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ON THE DEFICIENCIES OF MEROMORPHIC FUNCTIONS OF FINITE ORDER

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1. Introduction

Let $f(z)$ be a meromorphic function in $|z|<\infty$. The standard symbols of the Nevanlinna theory

$$\log^+,\ m(r, a),\ N(r, a),\ T(r, f),\ z=re^{i\theta},$$

are used throughout this note. Moreover, we use the following notations.

The order λ and lower order μ of $f(z)$ are defined by

$$\lambda=\limsup_{r\to\infty}\frac{\log T(r,f)}{\log r},\qquad \mu=\liminf_{r\to\infty}\frac{\log T(r,f)}{\log r}.$$

The deficiencies $\delta(a,f)$ in the sense of Nevanlinna and $\Delta(a,f)$ in the sense of Valiron, of the value a, are defined by, respectively:

$$\delta(a,f)=\liminf_{r\to\infty}\frac{m(r,a)}{T(r,f)},\qquad \Delta(a,f)=\limsup_{r\to\infty}\frac{m(r,a)}{T(r,f)},$$

where the total deficiency $\sum_a\delta(a,f)\leqq 2$.

The quantity $\kappa(f)$ is defined by

$$\kappa(f)=\limsup_{r\to\infty}\frac{N(r,0)+N(r,\infty)}{T(r,f)}\leqq 2-\delta(0,f)-\delta(\infty,f).$$

Now, one of the conjectures of F. Nevanlinna [5] is that $\delta(a_k, f)=q(k)/\lambda$ ($q(k)$: an integer) if $f(z)$ is a meromorphic function of finite order λ with $\sum_a\delta(a,f)=2$.

A. Pfluger [6] showed that the conjecture is valid for the entire functions, and A. Edrei [2, p. 54] pointed out without proof that it is also valid for meromorphic functions with $\sum_{a\neq\infty}\delta(a,f)=1$ and $\delta(\infty,f)=1$. We shall show that $q(k)$ is the number of asymptotic paths with the asymptotic value a_k (Theorem 1).

It is plausible that contributions, to a deficiency $\delta(a_k, f)$, of f near the paths are even, and considering a subdivision of deficiencies, we

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obtain some results that reflect this in a sense (Theorem 2 and its Corollaries).

The method of proofs in this note is mainly due to A. Edrei and W. H. J. Fuchs [4].

2. Statement and discussion of results.

We take $a_0=0$, $a_\infty=\infty$ and finite values a_k $(k=1, 2, \cdots, s)$.

Let $f(z)$ be a meremorphic function of finite order λ in $|z|<\infty$, and p the integer defined by

$$p-\frac{1}{2}\leqq \lambda < p+\frac{1}{2}.$$

From Theorem 6 of [3] (p. 298), Lemma 1 of [3] (pp. 298—299) and Theorem 3 of [1] (p. 173), we see that if

$$\sum_{a\neq\infty}\delta(a, f)=1 \quad \text{and} \quad \delta(\infty, f)=1, \tag{2.1}$$

then

$$\lambda=\mu \quad \text{and} \quad \lambda=p\geqq 1. \tag{2.2}$$

Frequently we use the following

Lemma A (Edrei and Fuchs [4], p. 279). *Let $f(z)$ be a meromorphic function of finite order λ in $|z|<\infty$. Give ε $(0<\varepsilon<1/16)$ and δ $(0<\delta<1/e)$ arbitrarily. If $\lambda(f)=0$, then $\lambda=p$ and there exists a sequence $\{c_n\}$ $(c_n=c(\alpha^n)$; $\alpha=e^{\frac{1}{p+1}})$ such that*

$|\log|f(z)|-\mathrm{Re}\, c_n z^p|<4\varepsilon|c_n|r^p$ for $z\in\Gamma_n-E_n$ $(|z|=r)$ if r is sufficiently large, $r>r_0$, where

$$\Gamma_n=\{z\,;\ \alpha^n\leqq|z|<\alpha^{n+\frac{3}{2}}\}$$

and the exceptional set E_n is contained in the finite number of discs in $\Gamma_{n-1}\cup\Gamma_n\cup\Gamma_{n+1}$, the sum of whose radii doesn't exceed $4e^2\delta\alpha^n$.

