B) and (II, B) and are two corresponding to  $u=0$  and  $u=\frac{\pi}{c}$  in (III). The metric form of  $M$  is given by

$$(2.6) \quad ds^2 = \begin{cases} \text{(I, B)} & du^2 + u^2\bar{ds}^2 \\ \text{(II, B)} & du^2 + (\sinh cu)^2\bar{ds}^2 \\ \text{(III)} & du^2 + (\sin cu)^2\bar{ds}^2. \end{cases}$$

These are the polar forms of the metrics of (I, B) a Euclidean space, (II, B) a hyperbolic space and (III) a sphere, respectively. Thus we have established the following

**Theorem 1.** *Let  $M$  be a 4-dimensional complete Riemannian manifold of constant scalar curvature  $\kappa$  and suppose that  $M$  admits a concircular scalar field  $\rho$ . If there exists no stationary point of  $\rho$ , then the manifold  $M$  is topologically the direct product of a straight line  $I$  and a 3-dimensional complete manifold  $\bar{M}$  of constant scalar curvature  $\bar{\kappa}$  and, the metric form of  $M$  is given by one of (2.5). If there exists a stationary point of  $\rho$ , then the manifold  $M$  is a Euclidean space, a hyperbolic space or a sphere.*

### 3. 4-dimensional Einstein manifolds

Let  $M$  be an  $n$ -dimensional Einstein manifold admitting a concircular scalar field  $\rho$ . Applying Ricci's formula to the equation (1.1), we have

$$(3.1) \quad -K_{\nu\mu\lambda}{}^{\kappa}\rho_{\kappa} = \phi_{\nu}g_{\mu\lambda} - \phi_{\mu}g_{\nu\lambda},$$

and contracting with  $g^{\mu\lambda}$ ,

$$(3.2) \quad -K_{\nu}{}^{\kappa}\rho_{\kappa} = (n-1)\phi_{\nu}.$$

Since  $M$  is an Einstein manifold, that is,  $K_{\nu}{}^{\kappa} = (n-1)\kappa\phi_{\nu}{}^{\kappa}$ , we have the equation

$$(3.3) \quad \phi_{\nu} = -\kappa\rho_{\nu}, \quad \text{or} \quad \phi = -\kappa\rho + b$$

and

$$(3.4) \quad \nabla_{\mu}\nabla_{\lambda}\rho = (-\kappa\rho + b)g_{\mu\lambda},$$

where  $b$  is an integral constant. Hence, in an Einstein manifold, a concircular field is special and the characteristic constant of  $\rho$  is equal to the constant scalar curvature  $\kappa$ .

**Theorem 2.** *If a 4-dimensional Einstein manifold  $M$  admits a concircular scalar field, then the manifold is of constant curvature.*

*Proof.* With respect to an adapted coordinate system  $(u^{\kappa})$  in a neighborhood  $U$  of any ordinary points of  $M$ , the equation (3.4) is reduced to the ordinary equation

$$(3.5) \quad \rho'' = -\kappa\rho + b \quad \text{and} \quad \rho''' = -\kappa\rho'.$$

Substituting these into the third equation of (1.7) for  $n=4$ , we have

$$(3.6) \quad K_{ji} = \bar{K}_{ji} - (2\rho^{111} + \rho^1 \rho^{11})f_{ji} = \bar{K}_{ji} - (2\rho^{12} - \kappa \rho^{12})f_{ji}.$$

Since  $M$  is an Einstein manifold, we substitute  $K_{ji} = 3\kappa g_{ji} = 3\kappa \rho^{12} f_{ji}$  into (3.6) and obtain the equations

$$(3.7) \quad \bar{K}_{ji} = (3\kappa \rho^{12} + 2\rho^{12} - \kappa \rho^{12})f_{ji} = 2(\kappa \rho^{12} + \rho^{12})f_{ji}.$$

