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ON GALOIS THEORY OF DIVISION RINGS

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Although several generalizations of Galois theory for fields have been undertaken for division rings and other rings under some finiteness assumptions ([2], [3], [10]), [1] there are few papers concerning non-commutative Galois theory for infinite cases. Recently N. Nobusawa has succeeded in extending Krull's Galois theory to division rings ([11], [12]). In his consideration, the principal assumption is the local finiteness of the total group, which is equivalent to that the total group is almost outer and the division ring considered is locally finite over its fixed subring ([9], [12]). For the outer case, the present authors hear that N. Jacobson has prepared his theory in his forthcoming book. But, as is easily seen, Nobusawa's theory is not yet satisfactory, because it does not completely contain the case of finite degree. One of the purposes of this paper is to generalize the theory in such a way that the generalization contains the theory of Nobusawa as well as that of the case of finite degree. Our generalization, which will be stated in §3, stands on the view point that, in either of above cases, the centralizer of the fixed subring is finite over the center, (or what is the same, the total group is locally compact.) And we can say roughly that our generalization is reduced to the outer case. Another of the purposes is to present some structural consequences with respect to local finite-dimensionality, which will be seen in §2. §4 contains some examples of Galois extensions.

Now we wish to begin our course with §**1**, which contains fundamental definitions, preliminary results and notations used frequently in this paper.

1. Preliminaries.

Notations in this section will be used throughout the paper.

Let K be a division ring, L be a division subring of K. K is *locally finite* over L when, for each finite subset S of K, $L(S)$ (the least division subring of K containing L and S) is finite over L (as a left L-module). Further, if K is locally finite over any intermediate division subring, then it is said to be *totally locally finite* over L. Let $\mathfrak{G}$ be a group of automorphisms of K. We denote by $\mathfrak{G}(L)$ the set of all L-automorphisms

1) Numbers in brackets refer to the references cited at the end of this paper.

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in $\mathfrak{G}$. If $\mathfrak{G}$ contains only a finite number of inner automorphisms [no inner automorphisms except the identity mapping], it will be said to be *almost outer* [*outer*]. Now we denote by $J(\mathfrak{G}, K)$[1] the fixed subring of $\mathfrak{G}$ in K, and we say that K is *Galois* over $J(\mathfrak{G}, K)$. In general, if there exists a group $\mathfrak{G}$ of automorphisms of K such that $L = J(\mathfrak{G}, K)$, then K is *Galois* over L (or K/L is Galois), and the totality $\mathfrak{G}(K/L)$ of all L-automorphisms of K is called the *total group* of K/L. Further a subgroup $\mathfrak{H}$ of $\mathfrak{G}(K/L)$ is called a *total subgroup* if $\mathfrak{H}$ is the total group of $K/J(\mathfrak{H}, K)$.

In case a group $\mathfrak{R}$ of automorphisms of K contains all $J(\mathfrak{R}, K)$-inner automorphisms, $\mathfrak{R}$ is called a ***regular group*** (of $K/J(\mathfrak{R}, K)$). *If the dimension $[K : J(\mathfrak{R}, K)]$ is finite, where the dimension will mean the left dimension, then the regular group $\mathfrak{R}$ is the total group of $K/J(\mathfrak{R}, K)$* ([11]).

When K is Galois over L, a division subring N containing L is said to be *normal* with respect to $\mathfrak{G}(K/L)$ (if there exists no confusion, we say also that N is normal over L) if each σ in $\mathfrak{G}(K/L)$ leaves N setwise invariant: $N^\sigma = N$. In case K/L is Galois, the total group $\mathfrak{G} = \mathfrak{G}(K/L)$ is said to be *locally finite-dimensional* (abbreviated, l. f. d.) [*locally finite*] if $L(S^{\mathfrak{G}})$[2] is finite over L [the set $S^{\mathfrak{G}}$ is finite] for each finite subset S of K. As is noted in [9] and [12], *$\mathfrak{G}$ is l. f. d. and K is totally locally finite over L when $\mathfrak{G}$ is locally finite.* If $\mathfrak{G}$ is l. f. d. then we can introduce a Hausdorff topology in it by making use of the same method as in Nobusawa's theory, and then $\mathfrak{G}$ becomes a topological group[3]. In particular, *if $\mathfrak{G}$ is locally finite, it is compact* ([11]).

We insert here the principal results of Nobusawa which will be required in the sequel ([11], [12]):

Let K be Galois over L and $\mathfrak{G} = \mathfrak{G}(K/L)$ be locally finite. Then there hold the following:

(i) $J(\mathfrak{G}(K'), K) = K'$ *for any intermediate subring K'.*

(ii) *If σ is an L-isomorphism of any intermediate subring K' into K, then σ can be extended to an automorphism in $\mathfrak{G}$.*

At last we shall gather here several notations used frequently in the sequel, where K be a division ring, L a subring of K, M, N and S be subsets of K, and $\mathfrak{M}$ be a set of mappings of K:

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1) This notation will be used for any set of automorphisms $\mathfrak{G}$ too.

2) $S^{\mathfrak{G}}$ denotes the set consisting of all images of S by $\mathfrak{G}$.

3) Here a fundamental system of neighbourhoods of the identity is defined as the totality of $\mathfrak{G}(N)$, where N are subrings which are finite and normal over L.

$L(M)$: the division subring generated by L and M.
$M\backslash N$: the complement of N in M.
$\tilde{M}$: the totality of inner automorphisms determined by non-zero (regular) elements of M.
$\tilde{k}$: the inner automorphism of K determined by a non-zero (regular) element k of K.
$V_K(M)$: the centralizer of M in K: $V_K(M) = \{k \in K;\ kx = xk$ for all $x \in M\}$.
$\mathfrak{M}_S$: the restriction of $\mathfrak{M}$ on S.
σ_S : the restriction of σ on S, where σ is a mapping of K.

2. Local finiteness and local finite-dimensionality.

Throughout this section we assume that K is a division ring which is Galois over a division subring L. And we denote the total group of K/L by $\mathfrak{G}$, which should be considered as a topological group in the sense of §1 whenever it is l. f. d.

Theorem 1. *Let $\mathfrak{G}$ be the total group of K/L which is l. f. d.*
(i) *The following conditions are equivalent to each other:*
(1) *$\mathfrak{G}$ is compact.*
(2) *$\mathfrak{G}$ is locally finite.*
(3) *$\mathfrak{G}$ is almost outer.*
(ii) *The following conditions are equivalent to each other:*
(1) *$\mathfrak{G}$ is discrete.*
(2) *K is finite over L.*

Proof. (i) It has been already proved in [9] and [12] that (2) is equivalent to (3). And by Nobusawa [11], (2) implies (1). Now we shall prove that (1) implies (2). We set here $S = L(\{a\}^{\mathfrak{G}})$, $a \in K$, which is normal and finite over L by our assumption, then by definition, $\mathfrak{G}(S)$ is an open invariant subgroup of $\mathfrak{G}$, and so the quotient group $\mathfrak{G}/\mathfrak{G}(S)$ is finite. Hence $\mathfrak{G}_S$ is finite, that is, the set $\{a\}^{\mathfrak{G}}$ is finite.

