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On some exponential sums involving the divisor function over arithmetical progressions

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ON SOME EXPONENTIAL SUMS INVOLVING THE DIVISOR FUNCTION OVER ARITHMETICAL PROGRESSIONS

To Professor SHIN-ICHI IZUMI on his seventieth birthday

SABURÔ UCHIYAMA

Let $d(n)$ denote as usual the number of positive divisors of the positive integer n. In 1916 G. H. Hardy [1] studied among other things the sum

$$S_N(t) = \sum_{1 \leqq n \leqq N} n^{-\frac{1}{2}} d(n) e^{-it\sqrt{n}} \qquad (t > 0)$$

and proved that, if $t \neq 4\pi q^{\frac{1}{2}}$ for any positive integer q, then

(1) $$S_N(t) = o(N^{\varepsilon})$$

and, if $t = 4\pi q^{\frac{1}{2}}$ for some positive integer q, then

(2) $$S_N(t) = \frac{2(1+i)\, d(q)}{q^{\frac{1}{4}}} N^{\frac{1}{4}} + o(N^{\varepsilon}),$$

where ε is any fixed positive number and N is tending to infinity.

The main purpose of this paper is to consider sums of the form

$$U(x, N) = U(x, N; k, l) \overset{\text{def}}{=} \sum_{\substack{1 \leqq n \leqq N \\ n \equiv l \ (\mathrm{mod}\ k)}} n^{-\frac{1}{2}} d(n) e^{2\pi i x\sqrt{n}} \qquad (x > 0),$$

where k and l are integers with $k \geqq 1$, $0 \leqq l < k$, and to obtain analogues of (1) and (2). It is not quite difficult to show that, if $x \neq (2q^{\frac{1}{2}})/k$ for any integer q, then one has for $N \to \infty$

(3) $$U(x, N) = O(\log N),$$

which is the best possible in the sense that the estimate $O(\log N)$ on the right-hand side of (3) cannot be improved to $o(\log N)$.[1)] On the other hand, if $x = (2q^{\frac{1}{2}})/k$ for some integer q, the situation becomes rather

1) The particular case of (3) with $k = 1$ and $l = 0$ has also been treated by Mr. T. Kano in a form rather different from ours.

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complicated, though it is in fact possible to find the corresponding result in its full generality.[2] In the simplest case of $(k, l) = 1$ our result takes the following form:

$$(4) \qquad U\left(\frac{2q^{\frac{1}{2}}}{k},\ N\right) = \frac{2(1-i)\,\sigma(q\,;\ k,\ l)}{k^{\frac{3}{2}}\,q^{\frac{1}{4}}}N^{\frac{1}{4}} + O(\log N)$$

with

$$\sigma(q\,; k,\ l) = \sum_{m|q} S\left(k\,; \frac{q}{m},\ ml\right),$$

where $S(k\,; u,\ v)$ is the Kloosterman sum,[3]

$$(5) \qquad S(k\,;\ u,\ v) = \sum_{\substack{a \bmod k \\ (a,k)=1}} \exp\frac{2\pi i}{k}(ua + v\bar{a}),$$

$\bar{a}$ being defined modulo k by $a\bar{a} \equiv 1 \pmod k$, $(a,\ k) = 1$. The O-term on the right side of (4) is again found to be the best possible one. Thus, our results (3) and (4) do not only generalize but also improve Hardy's (1) and (2).

Another way of generalizing (1) and (2) will be to consider the sum

$$S(x,\ N) = S(x,\ N;\ k,\ l) \overset{\text{def}}{=} \sum_{1 \leqq n \leqq N} n^{-\frac{1}{2}} d(n\,;\ k,\ l) e^{2\pi i x\sqrt{n}} \quad (x > 0),$$

where $d(n\,;\ k,\ l)$ denotes for $k \geqq 1$, $0 \leqq l < k$, the number of positive divisors d of n in the residue class $d \equiv l \pmod k$. This is the subject on which we discuss in [4].

