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SOME REMARKS ON π -REGULAR RINGS OF BOUNDED INDEX

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Recently, G. Azumaya introduced the concepts of right, left, and strong π -regularities of elements in a ring and obtained a sufficient condition that a right (left) π -regular element is strongly π -regular. Previously, R. Arens and I. Kaplansky ([1])¹⁾ studied rings in which all elements are right π -regular, and one of them continued his consideration in [6]. In this note, we shall prove several additional properties of π -regular rings of bounded index. As is well-known, Neumann's regularity is preserved under the construction of the complete matrix ring ([3, Lemma 2]). But the complete matrix ring over a strongly regular ring is not strongly regular except trivial cases. Our principal aim is to show the following fact: The $r \times r$ complete matrix ring over a π -regular ring of bounded index is also π -regular and of bounded index.

§1 is preparations of subsequent sections and contains some definitions and fundamental results without proofs. §2 deals with nil-rings of bounded index, and in §3 our principal theorem will be shown. In §4 we shall consider a ring of bounded index, and under this assumption several properties of the unique maximal π -regular ideal which are similar to those of the unique maximal regular ideal as in [3] will be investigated.

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1. Definitions and fundamental results. Let a be an element of a ring A . If there exists an element x such that $axa = a$, we say that a is *regular*. If there exists x such that $a^2x = a(xa^2 = a)$, a is said to be *right (left) regular*, and in case a is right as well as left regular a is *strongly regular*. A *regular (right regular, left regular, strongly regular) ring* will mean a ring in which all elements are regular (right regular, left regular, strongly regular)²⁾.

Now we introduce the following definitions. Let a be an element of a ring A . If some power of a is regular (right regular, left regular,

1) Numbers in brackets refer to the references cited at the end of this paper.

2) Cf. [10].

strongly regular), then we say that a is π -regular (*right π -regular, left π -regular, strongly π -regular*). And a π -regular (*right π -regular, left π -regular, strongly π -regular*) ring will be defined in the obvious way. A two-sided ideal in A is said to be a π -regular (*strongly π -regular*) ideal if each element of the ideal is π -regular (strongly π -regular).

As is easily verified, a is right (left) π -regular if and only if there exist an element x and an integer n such that $a^{n+1}x = a^n(xa^{n+1} = a^n)$. For any strongly π -regular element a of a ring A , the least integer n for which there holds $a^{n+1}x = a^n$ with some x is called the π -index of a . And the least upper bound of all π -indices of strongly π -regular elements is denoted as the π -index of A . On the other hand, the least upper bound of all indices of nilpotent elements of A is called the index of A . And in case the index of A is finite we say that the ring is of bounded index. Every nilpotent element is strongly π -regular, and moreover the index coincides with the π -index for such element.

G. Azumaya proved the following theorem¹⁾:

Theorem 1. *Let A be a ring of bounded index. If a is right (left) π -regular then it is strongly π -regular. And moreover, there exists an element x such that $ax = xa$ and $a^{n+1}x = a^n$, where n is the index of A .*

The proof was completed by making use of the elementary method, and as corollaries following results are obtained.

Corollary 1. *For any ring A the π -index of A coincides with the index of A .*

Corollary 2. *A right (left) π -regular element in a ring without nonzero nilpotent elements is strongly regular.²⁾*

In §§2-4, we shall restrict our attentions to the case of bounded index. Hence, in this case, right π -regularity, left π -regularity, and strong π -regularity are equivalent to each other. Further, the following theorem was given in [2] ([2, Theorem 5]):

Theorem 2. *Under the assumption that A is of bounded index, the following four conditions are equivalent to each other:*

- i) A is π -regular,
- ii) A is right π -regular,
- iii) A is left π -regular,

1) See [2].

2) Cf. [4] or [6, p. 7].

iv) A is strongly π -regular.

The next theorem proved by R. Arens and I. Kaplansky is fundamental ([1, Theorem 3.1]):

Theorem 3. *Let A be a ring in which for any a there exists an x such that $a^{n+1}x = a^n$ (with n independent of a). Then the radical¹⁾ of A is a nil-ideal with index of nilpotency at most n . If A is primitive, it is a matrix ring of degree at most n over a division ring.*

An extremal example of π -regular rings of bounded index is a nil-ring of bounded index, which will be considered in the next section.