(ii) Let $\mathfrak{G}$ be discrete, then there exists a division subring N which is normal and finite over L and for which $\mathfrak{G}(N)$ consists of only the identity mapping. If $K \supsetneqq N$, then there exists x such that $x \in K\backslash N$. Let N' be the least normal subring containing N and x, then $[N' : L] < \infty$. We obtain therefore $\mathfrak{G}(N'/L) = \mathfrak{G}_{N'}$, and there exists a one-to-one dual correspondence between regular subgroups of $\mathfrak{G}_{N'}$ and subrings of N' containing L([2], [5], [11]). From the assumption that $\mathfrak{G}(N)$ consists of only the identity, it follows that $N' = N$. This contradiction shows that $K = N$. The converse part is almost trivial.

Remark 1. For (i) in Theorem 1, we have obtained in [9] the following refinement:
(i) *If* C, *the center of* K, *is infinite then* $\mathfrak{G}$ *is outer.*
(ii) *If* $\mathfrak{G}$ *is non-outer then* $V_K(L)$ *is finite.*

Let $\mathfrak{H}$ be a regular group of K/L and H an intermediate division subring. Now we denote by $\mathfrak{M}_K(H)$ the H_r-K_r-module of all L-linear transformations of the left L-module H into K, where K_r means the totality of right multiplications by elements of K. The following lemma will be easily proved (cf. [12]).

Lemma 1. *If* $[H:L] = n < \infty$ *then* $[\mathfrak{M}_K(H):K_r] = n$, *accordingly* $\mathfrak{H}_H K_r = \sum_{i=1}^{m} \sigma_H^{(i)} K_r$[1] *with some* $\sigma^{(i)}$*'s in* $\mathfrak{H}$. *In particular, for each* L-*(ring-) isomorphism* σ *in* $(\mathfrak{H}K_r)_H$, *there holds that* $\sigma = (\sigma^{(j)}\iota)_H$ *with some* j *and some* L-*inner automorphism* ι.

By our assumption that $\mathfrak{H}$ is a regular group of K/L, it follows that the set of all $\mathfrak{H}K_r$-endomorphisms of K coincides with L_l, the totality of left multiplications by elements of L. As K is irreducible with respect to $\mathfrak{H}K_r$, by Jacobson's theorem ([4]), $\mathfrak{H}K_r$ is dense in $\mathfrak{M}_K(K)$ with respect to the so-called finite topology, that is, $\overline{\mathfrak{H}K_r} = \mathfrak{M}_K(K)$. If $[H:L] < \infty$, then $\mathfrak{M}_K(H)$ (the totality of L-linear transformations of H into K) is $(\mathfrak{M}_K(K))_H = (\overline{\mathfrak{H}K_r})_H = (\mathfrak{H}K_r)_H$. Hence, by Lemma 1, any L-isomorphism σ of H into K can be extended to an element of $\mathfrak{H}$. We state here this fact as a theorem, which will appear in the forthcoming book of Jacobson.

Theorem 2 (Jacobson). *Let* $\mathfrak{H}$ *be a regular group of* K/L, H *be an intermediate division subring with* $[H:L] < \infty$. *If* σ *is an* L-*isomorphism of* H *into* K *then* σ *is in* $\mathfrak{H}_H$.

The next is a generalization of Cartan's theorem.

Lemma 2.[2] *Let* R, S *be division subrings of a division ring* D. *If each inner automorphism determined by an element of* S *leaves* R *set-wise invariant, then either* $R \supset S$ *or* $R \subset V_D(S)$.

The proof is a slight modification of that of Theorem in [1], however,

1) Further, we shall obtain $\mathfrak{M}_K(H) = \mathfrak{H}_H K_r$ by the proof of Theorem 2.

2) In particular, taking D itself for S in the lemma, we obtain Cartan's theorem ([1], [2]): If a division subring R of a division ring D is transformed into itself by each non-zero element of D then R is D itself or contained in the center of D.

for caution's sake, we state here it.

Proof. Let r^*, s^* be in R and S respectively. Then there holds $r^*s^* = s^*r_1$ and $r^*(s^*+1) = (s^*+1)r_2$ with some r_1, r_2 in R. Subtracting the first from the second, we obtain $r^* - r_2 = s^*(r_2 - r_1)$. If s^* is not in R, this implies $r_1 = r_2$ and hence $r^* = r_2$, whence $r^*s^* = s^*r^*$. This shows that if $s^* \in S\backslash S \cap R$ then $s^* \in V_D(R)$.

Now we shall assume that $R \not\supset S$ (or $S \supsetneqq R \cap S$) and then show that $R \subset V_D(S)$. If $R \not\subset V_D(S)$, there exists an element r in R such that $rs \neq sr$ for some $s \in S$. And so, the above remark shows that $s \in S \cap R$. Since $S \supsetneqq S \cap R$, there exists an element s' in $S\backslash S \cap R$, whence $s + s'$ is also in $S\backslash S \cap R$. Therefore there holds that $r(s'+s) = (s'+s)r$ and $rs' = s'r$, and so we arrive to a contradiction that $rs = sr$.

We insert here several properties of $H = V_K(V_K(L))$ which will be used very often in the sequel: A brief computation shows that $V_K(H) = V_K(L)$ and $V_H(H) = V_H(L)$. And, in case K is Galois over L, H is normal over L, whence H is Galois over L. Needless to say, $\mathfrak{G}(K/L)$ leaves $V_K(L)$ setwise invariant.

Again let $\mathfrak{G}$ be the total group of K/L, $\mathfrak{J}$ be the group of all L-inner automorphisms of K, and K be locally finite over L. Now we consider the following conditions:

(I) *For each $a \in K$, the subspace spanned by $\{a\}^{\mathfrak{G}}$ over L is finite over L.*

(I′) *For each $a \in K$, the subspace spanned by $\{a\}^{\mathfrak{J}}$ over L is finite over L.*

(II) $[V_K(L) : V_L(L)] < \infty$.

(II′) $[L(V_K(L)) : L] < \infty$.

(III) $V_K(L) = V_K(K)$.

Clearly (III) is nothing but to say that $\mathfrak{G}$ is outer. And (I) is equivalent to the local finite-dimensionality of $\mathfrak{G}$, for K is locally finite over L.

Lemma 3. (II) $\rightarrow$ (II′) $\rightarrow$ (I′) $\leftrightarrow$ (I), (III) $\rightarrow$ (I).

Proof. (I′) $\rightarrow$ (I). As $L(a)$ is finite over L, by Lemma 1, there exists a finite set $\{\tau^{(i)}\,;\ i = 1, \ldots, k\}$ in $\mathfrak{G}$ such that, for each $\tau \in \mathfrak{G}$, $\tau_{L(a)} = \tau^{(i)}_{L(a)}\iota$ with some i, where ι denotes an element in $\mathfrak{J}$. Hence $L(\{a\}^{\mathfrak{G}}) \subset L(\sum_i L((\{a\}^{\tau^{(i)}})\mathfrak{J})$. The local finiteness of K over L and the condition (I′) imply that $L(\sum_i L((\{a\}^{\tau^{(i)}})\mathfrak{J}))$ is finite over L and so, (I) is satisfied.

(III) $\rightarrow$ (I). This is a special case of (I′) $\rightarrow$ (I).

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(I) $\rightarrow$ (I$'$). This is trivial.

(II) $\rightarrow$ (II$'$). Let $\{u_1, \dots, u_n\}$ be a $V_L(L)$-basis of $V_K(L)$. Then, we obtain that $L(V_K(L)) \subset L(u_1, \dots, u_n)$. The fact that K is locally finite over L implies that $L(V_K(L))$ is finite over L.

(II$'$) $\rightarrow$ (I$'$). This is clear from $L(\{a\}^{\mathfrak{S}}) \subset L(V_K(L), a)$ and $[L(V_K(L), a) : L] < \infty$.