1. Let k and l be fixed integers such that $k \geqq 1$, $0 \leqq l < k$. We use the letters m, n and q to represent positive integers and, x and N to denote real numbers with $x > 0$, $N \geqq 1$. The constants implied in the symbol O depend at most on k and l, whereas the constants in the symbol O_x may depend possibly on k, l and x.

Define

$$E(x,\ N) = E(x,\ N;\ k,\ l) = \sum_{\substack{1 \leqq n \leqq N \\ n \equiv l \pmod k}} e^{2\pi i x\sqrt{n}}.$$

2) In either case, we have to assume in reality that x should be greater than a constant multiple of k^3. See Theorem 2 below.

3) Here we adopt a notation for Kloosterman sums, slightly different from the usual one. Cf. [2].

The well-known Euler-Maclaurin summation formula (cf. [3; Theorem 2.1]) will yield

$$(6)\qquad E(x,\ N)=\frac{1}{k}\left(\frac{e^{2\pi ix\sqrt{N}}}{\pi ix}N^{\frac{1}{2}}+\frac{e^{2\pi ix\sqrt{N}}-1}{2\pi^2x^2}\right)+R(x,\ N)+O(1)$$

with

$$R(x,\ N)=P(x,N)-Q(x,\ N)+O\Big(x\sum_{m\neq\frac{kx}{2\sqrt{N}}}\frac{1}{|2mN^{\frac{1}{2}}-kx|^3}\Big)+O(xN^{-\frac{1}{2}}+1),$$

where

$$(7)\qquad P(x,\ N)=\frac{k^{\frac{1}{2}}x}{2^{\frac{1}{2}}}\sum_{m\geq\frac{kx}{2\sqrt{N}}}{}^{*}\frac{1}{m^{\frac{3}{2}}}\exp 2\pi i\left(\frac{kx^2}{4m}+\frac{ml}{k}-\frac{1}{8}\right)$$

and

$$(8)\qquad Q(x,\ N)=\frac{kx}{2\pi i}\sum_{m\neq\frac{kx}{2\sqrt{N}}}\frac{1}{m(2mN^{\frac{1}{2}}-kx)}\exp 2\pi i\left(xN^{\frac{1}{2}}-\frac{m(N-l)}{k}\right).$$

Here, on the right-hand side of (7), the summation $\sum_{m\geq u}^{*}$ indicates that the term which corresponds to $m=u$ (if existent) should be multiplied by the extra factor 1/2. For the detailed derivation of (6) we refer to [4].

2. We are now going to evaluate the sum

$$V(x,\ N)=V(x,\ N;\ k,\ l)=\sum_{\substack{1\leq n\leq N\\ n\equiv l(\mathrm{mod}\ k)}}d(n)e^{2\pi ix\sqrt{n}}.$$

We have

$$V(x,\ N)=\sum_{\substack{1\leq ab\leq N\\ ab\equiv l(\mathrm{mod}\ k)}}e^{2\pi ix\sqrt{ab}}$$

$$=2\sum_{1\leq a\leq\sqrt{N}}\ \sum_{\substack{a<b\leq N/a\\ ab\equiv l(\mathrm{mod}\ k)}}e^{2\pi ix\sqrt{ab}}+\sum_{\substack{1\leq a\leq\sqrt{N}\\ a^2\equiv l(\mathrm{mod}\ k)}}e^{2\pi ixa}$$

$$=2V_1+V_2$$

say, where we find

$$|V_2|\leq N^{\frac{1}{2}}$$

and

$$V_1 = \sum_{d\mid(k,l)} \sum_{\substack{1\le a\le\sqrt{N}/d\\(a,k_d)=1}} \sum_{\substack{ad<b\le N/ad\\ab\equiv l_d \pmod{k_d}}} e^{2\pi ix\sqrt{ab}}$$

with $k_d = k/d$ and $l_d = l/d$ for $d\mid(k, l)$.