2. Nil-rings of bounded index. It is the well-known result of Levitzki that a nil-ring of bounded index is semi-nilpotent, or equivalently, a finitely generated nil-ring of bounded index is nilpotent ([7, p. 1035]). From this fact we shall readily obtain the following theorem:

Theorem 4. *Let A be a nil-ring of bounded index generated by $a_1, \dots, a_g$. Then $A^{f(n, g)} = 0$, where n is the index of A and $f(n, g)$ is a positive integer depending solely on n and g .*

Although the proof by making use of Levitzki's result is easy (see Remark below), we shall, for caution's sake, give here a direct proof.

Proof. As $a^n = 0$ for all a in A , there holds $\sum_{(i_1, \dots, i_n)} x_{i_1} \cdots x_{i_n} = 0$ for any $x_1, \dots, x_n$ in A , where $(i_1, \dots, i_n)$ runs over all permutations of $1, \dots, n^2$.

Let a be an arbitrary element and let m be an integer greater than $[n/2]$. We consider here the $2m+1$ subrings A_i which are defined as follows³⁾:

$$\begin{aligned} A_{2j-1} &= a^{m-j+1}Ra^{j-1} & (j = 1, 2, \dots, m+1), \\ A_{2j} &= a^{m-j+1}Ra^j & (j = 1, 2, \dots, m), \end{aligned}$$

To be easily verified, if $s > t$ then $A_s A_t \subseteq Ra^{m+1}R$. Hence we obtain $A_{i_1} A_{i_2} \cdots A_{i_n} \subseteq Ra^{m+1}R$ for each $(i_1, \dots, i_n) \neq (1, \dots, n)$. Accordingly, by the remark stated at the beginning of this proof, we have $A_1 A_2 \cdots A_n \subseteq Ra^{m+1}R$, whence $\{(a^m)\}^{n+1} \subseteq Ra^{m+1}R$, where (a^m) de-

1) The terms "radical", "primitive ideals", and "semi-simple" will be in the sense of Jacobson [5].

2) Cf. [8, Lemma 3].

3) Cf. [8, Proof of Theorem 1].

notes the two-sided ideal generated by a^m . Hence if $m = [n/2] + 1$, then $\{(a^m)\}^{(n+1)^m} = 0$.

For $n = 1$ our assertion is clear. For $n = 2$, as $xy = -yx$ for any x, y in A , we can take $g + 1$ for $f(2, g)$.

Let $n > 2$ and assume the validity of our assertion for $m < n$. Now we set $m = [n/2] + 1$. Let F be the ring freely generated by g non-commutative indeterminates $x_1, \dots, x_g$, and let I_F be the two-sided ideal in F generated by all m -th powers of the elements of F . As the quotient ring F/I_F is a nil-ring of bounded index $m(< n)$ generated by g elements, by our induction hypothesis $F^{f(m, g)} \subseteq I_F$. We consider here the set $G_F = \{x_{i_1} \cdots x_{i_{f(m, g)}} \mid x_{i_j} \text{ is some } x_k (k = 1, \dots, g)\}$, and denote by $p(m, g)$ the least number of ideals of the form (y^m) such that the ideal sum of them contains G_F . As A is canonically homomorphic to F , the set $G = \{a_{i_1} \cdots a_{i_{f(m, g)}} \mid a_{i_j} \text{ is some } a_k (k = 1, \dots, g)\}$ is contained in some ideal B in A which can be represented as the sum of at most $p(m, g)$ ideals of the form (b^m) . Hence, we obtain $B^{p(m, g) \cdot (n+1)^m} = 0$, whence $a_{i_1} \cdots a_{i_{f(n, g)}} = 0$, where a_{i_j} is some a_k and $f(n, g) = f(m, g) \cdot p(m, g) \cdot (n+1)^m$. This completes our induction.

Corollary. *Let A be a nil-ring of bounded index. Then the $r \times r$ complete matrix ring $(A)_r$ over A is a nil-ring of bounded index too.*

Proof. Let $a^n = 0$ for all a in A . For any element $\alpha = [a_{ij}]$ in $(A)_r$, we consider the subring B generated by a_{ij} ($i, j = 1, \dots, r$). Then, by Theorem 4, $B^{f(n, r^2)} = 0$. Our assertion follows from this fact at once.