From Theorems 2 and 3 of [4] (pp. 263—264), we see that if (2.1) holds, there exist p asymptotic paths $\mathscr{L}^{(k)}(k=1, 2, \cdots, p)$ with finite asymtotic values such that each of $\mathscr{L}^{(k)}(k=2, \cdots, p)$ is the rotation of $\mathscr{L}^{(1)}$, around the origin, of angle $(k-1)2\pi/p$, and denoting by a_k $(k=1, 2, \cdots, s\,;\ s\leqq p)$ distinct values among the asymptotic values corresponding to these paths, then

$$\sum_{k=1}^{s} \delta(a_k, f) = 1. \tag{2.3}$$

As an immediate consequence of this result, we obtain the following

Theorem 1 (see. Edrei [2], p. 54). *Let $f(z)$ be a meromorphic function of finite order λ in $|z|<\infty$. If* (2.1) *holds, then*

$$\delta(a_k, f) = q(k)/\lambda,$$

where $q(k)$ is the number of asymptotic paths with the asymptotic value a_k $(k=1, 2, \cdots, s)$.

Now, we consider a subdivision of deficiencies. To do this we need the following definitions;

$$D_k = \left\{z\,;\, \frac{1}{|f(z)-a_k|} > 1\right\} \cap \{z\,;\, |z|>r_0\},$$

$$D_\infty = \{z\,;\, |f(z)|>1\} \cap \{z\,;\, |z|>r_0\} \qquad (r_0>0)$$

and $D_k^{(l)}(l=1, 2, \cdots, q(k))$ or $D_\infty^{(l)}(l=1, 2, \cdots, q(\infty))$ are the disjoint components of D_k or D_∞, respectively. We set

$$F_k(r) = \{\theta\,;\, z=re^{i\theta} \in D_k\} \quad \text{where} \quad F_k(r) \subset [0, 2\pi),$$

$$F_\infty(r) = \{\theta\,;\, z=re^{i\theta} \in D_\infty\} \quad \text{where} \quad F_\infty(r) \subset [0, 2\pi),$$

$$F_k^{(l)}(r) = \{\theta\,;\, z=re^{i\theta} \in D_k^{(l)}\} \quad (l=1, 2, \cdots, q(k)),$$

$$F_\infty^{(l)}(r) = \{\theta\,;\, z=re^{i\theta} \in D_\infty^{(l)}\} \quad (l=1, 2, \cdots, q(\infty)).$$

Further we set

$$m(r, a_k, f\,;\, F_k^{(l)}(r)) = \frac{1}{2\pi}\int_{F_k^{(l)}(r)} \log^+ \frac{1}{|f(z)-a_k|}\, d\theta \qquad (l=1, 2, \cdots, q(k))$$

$$m(r, \infty, f\,;\, F_\infty^{(l)}(r)) = \frac{1}{2\pi}\int_{F_\infty^{(l)}(r)} \log^+ |f(z)|\, d\theta \qquad (l=1, 2, \cdots, q(\infty))$$

and

$$\delta^{(l)}(a_k, f) = \liminf_{r\to\infty} \frac{m(r, a_k, f\,;\, F_k^{(l)}(r))}{T(r, f)}$$

$$\Delta^{(l)}(a_k, f) = \limsup_{r\to\infty} \frac{m(r, a_k, f\,;\, F_k^{(l)}(r))}{T(r, f)} \qquad (l=1, 2, \cdots, q(k)).$$

Then we have

$$\sum_l \delta^{(l)}(a_k, f) \leqq \delta(a_k, f) \leqq \Delta(a_k, f) \leqq \sum_l \Delta^{(l)}(a_k, f).$$

When we take $f'(z)$ for $f(z)$ in the above definitions, we use the symbols $G_0(r)$, $G_\infty(r)$, $G_0^{(l)}(r)$ and $G_\infty^{(l)}(r)$ in place of $F_0(r)$, $F_\infty(r)$, $F_0^{(l)}(r)$ and $F_\infty^{(l)}(r)$, respectively.