This shows that  $\bar{M}$  is a 3-dimensional Einstein manifold

$$(3.8) \quad \bar{K}_{ji} = 2 f_{ji}$$

and constant scalar curvature  $\bar{\kappa}$  is equal to

$$(3.9) \quad \bar{\kappa} = \kappa \rho^{12} + \rho^{12}.$$

As it is known that a 3-dimensional Einstein manifold is of constant curvature, the manifold  $\bar{M}$  is of constant curvature, that is,

$$(3.10) \quad \bar{K}_{kji}^h = \bar{\kappa}(\delta_k^h f_{ji} - \delta_j^h f_{ki}).$$

Substituting this and (3.9) into the third equation of (1.6), we have

$$(3.11) \quad K_{kji}^h = (\bar{\kappa} - \rho^{12})(\delta_k^h f_{ji} - \delta_j^h f_{ki}) = \kappa \rho^{12}(\delta_k^h f_{ji} - \delta_j^h f_{ki}) \\ = \kappa(\delta_k^h g_{ji} - \delta_j^h g_{ki}).$$

The first and second equations of (1.6) are rewritten as

$$(3.12) \quad K_{1ji}^h = -\delta_j^h = \kappa(\delta_1^h g_{ji} - \delta_j^h g_{1i}) \\ K_{1ji}^h = \kappa \rho^{12} f_{ji} = \kappa(\delta_1^1 g_{ji} - \delta_j^1 g_{1i})$$

by means of (1.5). The equations (3.11) and (3.12) together make the tensor equation  $K_{\nu\mu\lambda}^{\kappa} = \kappa(\delta_{\nu}^{\kappa} g_{\mu\lambda} - \delta_{\mu}^{\kappa} g_{\nu\lambda})$ . Therefore the manifold  $M$  is of constant curvature at ordinary points. Since the stationary point of  $\rho$  is isolated if there is any,  $M$  is of constant curvature.

Q. E. D.

By virtue of this theorem, we give a slight generalization of Kul-karni's theorem [2] in a different way.

**Corollary.** *Let  $M$  and  $M^*$  be 4-dimensional Einstein manifolds which are nowhere of constant curvature. Then every conformal map of  $M$  into  $M^*$  is a homothety.*

*Proof.* Let  $f$  be a conformal map of  $M$  into  $M^*$ , and denote the metric tensor  $f^*g$  by components  $g_{\mu\lambda}^*$ . Then they are related by the

equation  $g_{\mu\lambda}^* = \frac{1}{\rho^2} g_{\mu\lambda}$ , where  $\rho$  is a positive valued scalar field. We indicate by asterisking quantities of  $g_{\mu\lambda}^*$  corresponding to those of  $g_{\mu\lambda}$ . Then we obtain the transformation formulas

$$(3.13) \quad \left\{ \begin{smallmatrix} \kappa \\ \mu\lambda \end{smallmatrix} \right\}^* = \left\{ \begin{smallmatrix} \kappa \\ \mu\lambda \end{smallmatrix} \right\} - \frac{1}{\rho} (\delta_\mu^\kappa \rho_\lambda + \delta_\lambda^\kappa \rho_\mu - g_{\mu\lambda} \rho^\kappa),$$

$$(3.14) \quad K_{\nu\mu\lambda}^* = K_{\nu\mu\lambda} + \frac{1}{\rho} (\partial_\nu^\kappa \nabla_\mu \rho_\lambda - \delta_\mu^\kappa \nabla_\nu \rho_\lambda + g_{\mu\lambda} \nabla_\nu \rho^\kappa - g_{\nu\lambda} \nabla_\mu \rho^\kappa) - \frac{1}{\rho^2} \rho_\omega \rho^\omega (\delta_\nu^\kappa g_{\mu\lambda} - \delta_\mu^\kappa g_{\nu\lambda}),$$

$$(3.15) \quad K_{\mu\lambda}^* = K_{\mu\lambda} + \frac{1}{\rho} 2 \nabla_\mu \rho_\lambda + \frac{1}{\rho} g_{\mu\lambda} \nabla_\kappa \rho^\kappa - \frac{1}{\rho^2} 3 \rho_\kappa \rho^\kappa g_{\mu\lambda}.$$