Lemma 4. *If* $[K : L] < \infty$ *then* (II) *is satisfied.*

Proof. Clearly $\mathfrak{G}_{V_K(L)}$ is a regular group of $V_K(L)/V_L(L)$. Noting that $V_K(K) \subset V_{V_K(L)}(V_K(L))$, one can easily see that $[V_K(L) : V_L(L)]$ is finite by [2, Théorème 1].

Lemma 5. *If* $V_K(L) \supsetneqq C$, *the center of* K, *then* (I) *implies* (II).

Proof. Let a be in $V_K(L) \backslash C$, then there exists an element $b \in K$ such that $ab \neq ba$. Now we denote by N a subring normal, finite over L and containing $L(a, b)$. Since $\widetilde{N^{V_K(L)}} = N$ and $N \not\subset V_K(V_K(L))$, by Lemma 2, $N \supset V_K(L)$. Therefore $V_K(L) = V_N(L)$, and so $[V_K(L) : V_L(L)] = [V_N(L) : V_L(L)] < \infty$ by Lemma 4.

Combining Lemma 3 with Lemma 5, we shall readily obtain the following:

Theorem 3. *Let K be locally finite and Galois over L and $\mathfrak{G}$ be its total group. The condition* (I) *is satisfied if and only if one of the conditions* (II) *and* (III) *is satisfied, or what is the same, if one of the conditions* (II$'$) *and* (III) *is done*[1].

The proof of the implication (2) $\rightarrow$ (3) in Theorem 1, (i) is also an easy consequence of our theorem. We state here this fact as a corollary.

Corollary. *If* $\mathfrak{G}$ *is locally finite and* $V_K(L) \supsetneqq C$, *then* $V_K(L)$ *is finite.*

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1) We assume here that, under the assumption of Theorem 3, both (II$'$) and (III) are satisfied. If the condition (II) is not satisfied, there exists an infinite $C \cap L$-basis $\{c_1, c_2, \dots\}$ of C, where C denotes the center of K. As $[L(C) : L] < \infty$, there exists a (finite) maximal subset of $\{c_1, c_2, \dots\}$ whose members are linearly independent over L, say, $\{c_1, c_2, \dots, c_n\}$. There holds therefore, for some $l_1, \dots, l_n$ in L, $c_{n+1} = \sum_1^n c_i l_i$. Noting that c_i's are all in C, we obtain $\sum_1^n c_i l l_i = \sum_1^n c_i l_i l$ for each $l \in L$, which implies that l_i's are in $V_L(L) = C \cap L$, being contradictory to the linear independence of $c_1, \dots, c_{n+1}$ over $C \cap L$. This fact together with Theorem 3 shows the equivalence of (II) and (II$'$).

Proof. We consider the same N as in the proof of Lemma 5, then as a is in $V_N(L)\setminus V_N(N)$, by making use of the same method as in the proof of Lemma 1 of [9], one can easily see that $V_N(N)$ is finite. By Galois theory of finite degree, there holds that $[V_N(L) : V_N(N)] < \infty$, hence $V_N(L) = V_K(L)$ is finite too.

In the rest of this section, we shall restrict our attention to the case where K is Galois over L and $\mathfrak{G}(K/L)$ is l. f. d.

Lemma 6. *Let $\mathfrak{G}(K/L)$ be l. f. d. and non-outer. If an intermediate subring N containing $V_K(L)$ is normal and finite over L then there hold the following*:

(i) $V_K(T) = R'$, *where* $R' = V_N(N)$, $T = V_K(R')$.

(ii) $[V_T(L) : V_T(T)] < \infty$.

(iii) $[T: H] = [V_K(L) : R'] < \infty$, *where* $H = V_K(V_K(L))$.

Proof. (i) From $N \supset V_K(L)$, there holds that $V_N(N) = N \cap V_K(N) = V_K(N)$. Hence $V_K(V_K(R')) = V_K(V_K(V_K(N))) = V_K(N) = R'$.

(ii) Since N is normal and finite over L, and $\mathfrak{G}(K/L)_{R'}$ is a finite outer group of $R'/R' \cap L$, [2, Théorème 1] shows $[R' : R' \cap L] < \infty$. And the fact that N is Galois and finite over $L(R')$ implies $[V_N(L(R')) : R'] < \infty$. Further,

$$V_T(T) = T \cap V_K(T) \supset L \cap R', \text{ and}$$
$$V_T(L) = T \cap V_K(L) = V_K(L(R')) = V_K(L(R')) \cap V_K(L) \subset V_N(L(R')).$$

Hence we have

$$\infty > [V_N(L(R')) : R'] \cdot [R' : R' \cap L] = [V_N(L(R')) : R' \cap L] \geqq [V_T(L) : V_T(T)].$$

(iii) From (ii), we have $[T: V_T(V_T(L))] = [V_T(L) : V_T(T)] < \infty$ (see for example [2, Théorème 1, α]). Since $N \supset V_K(L)$ it follows that $T = V_K(R') \supset V_N(R') = N \supset V_K(L)$, whence $V_T(L) = V_K(L)$. Hence, $V_T(V_T(L)) = V_T(V_K(L)) \subset H$, and so $[T: H] < \infty$. On the other hand, $V_T(H) = T \cap V_K(H) = V_K(L) = V_T(L)$ and $V_T(T)$ and $V_T(T) = T \cap V_K(V_K(R')) = T \cap R' = R'$. We have therefore $[V_K(L) : R'] = [V_T(H) : V_T(T)] = [T: H]$.

Corollary. *Let $\mathfrak{G}(K/L)$ be l. f. d. and non-outer, S be a finite subset of K. Then there exists a subring T with the following properties*:

(1) *T contains $H(S)$ and is normal over L.*

(2) $[T: H] < \infty$, *where* $H = V_K(V_K(L))$.

(3) $[V_T(L) : V_T(T)] < \infty$[1].

Proof. Let N be a subring containing $L(S, V_K(L))$ which is normal and finite over L (see Theorem 3). Then $T = V_K(V_N(N))$ is a desired one by Lemma 6.

Theorem 4. *If K is Galois over L and $\mathfrak{G}(K/L)$ is l. f. d., then so is $\mathfrak{G}(K/H)$, where $H = V_K(V_K(L))$.*

Proof. In case $\mathfrak{G}(K/L)$ is non-outer, our assertion is clear from the above corollary. On the other hand, in case $\mathfrak{G}(K/L)$ is outer, $K = H$, for which there is nothing to prove.

Theorem 5. *If $\mathfrak{G}(K/L)$ is l. f. d., then $\overline{\widetilde{V_K(L)}} = \mathfrak{G}(K/H)$, where $\overline{\widetilde{V_K(L)}}$ is the topological closure of $\widetilde{V_K(L)}$ in $\mathfrak{G}(K/L)$ and $H = V_K(V_K(L))$.*

Proof. Let σ be in $\mathfrak{G}(K/L)$ and N be an arbitrary subring which is normal and finite over L. Clearly $H(N)$ is normal and finte over H (Theorem 4). Since $V_K(V_K(H)) = H$, there exists an element x in $V_K(H) = V_K(L)$ such that $\tilde{x}_{H(N)} = \sigma_{H(N)}$, and of course, that $\tilde{x}_N = \sigma_N$. We obtain therefore our assertion.