We set

$$\bar{\delta}(a_k, f) = \liminf_{r\to\infty} \frac{m(r, a_k, f; G_\infty(r))}{T(r, f)}, \quad \underline{\delta}(a_k, f) = \liminf_{r\to\infty} \frac{m(r, a_k, f; G_0(r))}{T(r, f)},$$

$$\bar{\delta}^{(l)}(a_k, f) = \liminf_{r\to\infty} \frac{m(r, a_k, f; G_\infty^{(l)}(r))}{T(r, f)}, \quad \underline{\delta}^{(l)}(a_k, f) = \liminf_{r\to\infty} \frac{m(r, a_k, f; G_0^{(l)}(r))}{T(r, f)},$$

$$\bar{\Delta}(a_k, f) = \limsup_{r\to\infty} \frac{m(r, a_k, f; G_\infty(r))}{T(r, f)}, \quad \underline{\Delta}(a_k, f) = \limsup_{r\to\infty} \frac{m(r, a_k, f; G_0(r))}{T(r, f)},$$

$$\bar{\Delta}^{(l)}(a_k, f) = \limsup_{r\to\infty} \frac{m(r, a_k, f; G_\infty^{(l)}(r))}{T(r, f)}, \quad \underline{\Delta}^{(l)}(a_k, f) = \limsup_{r\to\infty} \frac{m(r, a_k, f; G_0^{(l)}(r))}{T(r, f)}.$$

Using Lemma A, we obtain the following

Theorem 2. *Let $f(z)$ be a meromorphic function of finite order λ in $|z|<\infty$. If $\delta(0, f)=1$ and $\delta(\infty, f)=1$, then*

$$\delta(0, f) = \sum_{l=1}^{\lambda} \delta^{(l)}(0, f) = 1, \qquad \delta(\infty, f) = \sum_{l=1}^{\lambda} \delta^{(l)}(\infty, f) = 1,$$

$$\delta^{(l)}(0, f) = 1/\lambda, \qquad \delta^{(l)}(\infty, f) = 1/\lambda \qquad (l = 1, 2, \cdots, \lambda).$$

Corollary 1. *Let $f(z)$ be an entire function of finite order λ. If $\delta(0, f)=1$, then*

$$\Delta(\infty, f) = \sum_{l=1}^{\lambda} \Delta^{(l)}(\infty, f) = 1,$$

$$\delta^{(l)}(\infty, f) = \Delta^{(l)}(\infty, f) = 1/\lambda \qquad (l = 1, 2, \cdots, \lambda).$$

Corollary 2. *Let $f(z)$ be a meromorphic function of finite order λ in $|z| < \infty$. If* (2. 1) *holds, then*

$$\delta(\infty, f) = \bar{\delta}(\infty, f) = \sum_{l=1}^{\lambda} \bar{\delta}^{(l)}(\infty, f) = 1, \ \bar{\delta}^{(l)}(\infty, f) = 1/\lambda \ (l = 1, 2, \cdots, \lambda),$$

$$\underline{\delta}(\infty, f) = 0 \qquad (\text{so that } \ \underline{\delta}^{(l)}(\infty, f) = 0 \ \ (l = 1, 2, \cdots, \lambda)),$$

$$\bar{\Delta}(a_k, f) = 0 \qquad (\text{so that } \ \bar{\delta}^{(l)}(a_k, f) = 0 \ \ (l = 1, 2, \cdots, \lambda)),$$

$$\delta(a_k, f) = \Delta(a_k, f) = \underline{\Delta}(a_k, f), \ \ \underline{\delta}^{(l)}(a_k, f) \leqq 1/\lambda \ \ (l = 1, 2, \cdots, \lambda).$$

From Theorem 1 and Corollary 2, the author thinks that the following could be proved: *Under the condition* (2.1),

$$\delta(\infty, f) = \sum_{l=1}^{\lambda} \delta^{(l)}(\infty, f) = 1, \qquad \sum_{k=1}^{s}\sum_{l=1}^{q(k)} \delta^{(l)}(a_k, f) = 1,$$

$$\delta^{(l)}(\infty, f) = 1/\lambda \ (l=1,2,\cdots,\lambda), \qquad \delta^{(l)}(a_k, f) = 1/\lambda \quad (l=1,2,\cdots,q(k)).$$

Theorem 2 shows that this is valid in the special case that $\delta(0, f)=1$ and $\delta(\infty, f) = 1$.