Since  $M$  and  $M^*$  are Einstein manifolds, we substitute  $K_{\kappa\lambda}^* = 3\kappa^* g_{\mu\lambda}^* = 3\kappa^* \rho^{-2} g_{\mu\lambda}$  and  $K_{\mu\lambda} = 3\kappa g_{\mu\lambda}$  into (3.15), and obtain the following equation

$$(3.16) \quad \nabla_\kappa \rho_\lambda = \frac{\rho}{2} (3\rho^{-2} \kappa^* - 3\kappa - \frac{1}{\rho} \nabla_\kappa \rho^\kappa + \frac{3}{\rho^2} \rho^\kappa) g_{\mu\lambda}.$$

This equation means that  $\rho$  is a concircular scalar field if  $\rho$  would not be constant, and  $M$  would be a manifold of constant curvature by Theorem 2. This is a contradiction. Therefore  $\rho$  must be a constant, that is,  $f$  is a homothety.

Q. E. D.

We notice that  $\kappa^*$  need not be equal to  $\kappa$ .

#### 4. Manifolds of $H_{\sigma\omega\nu\mu\lambda}^* = 0$

E. Cartan defined the tensor  $H_{\sigma\omega\nu\mu\lambda}^*$  by the equation

$$(4.1) \quad H_{\sigma\omega\nu\mu\lambda}^* = K_{\sigma\omega\nu}^\tau K_{\tau\mu\lambda}^* + K_{\sigma\omega\mu}^\tau K_{\tau\nu\lambda}^* + K_{\sigma\omega\lambda}^\tau K_{\tau\nu\mu}^* - K_{\sigma\omega\tau}^* K_{\nu\mu\lambda}^\tau.$$

**Theorem 3.** *If  $M$  is a manifold with the property  $H_{\sigma\omega\nu\mu\lambda}^* = 0$  and admits a concircular scalar field  $\rho$  such that  $\phi$  is not identically constant, then the manifold is of constant curvature.*

*Proof.* We refer to an adapted coordinate system  $(u^*)$  in a neighborhood of any ordinary points of  $\rho$  and put the indices  $\sigma=1, \omega=k, \kappa=h, \lambda=i, \mu=j, \nu=1$  in the equation (4.1). Taking account of the components (1.6) of the curvature tensor, we have



$$\frac{\rho'''}{\rho'} \{ \bar{K}_{kji}{}^h - \rho'' (\delta_k^h f_{ji} - \delta_j^h f_{ki}) \} - \frac{\rho'''}{\rho'} (\rho' \rho''') (\delta_j^h f_{ki} - \delta_k^h f_{ji}) = 0$$

or

$$(4.2) \quad \frac{\rho'''}{\rho'} \{ \bar{K}_{kji}{}^h - (\rho'' - \rho' \rho''') (\delta_k^h f_{ji} - \delta_j^h f_{ki}) \} = 0.$$

As  $\rho''' \neq 0$ , we have from (4.2) the equation

$$(4.3) \quad \bar{K}_{kji}{}^h = (\rho'' - \rho' \rho''') (\delta_k^h f_{ji} - \delta_j^h f_{ki}),$$

from which follows  $\rho'' - \rho' \rho''' = \bar{\kappa}$ . This implies that  $\bar{M}$  is a manifold of constant curvature. Substituting (4.3) into the third of (1.6), we have

$$K_{kji}{}^h = -\frac{\rho'''}{\rho'} \delta_k^h g_{ji} - \delta_j^h g_{ki}.$$

From the first and second equations of (1.6) and the other components being zero, we can obtain the tensor equation

$$K_{\omega\mu\lambda}{}^\kappa = -\frac{\rho'''}{\rho'} (\delta_\nu^\kappa g_{\mu\lambda} - \delta_\mu^\kappa g_{\nu\lambda}).$$

Since a stationary point is isolated,  $M$  is a manifold of constant curvature.