Remark. In Theorem 4, A is a homomorphic image of F/I_F (n in place of m). Hence we can take, by Levitzki's result, the nilpotency index of F/I_F for $f(n, g)$.

In case A is a nil-algebra of bounded index n over a field K of characteristic 0 (not necessarily with a finite generating system), M. Nagata proposed that $A^{f(n)} = 0$ with some integer $f(n)$ depending solely on n . On the other hand, in case the characteristic of K is $p \neq 0$, he presented a non-nilpotent, nil-algebra of bounded index ([9]).

3. Matrix rings over π -regular rings of bounded index. A π -regular ring A is *homogeneous* if it is semi-simple and such that for every primitive ideal P , A/P is an $n \times n$ complete matrix ring over a division ring (n independent of P)¹⁾.

1) See [6, p. 67] and [6, Theorem 2.3].

In Theorem 4.2 of [6], I. Kaplansky announced that a homogeneous π -regular ring with an identity is a matrix ring over a strongly regular ring. The converse of this fact is also true, but we have more generally the following:

Lemma 1. *Let A be regular and of bounded index. Then the $r \times r$ complete matrix ring $(A)_r$ is also regular and of bounded index.*

Proof. As $(A)_r$ is regular ([3, Lemma 2]), it only remains to prove that $(A)_r$ is of bounded index. Let P be an arbitrary primitive ideal in A . By [6, Theorem 2.3] (or by Theorems 2 and 3), $(A)_r/(P)_r \cong (D)_{mr}$, where D is a division ring and $m \leq n =$ the index of A . $(A)_r$ is a subdirect sum of complete matrix rings of degree at most nr over division rings, for $(A)_r = (A)_r / \cap (P)_r$, where P runs over all primitive ideals in A . Hence $(A)_r$ is of bounded index.

Lemma 2. *If a is right π -regular modulo a π -regular ideal I of bounded index, then a is actually right π -regular. In particular, if A/I is π -regular and of bounded index, then A is π -regular and of bounded index too.*

Proof. Let m be the index of I . Then, for some element x, y and for some integer n , we obtain $(a^{mn+1}x - a^n)^{m+1}y = (a^{mn+1} - a^n)^m$. Hence $a^{mn+1}z = a^{mn}$ with some z .

Lemma 3. *If A is a homogeneous π -regular ring of bounded index, then so is the $r \times r$ complete matrix ring $(A)_r$.*

Proof. Let Z be the center of A . Then AZ is regular (and is of bounded index) and A/AZ is a nil-ring of bounded index ([6, Theorem 4.3]). Hence $(AZ)_r$ and $(A/AZ)_r$ are π -regular and of bounded index by Lemma 1 and Corollary to Theorem 4 respectively. As $(A)_r/(AZ)_r \cong (A/AZ)_r$, our assertion is an immediate consequence of Lemma 2.

Corollary. *If A is a semi-simple π -regular ring of bounded index, then so is $(A)_r$.*

Proof. Let $I_\alpha = \cap P$, where P runs over all primitive ideals such that A/P is isomorphic to some $m \times m$ complete matrix ring over a division ring with $m \leq \alpha$. If the index of A is n , then it is clear that $I_n = 0$. We may prove our assertion by making use of the induction with respect to the number i such that $I_i \neq 0$ and $I_{i+1} = 0$.

If $i = 0$, A is strongly regular. Hence our assertion is true. We assume now that it is true for $j < i$. As, to be easily seen, I_i is homogeneous¹⁾, by Lemma 3, $(I_i)_r$ is π -regular and of bounded index.

1) See [6, Theorem 3.1 and p. 67].

On the other hand, by the induction hypothesis, $(A/I_i)_r$ is also π -regular and of bounded index. Hence our induction is completed by means of Lemma 2.

Now we can prove readily the following principal theorem:

Theorem 5. *The $r \times r$ complete matrix ring $(A)_r$ over a π -regular ring of bounded index is π -regular and of bounded index too.*

Proof. Let N be the radical of A . Since $(A)_r/(N)_r \cong (A/N)_r$, our theorem is an immediate consequence of Theorem 3, Corollary to Theorem 4, and Corollary to Lemma 3.