3. Proof of Theorem 1

We put $\delta=1/(p+1)^{11}$, where $p=\lambda$ by (2.2), and take ε $(0<\varepsilon<1/16)$ arbitrarily.

Since $f(z)$ satisfies (2.1), $\kappa(f')=0$ by Lemma 1 of [3] (pp. 298—299). Hence, by Lemma A and Theorem 1 of [4] (pp. 261—262), there exist sequences $\{r_n\}$ and $\{\tilde{c}_n\}$ such that

$$\alpha^n \leqq r_n < \alpha^{n+\frac{1}{2}} \ (\alpha = e^{\frac{1}{p+1}}), \quad \{z\,;\,|z|=r_n\} \cap (\widetilde{E}_n \cup \widetilde{E}_{n-1}) = \emptyset,$$

(3.1) $$|\log|f'(z)| - \mathrm{Re}\,\tilde{c}_n z^p| < 4\varepsilon|\tilde{c}_n| r_n^p \qquad (|z|=r_n\,;\ n>n_0)$$

and

(3.2) $$|\tilde{c}_n| r_n^p = (1+o(1))\pi T(r_n, f')$$ on Γ_n, where $\widetilde{E}_n$ are the exceptional set for $f'(z)$.

Let $\tilde{\omega}_n$ be the argument of $\tilde{c}_n$. We put $\widetilde{G}_0(r_n) = \{\theta\,;\,\cos(p\theta+\tilde{\omega}_n) \leqq -5\varepsilon\}$. Then $\widetilde{G}_0(r_n) = \sum_{l=1}^{p} \widetilde{G}_0^{(l)}(r_n)$, where each $\widetilde{G}_0^{(l)}(r_n)$ is a component of $\widetilde{G}_0(r_n)$ contained in $\left[\frac{(4l-3)}{2p}\pi - \frac{\tilde{\omega}_n}{p}, \ \frac{(4l-1)}{2p}\pi - \frac{\tilde{\omega}_n}{p}\right]$ $(l=1,2,\cdots,p)$. We put $\mathscr{A}_n^{(l)} = \{r_n e^{i\theta}\,;\,\theta \in \widetilde{G}_0^{(l)}(r_n)\}$ $(l=1,2,\cdots,p)$. Let $q(1)$ be the number of asymptotic paths with an asymptotic value a_1. We may assume without loss of generality that $\mathscr{L}^{(l)}(l=1,2,\cdots,q(1))$ are these asymptotic paths.

As shown in [4] (pp. 289—290)

$$\lim_{n\to\infty} f(z) = a_1$$

uniformly on $\mathscr{A}_n^{(l)}$ for each $l=1,2,\cdots,q(1)$.

We put $G_0^1(r_n) = \sum_{l=1}^{q(1)} \widetilde{G}_0^{(l)}(r_n)$. Then

$$\sum_{k=2}^{s} \frac{1}{2\pi}\int_{G_0^1(r_n)} \log^+ \frac{1}{|f(z)-a_k|}\, d\theta = O(1) \qquad (|z| = r_n\ ;\ n > n_0).$$

In view of (3. 1) and (3. 2), we deduce from Lemma 7 of [4] (p. 285)

$$\sum_{k=2}^{s} \frac{1}{2\pi}\int_{\bar{G}_0(r_n)-G_0^1(r_n)} \log^+ \frac{1}{|f(z)-a_k|}\, d\theta \leqq \frac{1}{2\pi}\int_{\bar{G}_0(r_n)-G_0^1(r_n)} \log^+ \frac{1}{|f'(z)|}\, d\theta + O(\log r_n)$$

$$\leqq \frac{1}{2\pi}\, |\tilde{c}_n|\, r_n^p (p - q(1)) \int_{-\frac{\pi}{2p}}^{\frac{\pi}{2p}} \cos(p\theta)\, d\theta + 4\varepsilon\, |\tilde{c}_n|\, r_n^p + O(\log r_n)$$