In particular, if $H = L$, $\mathfrak{G}(K/L)$ coincides with $\overline{\widetilde{V_K(L)}}$ and furthermore, for an arbitrary intermediate proper subring N normal and finite over L, we obtain either $N \supset V_K(L)$ or $N \subset V_K(V_K(L)) = L$ by Lemma 2. But the latter case is impossible from our assumption, and so we have $V_K(L) = V_N(L)$, which shows that $\mathfrak{G}(N/L)$ is inner.

The next theorem is of interest. As one will see in the next section, in our generalization of Nobusawa's theory, the condition considered in the theorem plays an important rôle.

Theorem 6.[2] *Let K be Galois and $\mathfrak{G}(K/L)$ be l. f. d. $\mathfrak{G} = \mathfrak{G}(K/L)$ is locally compact if and only if the following condition is satisfied:*

$$(\beta) \qquad [V_K(L) : V_K(K)] < \infty.$$

Proof. Sufficiency. Clearly $[V_K(L) : V_K(K)] < \infty$ is equivalent to

1) The property 3) is important in our consideration in § 3. Combining Theorem 11 with this corollary, one will easily see that each L-automorphism of H can be extended to an L-isomorphism of any finite extension ring of H.

2) The authors are indebted to Professor M. Moriya who has given us continuous encouragement and valuable advices. He also pointed out this fact.

$[K:H]<\infty$ by [2, Théorème 1] and [11, Theorem 5], further in case either of these is satisfied, there holds $[V_K(L):V_K(K)]=[K:H]$, where $H=V_K(V_K(L))$. Hence there exists a finite H-basis $S=\{d_1,\dots,d_n\}$ of K. Since $\mathfrak{G}(K/L)$ is l. f. d. we can find a division subring N normal, finite over L and containing $L(S)$. Then we obtain $V_K(K)\subset V_K(N)\subset V_K(L(S))=V_K(L)\cap V_K(S)=V_K(H)\cap V_K(S)=V_K(H(S))=V_K(K)$, which shows $V_K(K)=V_K(N)$, i. e. $\mathfrak{G}(K/N)$ is outer. By Theorem 1, $\mathfrak{G}(K/N)$ is compact and it is a desired compact neighbourhood of the identity[1)].

Necessity. By assumption, there exists a division subring N of K which is normal and finite over L such that $\mathfrak{G}(K/N)$ is compact. If $\mathfrak{G}(K/N)$ is non-outer, then $V_K(N)$ is a finite field by Remark 1, (ii) in §2, and so $[V_K(N):V_K(K)]<\infty$. Clearly $\mathfrak{G}(K/N)$ is l. f. d. Therefore, as in the proof of sufficiency, we can find a division subring N^* which is normal and finite over N such that $\mathfrak{G}(K/N^*)$ is outer. Since N^* is finite over L, there exists a division subring N_0 containing N^* which is normal and finite over L. Clearly, $\mathfrak{G}(K/N_0)$ is outer (and of course closed in $\mathfrak{G}(K/L)$). Hence, without loss of generality, we may assume from the beginning that $\mathfrak{G}(K/N)$ is outer. Now let T be a finite set of elements forming an (independent) L-basis of N. Then $H(T)=H(L(T))=H(N)$, and so $H(T)$ is normal and finite over H by Corollary to Lemma 6. If $K\supsetneqq H(T)$, there exists a subring R properly containing $H(T)$ and normal, finite over H. Since $V_K(V_K(H))=H$, $\mathfrak{G}(R/H(N))$ is induced by $V_K(H(N))$, which means that there exists an $H(N)$-inner automorphism different from the identity. But this contradicts the assumption that $\mathfrak{G}(K/N)$ is outer. Hence $K=H(T)$, which is finite over H. This completes our proof.

If M is an arbitrary division subring of $H=V_K(V_K(L))$ containing L, then $V_M(M)=M\cap V_H(M)\supset M\cap V_H(H)=M\cap V_H(L)=V_M(L)\supset V_L(L)$, that is, the center of M contains the center of L.

Now we shall prove the following:

Theorem 7. *Let K be Galois over L, $\mathfrak{G}(K/L)$ be l. f. d., and $[L:V_L(L)]$ be finite. Then $[H:L]<\infty$ if and only if $[V_H(H):V_L(L)]<\infty$, where $H=V_K(V_K(L))$; moreover, in this case, $H=L(V_H(H))$.*

Proof. Necessity. H is Galois over L and $\mathfrak{G}(H/L)$ is outer.

1) In case $\mathfrak{G}(K/L)$ is l. f. d., there holds $J(\mathfrak{G}(K/L'), K)=L'$ for any intermediate subring L' finite over L (see the proof of Theorem 1, (ii)). And we see that the topology of $\mathfrak{G}(K/L')$ itself is equivalent with the relative topology of it as a subspace of $\mathfrak{G}(K/L)$.

Hence $\mathfrak{G}(H/L)$ is a finite group by our assumption. Clearly $\mathfrak{G}(H/L)_{V_H(H)} = \mathfrak{G}^*$ is an automorphism group of $V_H(H)$ and the fixed subring of $\mathfrak{G}^*$ in $V_H(H)$ is $V_H(H) \cap L = V_H(L) \cap L = V_L(L)$. This shows that $[V_H(H) : V_L(L)] =$ order of $\mathfrak{G}^* < \infty$.

Sufficiency. We consider $L(V_H(H))$. Then $L(V_H(H))$ is a division subring of H and finite over L by our assumption. If $L(V_H(H)) \neq H$, there exists an element h in $H \backslash L(V_H(H))$. We set $S = \{h, V_H(H)\}$. Then $N = L(S^{\mathfrak{G}(H/L)})$ is a subring of H which is finite over L and normal with respect to $\mathfrak{G}(H/L)$. Clearly $V_H(H) \subset V_N(N)$ and $V_N(N) \subset V_N(L) \subset V_H(L) = V_H(H)$, whence $V_H(H) = V_N(N)$. Since $[N : L] \cdot [L : V_L(L)] = [N : V_H(H)] \cdot [V_H(H) : V_L(L)]$, we obtain $[N : V_H(H)] < \infty$. Thus N is finite over its center $V_H(H)$[1]. As is well known, $\mathfrak{G}(N/V_H(H))$ is inner, accordingly so is $\mathfrak{G}(N/L(V_H(H))$. Since $N \supsetneqq L(V_H(H))$, there exists an L-inner automorphism different from the identity, being contrary to the fact that $\mathfrak{G}(H/L)$ is outer. Hence we have $L(V_H(H)) = H$.

Under the same assumption as in Theorem 7, we assume further that $\mathfrak{G}(K/L)$ is non-outer. Then, from Theorem 3, it follows that $[V_K(L) : V_L(L)] < \infty$. Since $V_H(H) \subset V_K(L)$ and $V_L(L) \subset V_H(H)$, we obtain $[V_H(H) : V_L(L)] < \infty$. Hence, as an easy consequence of Theorem 7, we obtain the next corollary.

Corollary. *Let K be Galois over L, $\mathfrak{G}(K/L)$ be l.f.d. and non-outer. If $[L : V_L(L)] < \infty$ then $[H : L] < \infty$.*

Remark 2. Combining Theorem 5 with the previous corollary, one will readily see that, in case $\mathfrak{G}(K/L)$ is l. f. d. and $[L : V_L(L)] < \infty$, $\mathfrak{G}(K/L)$ is outer or $[\mathfrak{G}(K/L) : \widetilde{\overline{V_K(L)}}] < \infty$. Hence we may say roughly that, in this case, $\mathfrak{G}(K/L)$ is either outer or essentially inner.