For $1 \leqq a \leqq N^{\frac{1}{2}}/d$, $(a, k_d) = 1$, where $d\mid(k, l)$, we define $\bar{a}$ (mod k_d) by the congruence $a\bar{a} \equiv 1$ (mod k_d); there is no loss in generality in assuming that $0 \leqq l_a \bar{a} < k_d$. It readily follows from (6), (7) and (8), with $k = k_d$ and $l = l_a \bar{a}$, that

$$\sum_{\substack{ad<b\le N/ad\\ab\equiv l_d \pmod{k_d}}} e^{2\pi ix\sqrt{adb}} = E\left(x(ad)^{\frac{1}{2}},\ \frac{N}{ad}\right) - E(x(ad)^{\frac{1}{2}},\ ad)$$

$$= \frac{e^{2\pi ix\sqrt{N}}}{\pi ikxa} N^{\frac{1}{2}} + P_d(a) - Q\left(x(ad)^{\frac{1}{2}},\ \frac{N}{ad}\right) + O_x(1),$$

where

$$P_d(a) = P\left(x(ad)^{\frac{1}{2}},\ \frac{N}{ad}\right) - P(x(ad)^{\frac{1}{2}},\ ad)$$

$$= \frac{k^{\frac{1}{2}}xa^{\frac{1}{2}}}{2^{\frac{1}{2}}} \sum^*_{\frac{kxa}{2\sqrt{N}}\le m\le\frac{kx}{2d}} \frac{1}{m^{\frac{3}{2}}} \exp 2\pi i\left(\frac{kx^2a}{4m} + \frac{ml\bar{a}}{k} - \frac{1}{8}\right).$$

Hence, noticing that

$$\sum_{\substack{1\le a\le M\\(a,k)=1}} \frac{1}{a} = \frac{\phi(k)}{k} \log M + O(1)$$

with Euler's totient function $\phi(k)$, we get

$$(9)\quad \begin{aligned} V_1 = {} & \frac{\Phi_k(l)}{k^2}\frac{e^{2\pi ix\sqrt{N}}}{2\pi ix} N^{\frac{1}{2}} \log N + \sum_{d\mid(k,l)} \sum_{\substack{1\le a\le\sqrt{N}/d\\(a,k_d)=1}} P_d(a) \\ & - \sum_{d\mid(k,l)} \sum_{\substack{1\le a\le\sqrt{N}/d\\(a,k_d)=1}} Q\left(x(ad)^{\frac{1}{2}},\ \frac{N}{ad}\right) + O_x(N^{\frac{1}{2}}), \end{aligned}$$

where

$$(10)\quad \Phi_k(l) = \sum_{d\mid(k,l)} d\phi\left(\frac{k}{d}\right).$$

We have therefore

(11) $$V(x, N) = \frac{\Phi_k(l)}{k^2} \frac{e^{2\pi i x\sqrt{N}}}{\pi i x} N^{\frac{1}{2}} \log N + P_0(x, N) + O_x(N^{\frac{1}{2}})$$

with

$$P_0(x, N) = 2 \sum_{d|(k,l)} \sum_{\substack{1 \leq a \leq \sqrt{N}/d \\ (a, k_d)=1}} P_d(a).$$

We note that the most delicate and difficult point in the derivation of (11) is in the proof of

(12) $$\sum_{d|(k,l)} \sum_{\substack{1 \leq a \leq \sqrt{N}/d \\ (a,k_d)=1 \\ a \notin A_d}} Q\left(x(ad)^{\frac{1}{2}}, \frac{N}{ad}\right) = O_x(N^{\frac{1}{2}}),$$

where A_d denotes, for each $d|(k, l)$, the set of those integers a with $1 \leq a \leq \sqrt{N}/d$, $(a, k_d) = 1$, which lie within the distance $N^{\frac{1}{4}}$ from some point $(2N^{\frac{1}{2}}m)/(kx)$ with $1 \leq m \leq (kx)/(2d)$; indeed, the proof of (12) can be carried out along the same lines just as in [4] on the assumption that $x \geq 4k^3$. We replace, as we may, the second double sum on the right side of (9) by the sum on the left side of (11), thus obtaining (11).