$$= (1+o(1))\ (1 - \frac{q(1)}{p} + 4\pi\varepsilon) T(r_n, f') + O(\log r_n),$$

$$\sum_{k=2}^{s} \frac{1}{2\pi}\int_{[0,2\pi)-\bar{G}_0(r_n)} \log^+ \frac{1}{|f(z)-a_k|}\, d\theta \leqq \frac{1}{2\pi}\int_{[0,2\pi)-\bar{G}_0(r_n)} \log^+ \frac{1}{|f'(z)|}\, d\theta + O(\log r_n)$$

$$\leqq 9(1+o(1))\pi\varepsilon T(r_n, f') + O(\log r_n) \quad (|z| = r_n).$$

Therefore

$$\sum_{k=2}^{s} m(r_n, a_k) \leqq (1+o(1))\ (1 - \frac{q(1)}{p} + 13\pi\varepsilon) T(r_n, f') + O(\log r_n).$$

By Lemma 1 of [3] (pp. 298—299)

$$\lim_{r\to\infty} \frac{T(r, f')}{T(r, f)} = 1. \tag{3. 3}$$

Hence

$$\sum_{k=2}^{s} \delta(a_k, f) \leqq \sum_{k=2}^{s} \liminf_{n\to\infty} \frac{m(r_n, a_k)}{T(r_n, f)} \leqq \liminf_{n\to\infty} \sum_{k=2}^{s} \frac{m(r_n, a_k)}{T(r_n, f)} \leqq 1 - \frac{q(1)}{p} + 13\pi\varepsilon.$$

Thus, we deduce from (2. 3), being ε $(0 < \varepsilon < 1/16)$ arbitrary,

$$\delta(a_1, f) \geqq q(1)/p.$$

Similarly, we have for $2 \leqq k \leqq s$,

$$\delta(a_k, f) \geqq q(k)/p.$$

Since $\sum_{k=1}^{s} q(k) = p$, we deduce from (2. 3)

$$\delta(a_k, f) = q(k)/p \qquad (k = 1, 2, \cdots, s).$$

Since $p = \lambda$, the proof of Theorem 1 is now complete.

4. Proof of Theorem 2

We take ε $(0<\varepsilon<1/16)$ and δ $(0<\delta<1/e)$ arbitrarily. By Lemma A and Theorem 1 of [4] (pp. 261—262), there exist sequences $\{r_n\}$ and $\{c_n\}$ such that

$$\alpha_n \leqq r_n < \alpha^{n+2}, \qquad \{z\,;\ |z|=r_n\} \cap (E_n \cup E_{n-1}) = \emptyset$$

(4. 1) $$|\log|f(z)| - \mathrm{Re}\ c_n z^p| < 4\varepsilon |c_n| r_n^p \qquad (|z|=r_n)$$

and

(4. 2) $$|c_n| r^p = (1+o(1))\pi T(r, f) \qquad (|z|=r,\ z\in\Gamma_n).$$

We put $c_n = |c_n|e^{i\omega_n}$ $(c_n \neq 0)$ and $\widetilde{F}_\infty(r, n) = \{\theta\,;\ \cos(p\theta+\omega_n) \geqq 4\varepsilon\}$. Then we have $\widetilde{F}_\infty(r, n) = \sum_{l=1}^{p} \widetilde{F}_\infty^{(l)}(r, n)$, where each $\widetilde{F}_\infty^{(l)}(r, n)$ is a component of $\widetilde{F}_\infty(r, n)$ contained in $\left[\frac{(4l-1)}{2p}\pi - \frac{\omega_n}{p},\ \frac{(4l+1)}{2p}\pi - \frac{\omega_n}{p}\right]$ $(l=1, 2, \cdots, p)$.

In view of (4. 1) and (4. 2)

$$\begin{aligned} m(r_n, \infty, f\,;\ \widetilde{F}_\infty^{(l)}(r_n)) &\leqq \frac{1}{2\pi}|c_n| r_n^p \int_{\widetilde{F}_\infty^{(l)}(r_n)} \{\cos(p\theta+\omega_n)+4\varepsilon\}\, d\theta \\ &\leqq \frac{1}{2\pi}|c_n|\, r_n^p \int_{-\frac{\pi}{2p}}^{\frac{\pi}{2p}} \cos(p\theta)\ d\theta + 4\varepsilon |c_n| r_n^p \\ &= (1+o(1))\left(\frac{1}{p} + 4\pi\varepsilon\right) T(r_n, f), \end{aligned}$$