We shall conclude this section by giving a theorem which is concerned with some special case of locally finite total groups.

If $\mathfrak{G}(K/L)$ is locally finite, then $\mathfrak{G}(K/L)$ is l. f. d. ([9, p. 657]). Suppose that $[L : Z] < \infty$, where $Z = V_L(L)$. Now if $V_K(L) \supsetneqq V_K(K)$, $V_K(L)$ is a finite field, accordingly so is H by Theorem 7. And evidently K is a finite field too. But this is a contradiction. Therefore, if $\mathfrak{G}(K/L)$ is locally finite and $[L : Z] < \infty$, $\mathfrak{G}(K/L)$ is outer.

Lemma 7. *Let K be Galois and finite over L, L be finite over $Z = V_L(L)$. If $\mathfrak{G}(K/L)$ is locally finite, then $K = L \times_Z C$, where $C =$*

1) The finiteness of $[N : V_N(N)]$ is only a consequence of $[N : L] < \infty$ and $[L : V_L(L)] < \infty$ ([6, Theorem 1]).

$V_K(K)$.

Proof. Since $\mathfrak{G}(K/L)$ is outer, H coincides with K. We obtain therefore $K=L(C)$ by Theorem 7, whence $K=L\times_Z C$.

Theorem 8. *Let K be Galois over L, L be finite over $Z=V_L(L)$, and $\mathfrak{G}(K/L)$ be locally finite. Then,*

(i) $K=L\times_Z C$.

(ii) *If $K\supset K'\supset L$ then $K'=L\times_Z C'$, where C' is the center of K'. And by this relation, there exists a one-to-one correspondence between division subrings of K containing L and subfields of C containing Z.*

Proof. (i) By assumption, $\mathfrak{G}(K/L)$ is l. f. d. and outer. Hence $V_K(L)=C$, and so $V_{K'}(K')\subset C$ for any intermediate division subring K'. Let a be an arbitrary element of K, and we denote by N the least normal (over L) subring containing $L(a)$. Then N/L is finite and Galois, and by Lemma 7, $N=L\times_Z C'$, where C' is the center of N. Hence we have $K=L\times_Z C$.

(ii) Let $\{d_1,\dots,d_n\}$ be a Z-basis of L, and $a=d_1c_1+\cdots+d_nc_n$, $c_i\in C$, be an arbitrary element of K. Then N is Galois and finite over $L(a)$, where N is the least normal (over L) subring containing $\{c_1,\dots,c_n\}$. If $\sigma\in\mathfrak{G}(N/L(a))$, then $a^\sigma=d_1c_1^\sigma+\cdots+d_nc_n^\sigma=a$. Hence $c_i^\sigma=c_i$ $(i=1,\dots,n)$, whence $c_i\in L(a)$. Therefore $L(a)=L\times_Z Z(c_1,\dots,c_n)$, and so, for any division subring K' with $K\supset K'\supset L$, we obtain that $K'=L\times_Z C'$, where C' is the center of K'. Conversely, if $C\supset C'\supset Z$, then $L(C')=L\times_Z C'$, and its center is C'.

3. A generalization of Nobusawa's theory.

The purpose of this section is to generalize Nobusawa's theory in such a way that the generalization contains also the case of finite degree. As is shown in Theorem 1, the cases correspond to the compact case and discrete case respectively. Hence, it seems that our next step is to investigate the case where the total group is locally compact. In fact, in this case, we shall see that there exists a one-to-one dual correspondence between closed regular subgroups and intermediate subrings in the usual sense of Galois theory (Theorem 12). But we shall deal, at first, with a more (really) general case, and at last come back to the locally compact case.

Throughout this section, let K be Galois over L, $\mathfrak{G}$ be the total group of K/L, H signify $V_K(V_K(L))$ and C be the center of K.

Our first lemma is the next, which has been already used in the preceding section.

Lemma 8. *H is normal, whence Galois over L, and $\mathfrak{G}(H/L)$ is outer*[1].

Now we consider the following condition:

$$(\beta) \qquad [V_K(L):C] < \infty.$$

Clearly, if $\mathfrak{G}$ is l. f. d. and locally compact, then the condition (β) is satisfied by Theorem 6. In case the condition (β) is satisfied, there holds that $[K:H]=[V_K(L):C]$. Hence in below, whenever the condition (β) is satisfied, $\{d_1, \ldots, d_n\}$ will mean a (fixed) H-basis of K, L_1 mean $L(d_1, \ldots, d_n)$, and H_1 denote $L_1 \cap H$.

Our first consideration will be undertaken under the following conditions which should be satisfied in case $\mathfrak{G}$ is l. f. d. and locally compact (Theorem 6):

(α) *H is locally finite over L.*
(β) $[V_K(L):C] < \infty$.
(γ) *L_1 is finite over L.*
(δ) *K is Galois over L_1.*

Lemma 9. *Under the conditions (α)-(δ), there hold the following:*
(i) *K is locally finite over L.*
(ii) *There exists a one-to-one correspondence between subrings H_2 of H with $[H_2:H_1] < \infty$ and subrings L_2 of K with $[L_2:L_1] < \infty$ in the relations $H_2 = L_2 \cap H$ and $L_2 = \sum_{i=1}^{n} \oplus H_2 d_i$. In particular, if L_2 is normal and finite over L_1, then H_2 is finite over H_1 and normal with respect to $\mathfrak{G}(H/H_1)$, and conversely.*
(iii) $\mathfrak{G}(H/L) = \mathfrak{G}_H$.

Proof. (i) Clearly there holds that $V_K(L_1) = V_K(L) \cap V_K(d_1, \ldots, d_n)$ $= V_K(H) \cap V_K(d_1, \ldots, d_n) = V_K(K) = C$, hence $\mathfrak{G}^{(1)} = \mathfrak{G}(K/L_1)$ is outer. Now we shall prove that $H_1 = J(\mathfrak{G}_H^{(1)}, H)$ and that $\{d_1, \ldots, d_n\}$ forms an

1) For any H' contained in H and finite over L, it is normal with respect to $\mathfrak{G}(K/L)$ if (and only if) it is so with respect to $\mathfrak{G}(H/L)$. Hence, in case H is locally finite over L, there exists the least subring containing H' and normal, finite over L. This fact will be used later.

H_1-basis of L_1. Evidently $H_1 = L_1 \cap H = J(\mathfrak{G}(L_1), K) \cap H = J(\mathfrak{G}_H^{(1)}, H)$. The second assertion is proved as follows. If $\sum_i h_i d_i$ is in L_1, where $h_i \in H$, then $\sum_i h_i^\sigma d_i = \sum_i h_i d_i$ for each $\sigma \in \mathfrak{G}^{(1)}$. Hence $h_i^\sigma = h_i$ ($i = 1, \dots, n$), and so h_i is in H_1, which shows that $L_1 = \sum_{i=1}^n \oplus H_1 d_i$.

Let $\{a_j = \sum_{i=1}^n h_{ji} d_i \, ; \, j = 1, \dots, m\}$ be an arbitrary finite subset of K, where $h_{ji} \in H$, then $[H_1(\{h_{ji}\text{'s}\}) : L] < \infty$ ($i = 1, \dots, n$; $j = 1, \dots, m$) by assumption. Now we set $H_2 = H_1(\{h_{ji}\text{'s}\})$ and $\mathfrak{G}^{(2)} = \mathfrak{G}^{(1)}(H_2)$. Then $J(\mathfrak{G}_H^{(2)}, H) = H_2$, for $\mathfrak{G}_H^{(1)}$ is a (regular) outer group of H/H_1, from which we obtain $J(\mathfrak{G}^{(2)}, K) = \sum_{i=1}^n \oplus H_2 d_i$. Hence $\sum_i \oplus H_2 d_i$ is a division subring which is finite over L, accordingly $L(a_1, \dots, a_m)$ is finite over L as a division subring of $\sum_i \oplus H_2 d_i$. This shows that K is locally finite over L.