4. π -regular ideals in a ring of bounded index. We begin this section with the following lemma.

Lemma 4. *Let A be a ring of bounded index. Then there exists the unique maximal (strongly) π -regular ideal.*

Proof. We set $\Pi(A) = \{a \in A \mid (a) \text{ is } \pi\text{-regular}\}$. Clearly, $\Pi(A)$ is closed under the right as well as left multiplications by the elements of A . Now, if a_1, a_2 are in $\Pi(A)$, then $(a_1 - a_2) \subseteq (a_1) + (a_2)$. Hence, for any element $a \in (a_1 - a_2)$, we have $a = u_1 + u_2$, where $u_i \in (a_i)$ ($i = 1, 2$). As $u_1^{n+1}r = u_1^n$ with some $r \in A$ and some integer n , it follows that $a^{n+1}r - a^n = (u_1 + u_2)^{n+1}r - (u_1 + u_2)^n = u_1^{n+1}r - u_1^n + u_2^* = u_2^*$ with some $u_2^* \in (a_2)$. By Lemma 2, a is π -regular. Thus $\Pi(A)$ defined above is a two-sided ideal.

The next properties of $\Pi(A)$ are similar to those of $M(A)$ in Theorems 2, 3, and 4 of [3].

Theorem 6. *Let A be a ring of bounded index. Then*

- i) $\Pi(A/\Pi(A)) = 0$.
- ii) *If I is a two-sided ideal in A , then $\Pi(I) = I \cap \Pi(A)$.*
- iii) *Let $(A)_r$ be the $r \times r$ complete matrix ring over A , then $(\Pi(A))_r$ is the unique maximal strongly π -regular ideal.*

Proof. i) It is easy to see that $A/\Pi(A)$ is of bounded index (Corollary 1 to Theorem 1 and Lemma 2). Let $\bar{b} \in \Pi(A/\Pi(A))$, where $\bar{b}$ denotes the residue class modulo $\Pi(A)$ containing b . For each element a in $(\bar{b})$, there holds that $\bar{a}^{n+1}\bar{x} = \bar{a}^n$ with some x and n . Hence $a^{n+1}x - a^n \in \Pi(A)$. By Lemma 2, a is π -regular, whence b is contained in $\Pi(A)$.

ii) Let $a \in \Pi(I)$ and $b \in (a)$. As is easily verified by a brief computation, $b^3 \in (a)'$, where $(a)'$ is the two-sided ideal generated by a in I . Hence b^3 is π -regular, accordingly b is so. Thus we have proved that $\Pi(I) \subseteq I \cap \Pi(A)$. The converse relation is clear.

iii) We consider here the ring $A^* = (1) + A$, which consists of all pairs (n, a) with integers n and a in A , and in which the addition and the multiplication are defined as follows: $(n, a) + (n', a') = (n + n', a + a')$, $(n, a)(n', a') = (nn', na' + n'a + aa')$. Let ε_{ij} be a matrix in $(A^*)_r$ with an identity in the (i, j) position and zeros elsewhere. If $\alpha = [a_{ij}]$ is a matrix in $(A)_r$, then $\alpha_{ij} = \varepsilon_{it} \alpha \varepsilon_{jt}$ is a matrix with a_{ij} in the $(1, 1)$ position and zeros elsewhere, and it is in $(\alpha)^*$, where $(\alpha)^*$ denotes the two-sided ideal generated by α in $(A^*)_r$. For any element $a' \in (a_{ij})$, let α' be a matrix with a' in the $(1, 1)$ position and zeros elsewhere. Then α' also belongs to $(\alpha)^*$. As is easily seen, α'^3 is in (α) . If (α) is a strongly π -regular ideal, then there exists a matrix $\beta = [b_{ij}] \in (A)_r$ such that $\alpha'^{3(n+1)} \cdot \beta = \alpha'^{3n}$. But this implies that $\alpha'^{3n+3} \cdot b_{11} = \alpha'^{3n}$, and hence a' is π -regular, that is, (a_{ij}) is a π -regular ideal. Hence, $(A)_r$ possesses the unique maximal strongly π -regular ideal, which coincides with $(\Pi(A))_r$ by Theorem 5.

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1) We consider here A as a subring of A^* .