3. We are now in a position to evaluate the sum

$$P_0(x, N) = 2 \sum_{d|(k,l)} \sum_{\substack{1 \leq a \leq \sqrt{N}/d \\ (a, k_d)=1}} P_d(a),$$

where

$$\sum_{\substack{1 \leq a \leq \sqrt{N}/d \\ (a, k_d)=1}} P_d(a) = \frac{(1-i)k^{\frac{1}{2}}x}{2} \sum^*_{0<m\leq \frac{kx}{2d}} \frac{1}{m^{\frac{3}{2}}}$$

$$\sum^*_{\substack{0<a\leq \frac{2\sqrt{N}}{kx}m \\ (a, k_d)=1}} a^{\frac{1}{2}} \exp 2\pi i\left(\frac{kx^2 a}{4m} + \frac{ml\bar{a}}{k}\right)$$

with $\bar{a}$ defined modulo k_d.

Suppose first that $x \neq (2q^{\frac{1}{2}})/k$ for any integer q. Then, since

$$\exp\left(2\pi i \frac{kx^2}{4m}\right) \neq 1 \quad \text{for all } m,$$

we have for any integer c, $(c, k_d) = 1$,

$$\sum_{\substack{1 \leq a \leq M \\ a \equiv c \pmod{k_d}}} \exp 2\pi i \left(\frac{kx^2 a}{4m} + \frac{ml\bar{a}}{k} \right)$$

$$= \exp\left(2\pi i \frac{ml\bar{c}}{k}\right) \sum_{\substack{1 \leq a \leq M \\ a \equiv c \pmod{k_d}}} \exp\left(2\pi i \frac{kx^2 a}{4m}\right) = O_x(1)$$

uniformly in M and in m, $1 \leqq m \leqq (kx)/(2d)$, $d \mid (k, l)$. It follows from this by partial summation that

$$\sum\nolimits^*_{0<m\leq \frac{kx}{2d}} \frac{1}{m^{\frac{3}{2}}} \sum\nolimits^*_{0<a\leq \frac{2\sqrt{N}}{kx}m} a^{\frac{1}{2}} \exp 2\pi i \left(\frac{kx^2 a}{4m} + \frac{ml\bar{a}}{k} \right) = O_x(N^{\frac{1}{4}}),$$

whence

$$P_0(x, N) = O_x(N^{\frac{1}{4}})$$

in this case.

Next suppose that x is of the form $(2q^{\frac{1}{2}})/k$, where q is an integer. Let d be a divisor of (k, l). If $d \nmid q$ then

$$\exp\left(2\pi i \frac{kk_d x^2}{4m}\right) = 1 \quad \text{for all } m,$$

so that we have for any integer c with $(c, k_d) = 1$

$$\sum_{\substack{1 \leq a \leq M \\ a \equiv n \pmod{k_d}}} \exp 2\pi i \left(\frac{kx^2 a}{4m} + \frac{ml\bar{a}}{k} \right) = O_x(1)$$

uniformly in M and in m, $1 \leqq m \leqq (kx)/(2d)$. Hence, we find as before

$$\sum\nolimits^*_{0<m\leq \frac{kx}{2d}} \frac{1}{m^{\frac{3}{2}}} \sum\nolimits^*_{0<a\leq \frac{2\sqrt{N}}{kx}m} a^{\frac{1}{2}} \exp 2\pi i \left(\frac{kx^2 a}{4m} + \frac{ml\bar{a}}{k} \right) = O_x(N^{\frac{1}{4}})$$

for such d. The same is also true for $d \mid q$, provided that the range of m is restricted to $1 \leqq m \leqq (kx)/(2d)$, $m \nmid q/d$.