$$m(r_n, \infty, f\,;\ F_\infty^{(l)}(r_n) - \widetilde{F}_\infty^{(l)}(r_n)) \leqq 8(1+o(1))\pi\varepsilon\, T(r_n, f) \qquad (n>n_0),$$

where $\widetilde{F}_\infty^{(l)}(r_n) = \widetilde{F}_\infty^{(l)}(r_n, n)$, so that we have

$$m(r_n, \infty, f\,;\ F_\infty^{(l)}(r_n)) \leqq (1+o(1))\left(\frac{1}{p} + 12\pi\varepsilon\right) T(r_n, f).$$

Hence, we have, being ε $(0<\varepsilon<1/16)$ arbitrary,

(4. 3) $$\delta^{(l)}(\infty, f) \leqq \liminf_{n\to\infty} \frac{m(r_n, \infty, f\,;\ F_\infty^{(l)}(r_n))}{T(r_n, f)} \leqq \frac{1}{p} \qquad (l=1, 2, \cdots, p).$$

Let $J(r, n)$ be the part of the exceptional set E_n on $|z|=r$, i. e., $J(r, n) = \{\theta\,;\ z=re^{i\theta}\in E_n\}$. Then we have

(4. 4) $$\text{meas } J(r, n) \leqq 8\pi e^2\delta.$$

In view of Lemma A, (4. 2) and (4. 4)

$$m(r, \infty, f; F_\infty^{(l)}(r)) = \frac{1}{2\pi}\int_{F_\infty^{(l)}(r)} \log^+|f(z)|\,d\theta \geqq \frac{1}{2\pi}\int_{\tilde{F}_\infty^{(l)}(r,n)-J(r,n)} \log^+|f(z)|\,d\theta$$

$$\geqq \frac{1}{2\pi}|c_n|r^p \int_{\tilde{F}_\infty^{(l)}(r,n)-J(r,n)} \{\cos(p\theta+\omega_n)-4\varepsilon\}\,d\theta$$

$$\geqq |c_n|r^p\left\{\frac{1}{2\pi}\int_{-(1-4\varepsilon)\frac{\pi}{2p}}^{(1-4\varepsilon)\frac{\pi}{2p}} \cos(p\theta)\,d\theta - \frac{1}{2\pi}\text{meas } J(r, n) - 4\varepsilon\right\}$$

$$\geqq (1+o(1))\left(\frac{1-4\varepsilon}{p} - 4\pi e^2\delta - 4\pi\varepsilon\right) T(r, f),$$

so that we have

(4. 5) $$\delta^{(l)}(\infty, f) \geqq \frac{1}{p} - \frac{4\varepsilon}{p} - 4\pi\varepsilon - 4\pi e^2\delta.$$

We deduce from (4. 3) and (4. 5), being ε $(0<\varepsilon<1/16)$ and δ $(0<\delta<1/e)$ arbitrary,

$$\delta^{(l)}(\infty, f) = 1/p \qquad (l=1, 2, \cdots, p).$$

Thus, we have

$$\delta(\infty, f) = \sum_{l=1}^{p} \delta^{(l)}(\infty, f) = 1.$$

Next, taking $1/f$ for f, we have by the same calculation as the above one

$$\delta^{(l)}(0, f) = 1/p \qquad (l=1, 2, \cdots, p).$$

Thus, we have

$$\delta(0, f) = \sum_{l=1}^{p} \delta^{(l)}(0, f) = 1.$$

Since $p=\lambda$, the proof of Theorem 2 is now complete.