Q. E. D.

We put  $J = \{u \mid \rho'''(u) = 0\}$ . If  $J$  contains intervals, we have  $\rho = Au^2 + Bu + C$ , where  $A, B$  and  $C$  are constant. At the points of the complement  $J^c$  of  $J$ , the equation  $\rho'' - \rho' \rho''' = \bar{\kappa}$  is satisfied under initial conditions  $\rho''(0) = 2A$ ,  $\rho'(0) = B$  and  $\rho(0) = C$  by a suitable choice of arc-length  $u$ . Then the solution is given by

$$\rho(u) = \frac{B^2}{(4A^2 - \bar{\kappa})} \left( 2A \cosh \frac{\sqrt{4A^2 - \bar{\kappa}}}{B} u + \sqrt{4A^2 - \bar{\kappa}} \sinh \frac{\sqrt{4A^2 - \bar{\kappa}}}{B} u - 2A \right) + C.$$

So the differentiability is broken at the point of  $u=0$ . Therefore,  $J$  is equal to the whole straight line  $I$  or discrete. When  $J$  is the straight line,  $\rho'''(u)=0$  for every point of  $I$ , that is,  $\rho''=b$  and  $\nabla_\mu \rho_\lambda = b g_{\mu\lambda}$ ,  $b$  being constant. It follows that  $\rho_\lambda$  is concurrent or parallel. If  $J$  is discrete,  $M$  is of constant curvature at any point.

## 5. Manifolds of $\bar{H}_{\sigma\omega\mu\lambda}^* = 0$

We put the tensor,

$$(5.1) \quad H_{\sigma\omega\mu\lambda} = H_{\sigma\omega\nu\mu\lambda}{}^\nu = K_{\sigma\omega\mu}{}^\nu K_{\nu\lambda} + K_{\sigma\omega\lambda}{}^\nu K_{\mu\nu}.$$

**Theorem 4.** *If  $M$  is a manifold having the property  $\bar{H}_{\sigma\omega\mu\lambda}^* = 0$  and admits a concircular scalar field  $\rho$  such that  $\phi$  is not identically constant, then  $M$  is an Einstein manifold.*

*Proof.* Referring to an adapted coordinate system  $(u^e)$  in a neighborhood of any ordinary point of  $\rho$ , and putting the indices  $\sigma=1$ ,  $\omega=j$ ,  $\lambda=i$ ,  $\mu=1$  in the equation (5.1), we have

$$(5.2) \quad \frac{\rho'''}{\rho'} \{ \bar{K}_{j1} - (n-2)(\rho''^2 - \rho' \rho''') f_{j1} \} = 0.$$

As  $\rho''' \neq 0$ , it follows from (5.2) that

$$(5.3) \quad \bar{K}_{j1} = (n-2)(\rho''^2 - \rho' \rho''') f_{j1}.$$

This implies that  $\bar{M}$  is an Einstein manifold and the scalar curvature is equal to  $\bar{\kappa} = \rho''^2 - \rho' \rho'''$ . Substituting (5.3) into the third of (1.7), we have

$$(5.4) \quad K_{j1} = -(n-1) \frac{\rho'''}{\rho'} g_{j1}.$$

From the first and second equations of (1.7), we have the tensor equation  $K_{\mu\lambda} = -(n-1) \frac{\rho'''}{\rho'} g_{\mu\lambda}$  and hence  $M$  is an Einstein manifold.

Q. E. D.

When  $\rho'''$  vanishes, the same argument as that of 4 is applicable.

#### REFERENCES

- [1] Y. TASHIRO: Conformal transformations in complete Riemannian manifolds, the Study Group of Geometry, 1967.
- [2] R. S. KULKARNI: Curvature structures and conformal transformations, J. Differential Geometry, 4 (1967), 425-451.

TSUYAMA TECHNICAL COLLEGE

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