On the other hand, if $d \mid q$ and $m \mid q/d$ (i. e. if $dm \mid q$), then

$$\sum\nolimits^*_{\substack{0<a\leq \frac{2\sqrt{N}}{kx}m \\ (a, k_d)=1}} a^{\frac{1}{2}} \exp 2\pi i \left(\frac{kx^2 a}{4m} + \frac{ml\bar{a}}{k} \right)$$

$$= \sum_{\substack{c \bmod k_d \\ (c,\, k_d)=1}} \exp 2\pi i \left(\frac{qc}{mk} + \frac{ml\bar{c}}{k}\right) \sum^*_{0<a\leq\sqrt{(N/q)m}} a^{\frac{1}{2}}$$

$$= S\left(\frac{k}{d};\ \frac{q}{dm},\ \frac{ml}{d}\right)\left\{\frac{2}{3k_d}\left(\left(\frac{N}{q}\right)^{\frac{1}{2}}m\right)^{\frac{3}{2}} + O\left(\left(\left(\frac{N}{q}\right)^{\frac{1}{2}}m\right)^{\frac{1}{2}}\right)\right\},$$

where $S(k;\ u,\ v)$ is the Kloosterman sum defined in (5). It follows that

$$\sum^*_{\substack{0<m\leq\frac{kx}{2d} \\ m|q/d}} \frac{1}{m^{\frac{3}{2}}} \sum^*_{0<a\leq\frac{2\sqrt{N}}{kx}m} a^{\frac{1}{2}} \exp 2\pi i\left(\frac{kx^2a}{4m} + \frac{ml\bar{a}}{k}\right)$$

$$= \frac{2N^{\frac{3}{4}}}{3k_dq^{\frac{3}{4}}} \sum_{\substack{0<m\leq\sqrt{q}/d \\ m|q/d}} S\left(\frac{k}{d};\ \frac{q}{dm},\ \frac{ml}{d}\right) + O_x(N^{\frac{1}{4}})$$

Hence

$$P_0\left(\frac{2q^{\frac{1}{2}}}{k},\ N\right) = \frac{2(1-i)\sigma(q\,;k,\ l)}{3k^{\frac{3}{2}}\,q^{\frac{1}{4}}}\,N^{\frac{3}{4}} + O_x(N^{\frac{1}{4}}),$$

where

(13) $$\sigma(q\,;k,\ l) = 2\sum_{d|D} d \sum^*_{\substack{0<m\leq\sqrt{q}/d \\ m|q/d}} S\left(\frac{k}{d};\ \frac{q}{dm},\ \frac{ml}{d}\right).$$

with $D=(q,\ k,\ l)$.

We thus have established the following

Theorem 1. *We have for* $N\to\infty$

$$V(x,\ N) = \frac{\Phi_k(l)}{k^2}\frac{e^{2i\pi x\sqrt{N}}}{\pi ix}N^{\frac{1}{2}}\log N + O_x(N^{\frac{1}{2}})$$

if $x\neq(2q^{\frac{1}{2}})/k$ *for any integer* q, *and*

$$V(x,\ N) = \frac{2(1-i)\sigma(q\,;\ k,\ l)}{3k^{\frac{3}{2}}\,q^{\frac{1}{4}}}N^{\frac{3}{4}} + \frac{\Phi_k(l)}{k^2}\frac{e^{2\pi ix\sqrt{N}}}{\pi ix}N^{\frac{1}{2}}\log N + O_x(N^{\frac{1}{2}})$$

if $x=(2q^{\frac{1}{2}})/k$ *for some integer* q, *provided that* $x\geqq 4k^3$. *The quanti-*

ties $\Phi_k(l)$ and $\sigma(q\,;\,k,\,l)$ *are the ones defined respectively by* (10) *and* (13).

4. Again, let k and l be fixed integers with $k\geqq 1$, $0\leqq l<k$. It is a simple task to deduce, by partial summation, the following result from Theorem 1.