(ii) If $H \supset H_2 \supset H_1$ and $[H_2 : H_1] < \infty$, then $L_2 = \sum_i H_2 d_i$ is a division ring (see the last part of the proof (i)). Conversely, let L_2 be a division subring with $[L_2 : L_1] < \infty$ and we set $\mathfrak{G}^{(2)} = \mathfrak{G}(L_2)$. Then, as $\mathfrak{G}^{(1)}$ is outer, by making use of the same method as at the beginning of the proof of (i), we can readily show that $H_2 = H \cap L_2 = J(\mathfrak{G}^{(2)}{}_H, H)$, $L_2 = \sum_i \oplus H_2 d_i$ and that H_2 is finite over H_1.

Now if H_2 is finite over H_1 and normal with respect to $\mathfrak{G}(H/H_1)$ then, for any $\sigma \in \mathfrak{G}^{(1)}$, σ_H belongs to $\mathfrak{G}(H/H_1)$. We know therefore that $L_2 = \sum_i \oplus H_2 d_i$ is normal with respect to $\mathfrak{G}^{(1)}$. Conversely, if L_2 is normal and finite over L_1 then $H_2 = L_2 \cap H$ is left set-wise invariant by $\mathfrak{G}_H^{(1)}$. Since $\mathfrak{G}(H/H_1)$ is outer, $\mathfrak{G}_H^{(1)}$ is a regular group of H/H_1, accordingly H_2 is (finite over H_2 and) normal with respect to $\mathfrak{G}(H/H_1)$.

(iii) Let $\mathfrak{G}^{(1)}$ and $\mathfrak{H} = \mathfrak{G}(H/H_1)$ be topologized as in §1. We consider the (univalent) mapping $\varphi : \sigma \to \sigma_H$ of $\mathfrak{G}^{(1)}$ into $\mathfrak{H}$. If H' is an arbitrary division subring of H which is normal and finite over H_1, then $\mathfrak{G}^{(1)}(\sum_i \oplus H' d_i)$ is an (open) neighbourhood of the identity in $\mathfrak{G}^{(1)}$ (see the last part of (ii)). And $\varphi(\mathfrak{G}^{(1)}(\sum_i \oplus H' d_i)) = \mathfrak{G}_H^{(1)}(H') \subset \mathfrak{H}(H')$, which shows the continuity of φ. Since the topological groups $\mathfrak{G}^{(1)}$ and $\mathfrak{H}$ are compact, φ is a closed mapping, accordingly $\mathfrak{G}_H^{(1)}$ is closed in $\mathfrak{H}$. On the other hand, as $\mathfrak{G}_H^{(1)}$ is a regular group of H/H_1, $\mathfrak{G}_H^{(1)}$ is dense in $\mathfrak{H}$. Combining these facts, we obtain that $\mathfrak{G}_H^{(1)}$ coincides with $\mathfrak{H}$.

Now let $L' = \sum_i \oplus H' d_i$ and $\mathfrak{G}' = \mathfrak{G}(L')$, where $H_1 \subset H' \subset H$ and H'

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is finite and normal over L. If the finite group $\mathfrak{G}(H'/L)$ is induced by $\{\sigma^{(1)}, \ldots, \sigma^{(m)}\} \subset \mathfrak{G}$, for each $\sigma \in \mathfrak{G}(H/L)$, there holds that $\sigma_{H'} = \sigma_{H'}^{(i)}$ with some i, that is, $\sigma_{H'}^{(i)-1}\sigma$ is in $\mathfrak{H}(H')$. Since $\mathfrak{G}_{H'}^{(1)} = \mathfrak{H}$, we have $\mathfrak{H}(H') \subset \mathfrak{G}'_{H}$, which shows that $\mathfrak{G}(H/L)$ is induced by $\{\sigma^{(1)}\mathfrak{G}', \cdots, \sigma^{(m)}\mathfrak{G}'\}$, and so $\mathfrak{G}(H/L) = \mathfrak{G}_H$[1].

Corollary. *If $H \supset L' \supset L$ then $J(\mathfrak{G}(L'), K) = L'$.*

Proof. Since $\mathfrak{G}(H/L) = \mathfrak{G}_H$ is outer, $H \cap J(\mathfrak{G}(L'), K) = J(\mathfrak{G}_H(L'), H) = L'$ by §1, (i). On the other hand $H = V_K(V_K(H))$ implies that $J(\mathfrak{G}(L'), K) \subset H$. Hence we obtain our assertion.

Lemma 10. *Under the conditions (α)-(δ), there exists a subgroup $\mathfrak{G}^{(0)}$ of $\mathfrak{G}$ such that $J(\mathfrak{G}^{(0)}, K) = L$ and such that, for each finite subset S of K, $[L(S^{\mathfrak{G}^{(0)}}) : L] < \infty$.*

Proof. Let $\{e_1, \ldots, e_n\}$ be a C-basis of $V_K(L)$. Then, by Lemma 9 (ii), $L_1(e_1, \ldots, e_n) = \sum_i \oplus H_2 d_i$ with some H_2 finite over H_1. Now let N_2 be a division subring of H containing H_2 and normal, finite over L, and we set $\mathfrak{G}^{(2)} = \mathfrak{G}(\sum_i \oplus N_2 d_i)$. If the finite group $\mathfrak{G}(N_2/L)$ is induced by $\{\sigma^{(1)}, \ldots, \sigma^{(t)}\} \subset \mathfrak{G}$, then $\mathfrak{G}(H/L)$ is induced by $\{\sigma^{(1)}\mathfrak{G}^{(2)}, \ldots, \sigma^{(t)}\mathfrak{G}^{(2)}\}$ (see the last part of the proof of Lemma 9 (iii)). We set here $d_j^{\sigma^{(i)}} = \sum_k h_{ijk} d_k$ ($i = 1, \ldots, t$; $j = 1, \ldots, n$). Let N be a division subring of H containing N_2 and all the h_{ijk}'s which is normal and finite over L, and we set $M = \sum_i \oplus N d_i$. Then $\mathfrak{G}^{(0)} = \{\tau \in \mathfrak{G} ; M^{\tau} = M\}$ contains $\{\sigma^{(1)}\mathfrak{G}^{(2)}, \ldots, \sigma^{(t)}\mathfrak{G}^{(2)}\}$ as well as $\widetilde{V_M(L)}$ ($\supset \widetilde{\{e_i\text{'s}\}}$). Hence $J(\mathfrak{G}^{(0)}, K) \subset J(\{\sigma^{(1)}\mathfrak{G}^{(2)}, \ldots, \sigma^{(t)}\mathfrak{G}^{(2)}\}, K) \cap J(\widetilde{V_M(L)}, K) = L$, whence $J(\mathfrak{G}^{(0)}, K) = L$. Let $S = \{a = \sum_j h_{ij} d_j ; i = 1, \ldots, p\}$ be an arbitrary finite subset of K, and we set $E = \{h_{ij}^{\tau} ; i = 1, \ldots, p ; j = 1, \ldots, n ; \tau \in \mathfrak{G}^{(0)}\}$, which is finite. Then $L(S^{\mathfrak{G}^{(0)}}) \subset M(E)$ and $M(E)$ is finite ovsr L, which completes the proof.