5. Proofs of corollaries

(1). **Proof of Corollary 1.** By Lemma A for an entire function $f(z)$,

$$\log|f(z)| < \mathrm{Re}\, c_n z^p + 4\varepsilon|c_n|r^p \qquad \text{on } \Gamma_n. \tag{5.1}$$

In view of (4.2) and (5.1)

$$m(r, \infty, f;\ F_\infty^{(l)}(r)) \leqq (1+o(1))\left(\frac{1}{p} + 12\pi\varepsilon\right)T(r, f),$$

ε $(0<\varepsilon<1/16)$ being arbitrary, so that we have,

$$\Delta^{(l)}(\infty, f) \leqq 1/p \qquad (l=1, 2, \cdots, p).$$

As $\delta^{(l)}(\infty, f) \leqq \Delta^{(l)}(\infty, f)$, we deduce from Theorem 2

$$\delta^{(l)}(\infty, f) = \Delta^{(l)}(\infty, f) = 1/p \qquad (l=1, 2, \cdots, p)$$

and

$$\Delta(\infty, f) = \sum_{l=1}^{p} \Delta^{(l)}(\infty, f) = 1.$$

(II). **Proof of Corollary 2.** We deduce from [7] (pp. 23—24), (2.2) and Theorem 2

$$\delta^{(l)}(0, f') = 1/p \qquad (l=1, 2, \cdots, p), \tag{5.2}$$

$$\delta^{(l)}(\infty, f') = 1/p \qquad (l=1, 2, \cdots, p). \tag{5.3}$$

Since $\log^+|f'| \leqq \log^+|f| + \log^+\left|\frac{f'}{f}\right|$,

$$\begin{aligned} m(r, \infty, f';\ G_\infty^{(l)}(r)) &\leqq m(r, \infty, f;\ G_\infty^{(l)}(r)) + m\left(r, \frac{f'}{f}\right) \\ &= m(r, \infty, f;\ G_\infty^{(l)}(r)) + O(\log r) \qquad (r > r_0) \end{aligned}$$

and hence, we deduce from (3.3) and (5.3)

$$\bar{\delta}^{(l)}(\infty, f) \geqq \delta^{(l)}(\infty, f') = 1/p \qquad (l=1, 2, \cdots, p).$$

As $\sum_{l=1}^{p} \bar{\delta}^{(l)}(\infty, f) \leqq \bar{\delta}(\infty, f) \leqq \delta(\infty, f) = 1$, we have

$$\delta(\infty, f) = \bar{\delta}(\infty, f) = \sum_{l=1}^{p} \bar{\delta}^{(l)}(\infty, f) = 1$$

and

$$\bar{\delta}^{(l)}(\infty, f) = 1/p \qquad (l=1, 2, \cdots, p).$$

As $\bar{\delta}(\infty, f) + \underline{\delta}(\infty, f) \leqq \delta(\infty, f)$, we have

$$\underline{\delta}(\infty, f) = 0 \qquad (\text{so that } \underline{\delta}^{(l)}(\infty, f) = 0 \qquad (l=1, 2, \cdots, p)).$$

Since we have

$$\text{(5. 4)} \qquad \log^+\left|\frac{1}{f-a_k}\right| \leqq \log^+\left|\frac{1}{f'}\right| + \log^+\left|\frac{f'}{f-a_k}\right|,$$

$$m(r, a_k, f;\ G_\infty(r)) \leqq m(r, 0, f';\ G_\infty(r)) + m\left(r, \frac{f'}{f-a_k}\right)$$

$$= O(\log r) \qquad (r > r_0).$$

Hence

$$\bar{\Delta}(a_k, f) = 0 \qquad (\text{so that } \bar{\delta}^{(l)}(a_k, f) = 0 \qquad (l=1, 2, \cdots, p)).$$

Therefore

$$\Delta(a_k, f) \leqq \bar{\Delta}(a_k, f) + \underline{\Delta}(a_k, f) = \underline{\Delta}(a_k, f) \leqq \Delta(a_k, f)$$

and hence, we deduce from Lemma A of [2] (p. 59)

$$\delta(a_k, f) = \Delta(a_k, f) = \underline{\Delta}(a_k, f).$$

By (5. 4)

$$m(r, a_k, f;\ G_0^{(l)}(r)) \leqq m(r, 0, f';\ G_0^{(l)}(r)) + m\left(r, \frac{f'}{f-a_k}\right)$$

$$= m(r, 0, f';\ G_0^{(l)}(r)) + O(\log r),$$

so that, in view of (3. 3) and (5. 2), we have

$$\underline{\delta}^{(l)}(a_k, f) \leqq \delta^{(l)}(0, f') = 1/p \qquad (l=1, 2, \cdots, p).$$

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