Theorem 2. *If* $x\neq(2q^{\frac{1}{2}})/k$ *for any integer* q, *then*

$$U(x,\ N)=O_x(\log N),$$

and if $x=(2q^{\frac{1}{2}})/k$ *for some integer* q, *then*

$$U(x,\ N)=\frac{2(1-i)\sigma(q\,;\,k,\,l)}{k^{\frac{3}{2}}q^{\frac{1}{4}}}N^{\frac{1}{4}}+O_x(\log N),$$

provided that $x\geqq 4k^3$.

We now define

$$W(x,\ N)=W(x,\ N;\ k,\ l)=\sum_{\substack{1\leq n\leq N\\ n\equiv l(\mathrm{mod}\ k)}}n^{\frac{1}{4}}d(n)e^{2i\pi\sqrt{n}}.$$

Then, there holds the following

Theorem 3. *If* $x\neq(2q^{\frac{1}{2}})/k$ *for any integer* q, *then*

$$W(x,\ N)=\frac{\Phi_k(l)}{k^2}\frac{e^{2\pi ix\sqrt{N}}}{\pi ix}N\log N+O_x(N),$$

whereas if $x=(2q^{\frac{1}{2}})/k$ *for some integer* q, *then*

$$W(x,\ N)=\frac{2(1-i)\sigma(q\,;\,k,\,l)}{5k^{\frac{3}{2}}q^{\frac{1}{4}}}N^{\frac{5}{4}}+\frac{\Phi_k(l)}{k^2}\frac{e^{2\pi ix\sqrt{N}}}{\pi ix}N\log N+O_x(N)$$

provided that $x\geqq 4k^3$.

This follows again from Theorem 1 by partial summation.

Taking into account of Theorem 3, we can argue just as in [4] to show that neither of the O-terms in the formulae in Theorem 2 can be replaced by $o(\log N)$.

5. In conclusion we should like to examine some properties of the

quantity $\sigma(q\,;\,k,\,l)$ defined in (13), where $k \geqq 1$, $0 \leqq l < k$, and $q \geqq 1$. At first glance the definition (13) seems to be rather unsymmetrical in respect of the inner sum therein contained. However, in certain special but important cases we can express $\sigma\,(q\,;\,k,\,l)$ in a concise form. Indeed, if $(k,\,l) = 1$ and hence $D = (q,\,k,\,l) = 1$, then

$$\sigma\,(q\,;\,k,\,l) = 2 \sum_{\substack{0<m\leqq\sqrt{q} \\ m\mid q}}^{*} S\Big(k\,;\,\frac{q}{m},\,ml\Big)$$

$$= 2 \sum_{\substack{0<m\leqq\sqrt{q} \\ m\mid q}}^{*} S\Big(k\,;\,m,\,\frac{q}{m}\,l\Big)$$

$$= \sum_{m\mid q} S\Big(k\,;\,\frac{q}{m},\,ml\Big),$$

whereas if $(q,\,k) = 1$ then we have $D = (q,\,k,\,l) = 1$ again and

$$\sigma(q\,;\,k,\,l) = 2 \sum_{\substack{0<m\leqq\sqrt{q} \\ m\mid q}}^{*} S\Big(k\,;\,\frac{q}{m},\,ml\Big)$$

$$= S(k\,;\,1,\,ql)\cdot 2 \sum_{\substack{0<m\leqq\sqrt{q} \\ m\mid q}}^{*} 1$$

$$= S(k\,;\,q,\,l)\,d(q)\,.$$

Since every Kloosterman sum has a real value, our $\sigma(q\,;\,k,\,l)$ composed thereof by (13) has always a real value, too.

We have $\sigma(q\,;\,k,\,l) \neq 0$ in general. However, for certain particular values of k, l and q $\sigma(q\,;\,k,\,l)$ may vanish, of course. Thus we have, for instance,

$$\sigma(q\,;\,2^r,\,1) = 0$$

if either $q \equiv 3 \pmod 4$ and $r \geqq 4$, or $q \equiv 5 \pmod 8$ and $r \geqq 6$.

Various properties of $\sigma(q\,;\,k,\,l)$ can be derived from the properties of Kloosterman sums, for which one may refer to an extensive study [2] by H. Salié.

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