Lemma 11. *Let $\mathfrak{G}^{(0)}$ be a group of automorphisms of a division ring K, and $D = J(\mathfrak{G}^{(0)}, K)$. If, for each finite subset S of K, $[D(S^{\mathfrak{G}^{(0)}}) : D] < \infty$, then $J(\mathfrak{G}(L'), K) = L'$ for any division subring L' such that $[L' : D] < \infty$, where $\mathfrak{G} = \mathfrak{G}(K/D)$.*

Proof. Let $\{f_1, \ldots, f_q\}$ be a D-basis of L', and b be in $K \backslash L'$. We set $L'' = D(\{f_1, \ldots, f_q, b\}^{\mathfrak{G}^{(0)}})$, which is finite over D. As $L''^{\mathfrak{G}^{(0)}} \subset L''$,

1) The proof of the last part is also given by making use of Theorem 2.

$J(\mathfrak{G}_{L''}^{(0)}, L'') = D$. And, by Theorem 2, $\mathfrak{G}(L''/D)$ is induced by $\mathfrak{G}^* = \{\sigma \in \mathfrak{G}(K/D)\,;\ L''^{\sigma} = L''\}$. As L'' is finite and Galois over D, so is L'' over L'. And $\mathfrak{G}(L''/L')$ is induced by some subgroup of $\mathfrak{G}^*$. Hence there exists an automorphism τ in $\mathfrak{G}^*$ such that $b^{\tau} \neq b$ and $x^{\tau} = x$ for all $x \in L'$. Therefore $J(\mathfrak{G}(L'), K) = L'$.

Combining Lemmas 9, 10 and 11, we obtain the following principal theorem.

Theorem 9. *Let K be Galois over L. If the conditions (α)-(δ) are satisfied, then $J(\mathfrak{G}(K'), K) = K'$ for each intermediate subring K', and K is locally finite over K'. (Hence K is totally locally finite over L.)*

Proof. At first, in virtue of Lemmas 10, 11, $J(\mathfrak{G}(L'), K) = L'$ for each L' finite over L. If $V_K(L) = V_K(K')$, then there holds that $L \subset \subset K' \subset H$. Thus, in this case, our assertion is clear from Corollary to Lemma 9. Hence, we may, and shall, assume that $V_K(L) \supsetneqq V_K(K')$. In this case, again by Corollary to Lemma 9, it suffices to show that there exists some K'' such that $K' \supset K'' \supset L$ and $[K'' : L] < \infty$ and such that $V_K(K'') = V_K(K')$. (Then we can take K'' and $V_K(V_K(K''))$ instead of L and H respectively, by the remark at the beginning.) Under our assumption, there exists an element $b_1 \in K'$ such that $a_1 b_1 \neq b_1 a_1$ for some $a_1 \in V_K(L) \setminus V_K(K')$, and so $V_K(L) \supsetneqq V_K(L(b_1)) \supset V_K(K')$. Since $[V_K(L) : V_K(K')] \leqq [V_K(L) : C] < \infty$, repeating a finite number of above procedures, we have a finite subset $\{b_1, \ldots, b_t\}$ of K such that $V_K(L(b_1, \ldots, b_t)) = V_K(K')$.

Next, we are going to prove the second part. The preceding argument enables us to find a subring K'' finite over L such that $K'' \subset K' \subset V_K(V_K(K''))$. Since all the conditions in the theorem are satisfied with respect to K/K'', we may take K'' and $V_K(V_K(K''))$ instead of L and H respectively. Hence, it suffices to prove our assertion in the case where K' is contained in H. Let $\{h_1, \ldots, h_s\}$ be an L-basis of H_1, and we set $H' = K'(h_1, \ldots, h_s)$. Then, as is easily seen[1], $\sum_i \oplus H'd_i = J(\mathfrak{G}^{(1)}(H'), K)$, which shows that $\sum_i \oplus H'd_i$ is a division subring containing K' as well as L_1, where $\mathfrak{G}^{(1)} = \mathfrak{G}(K/L_1)$. Noting that $\mathfrak{G}(H/L)$ is outer, we know that H is locally finite over K' (see §1), accordingly $[\sum_i \oplus H'd_i : K'] = n[H' : K'] > \infty$. Since $\mathfrak{G}(K/L_1)$ is outer also, K is locally finite over $\sum_i \oplus H'd_1$, and hence so is over K'.

The next is an easy consequence of Theorem 9.

Theorem 10. *Under the conditions* (α)-(δ), *there exists a one-to-one dual correspondence between total subgroups* $\mathfrak{H}$ *of* $\mathfrak{G}=\mathfrak{G}(K/L)$ *and intermediate subrings* K' *in the usual sense of Galois theory.*

Remark 3. If K is Galois over L and $\mathfrak{G}(K/L)$ is l. f. d. and the condition (β) is satisfied, (or what is the same, it is l. f. d. and locally compact,) then clearly all the assumptions in Theorem 9 are fulfilled. It is our conjecture that, in this case, the condition (β) may be needless to obtain the one-to-one dual correspondence.

In the rest of this section except the last part, we assume the conditions (α)-(δ). Let N be a division subring with $H\supset N\supset L$. If ρ is an L-isomorphism of N into K, then $N^\rho\subset H$. For, if not, N contains a division subring F finite over L such that $F^\rho\not\subset H$. Then, by Theorem 2, ρ_F can be extended to an automorphism in $\mathfrak{G}$. But this is contrary to the normality of H. And the fact that $\mathfrak{G}(H/L)$ is outer shows that ρ can be extended to an automorphism in $\mathfrak{G}(H/L)=\mathfrak{G}_H$ (see §1, (ii)), and so to an automorphism in $\mathfrak{G}$.

Now we are arrived at the position to prove the following theorem which corresponds to Theorem 3 in [12].

Theorem 11. *Under the conditions* (α)-(δ), *each* L*-isomorphism* ρ *of an arbitrary intermediate subring* K' *into* K *can be extended to an automorphism in* $\mathfrak{G}$.

Proof. If $V_K(L)=V_K(K')$ then there holds that $L\subset K'\subset H$. Hence, in this case, our assertion is clear from the above remark. Hence, we may, and shall, assume that $V_K(L)\supsetneqq V_K(K')$. In this case, by making use of the same method as in the proof of Theorem 9, we obtain that $K''\subset K'\subset V_K(V_K(K''))$ for some K'' finite over L. By Theorem 2, $\rho_{K''}$ can be extended to some σ in $\mathfrak{G}$. Since $\rho\sigma^{-1}$ is a K''-isomorphism of K' into K, we can extend it to some τ in $\mathfrak{G}(K'')$ (of course, in $\mathfrak{G}$). Clearly $\tau\sigma$ is a required one.

At last we shall come back to the case where $\mathfrak{G}$ is l. f. d. and locally compact. To prove the Galois correspondence previously mentioned, it suffices to show the following:

Lemma 12. *If* $\mathfrak{G}(K/L)$ *is l. f. d. and locally compact then any closed regular subgroup of* $\mathfrak{G}(K/L)$ *is a total subgroup, and conversely.*

1) Note that $\mathfrak{G}(H/H_1)=\mathfrak{G}_H^{(1)}$.

Proof. By the latter part of Theorem 9, K is totally locally finite over L. Hence our assertion can be proved in the same manner as in the proof of Theorem 7 in [11].

Combining Lemma 12 with Theorem 10, we obtain the following theorem.

Theorem 12. *Let K be Galois over L. If $\mathfrak{G}(K/L)$ is l.f.d. and locally compact then there exists a one-to-one dual correspondence between closed regular subgroups of $\mathfrak{G}(K/L)$ and intermediate subrings in the usual sense of Galois theory. In particular, if $\mathfrak{G}(K/L)$ is outer then we have the one-to-one correspondence between closed subgroups of $\mathfrak{G}(K/L)$ and intermediate subrings.*

For an arbitrary subgroup $\mathfrak{H}$ of $\mathfrak{G}=\mathfrak{G}(K/L)$, the composite of $\mathfrak{H}$ and $\widetilde{V_K(J(\mathfrak{H}, K))}$ is denoted by $\tilde{\mathfrak{H}}$. On the other hand, for any intermediate subring M, $\mathfrak{G}^M$ will signify the totality of automorphisms σ in $\mathfrak{G}$ such that $M^\sigma = M$. Our last theorem is the next, which corresponds to Theorem 5 in [12].

Theorem 13. *Let K be Galois over L, $\mathfrak{G}=\mathfrak{G}(K/L)$ be l.f.d. and locally compact, and K' be an intermediate subring. K' is Galois over L if and only if $\widetilde{\mathfrak{G}^{K'}}$ is dense in $\mathfrak{G}$.*

Proof. At first, we assume that K' is Galois over L, then by Theorem 11, $\mathfrak{G}(K'/L)$ coincides with $(\mathfrak{G}^{K'})_{K'}$, and Theorem 9 secures the equality $J(\mathfrak{G}^{K'}, K)=L$. Hence $\widetilde{G^{K'}}$ is dense in $\mathfrak{G}$ by Lemma 12. Conversely we assume that $\overline{\widetilde{\mathfrak{G}^{K'}}}=\mathfrak{G}$. Again by Theorem 9, we have $J((\mathfrak{G}^{K'})_{K'}, K')=J(\mathfrak{G}^{K'}, K)=J(\overline{\widetilde{\mathfrak{G}^{K'}}}, K)=L$, which shows that K' is Galois over L.

4. Examples.

(a) *An infinite Galois extension whose total group is outer.*

In pp. 23—24 of [7], G. Köthe proved that there exists a (countably) infinite number of normal extensions over the rational number field Z of which the degrees are prime to each other, and so, that there exists a (countably) infinite number of division algebras over Z: $\{K_1, K_2, \dots\}$, where K_i contains a maximal subfield M_i which is normal over Z and $([K_i:Z], [K_j:Z])=1$ for $i\neq j$. Then, as is well-known, $A_i=K_1\times_Z K_2 \times_Z\cdots\times_Z K_i$ is a division ring. If $i<j$, by the canonical isomorphism,

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A_i can be considered as a division subalgebra of A_j, and $K = \bigcup_{i=1}^{\infty} A_i$, which will be denoted as $K_1 \times K_2 \times \cdots \times K_n \times \cdots$, may be considered. Throughout this section, whenever we consider $H = H_1 \times H_2 \times \cdots$ with $K_i \supset H_i \supset Z$, H would have the same content as in the above. (Note here that H is a division subring of K.)

Now we consider the division ring $K_0 = K_1 \times M_2 \times M_3 \times \cdots$. Then $V_{K_0}(K_1) = M_2 \times M_3 \times \cdots = C = V_{K_0}(K_0)$. It is easily shown that C/Z is Galois. Let $\mathfrak{G}$ be the Galois group of C/Z, and $\{d_1 = 1, d_2, \ldots, d_n\}$ be a Z-basis of K_1. Then we can extend each automorphism in $\mathfrak{G}$ to some K_1-automorphism of K_0 by the following definition :

$$a \to \sum_{i=1}^{n} d_i c_i^{\sigma}, \text{ where } a = \sum_{i=1}^{n} d_i c_i \in K_0,\ c_i\text{'s} \in C,\ \sigma \in \mathfrak{G}.$$

And conversely, each K_1-automorphism of K_0 is obtained in this way. Clearly $J(\mathfrak{G}_0, K_0) = K_1$, where $\mathfrak{G}_0$ is the set of all K_1-automorphisms of K_0. Hence K_0/K_1 is Galois and $\mathfrak{G}(K_0/K_1) = \mathfrak{G}_0$. And $\mathfrak{G}_0$ is locally finite. For, let $a = \sum_i d_i c_i$, c_i's in C, be an arbitrary element of K_0, then $a^{\sigma} = \sum_i d_i c_i^{\sigma}$ for each $\sigma \in \mathfrak{G}_0$. Since $\{c_i\}^{\mathfrak{G}_0} = \{c_i\}^{\mathfrak{G}}$ $(i = 1, \ldots, n)$ is finite, $\{a\}^{\mathfrak{G}_0}$ is finite too. Obviously the center of K_0 is infinite hence $\mathfrak{G}_0$ is outer.

(b) *A Galois extension whose total group is not outer but almost outer.*

Let F be the algebraic closure of $GF(p)$, p is a non-zero prime number, then F is *ideally cyclic* (ideal zyklisch) in Krull's sense, that is there exists an automorphism σ of F such that its restriction to any finite subfield is a generating element of the Galois group[1] ([8]).

Now we set $I = [t, \sigma]$, where $[t, \sigma]$ is the principal ideal domain consisting of all forms $\sum_i t^i \alpha_i$, $\alpha_i \in F$, with the usual addition and the multiplication defined by $\alpha t = t\alpha^{\sigma}$. And let K be the quotient division ring of I, then the center of K is $GF(p)$. For, if $0 \neq x^{-1}y = yx^{-1}$ is an element of the center of K, then for any $\gamma \in F$, $y\gamma x = x\gamma y$ and $ytx = xty$. We set $x = \sum^{m} t^i \alpha_i$, $\alpha_m \neq 0$, $y = \sum^{n} t^j \beta_j$, where we may assume that $\beta_n = 1$. Then the above two equations come to the following :

$$\sum_{i,j} t^{i+j} \alpha_i^{\sigma^j} \gamma^{\sigma^j} \beta_j = \sum_{i,j} t^{i+j} \alpha_i \gamma^{\sigma^i} \beta_j^{\sigma^i}, \tag{1}$$

1) In fact, $\sigma : \alpha \to \alpha^p$ $(\alpha \in F)$ is such an automorphism.

$$\sum_{i,j} t^{i+j+1} \alpha_i^{\sigma^{j+1}} \beta_j = \sum_{i,j} t^{i+j+1} \alpha_i \beta_j^{\sigma^{i+1}}. \tag{2}$$

Comparing the coefficients of the heighest degree, we obtain $\alpha_m^{\sigma^n}\gamma^{\sigma^n} = \alpha_m\gamma^{\sigma^m}$ and $\alpha_m^{\sigma^{n+1}} = \alpha_m$. In particular, from the first, we get $\alpha_m^{\sigma^n} = \alpha_m$ so that, from the latter, $\alpha_m^{\sigma} = \alpha_m$, whence $\alpha_m \in GF(p)$, hence, again from the first, $\alpha_m(\gamma^{\sigma^m} - \gamma^{\sigma^n}) = 0$ for all γ in F. If $m \neq n$ then there exists an element γ such that $\gamma^{\sigma^m} \neq \gamma^{\sigma^n}$, whence $\alpha_m = 0$. This contradiction implies that $m = n$. We now proceed by induction and assume that $\alpha_i = \alpha_m\beta_i$ for $m \geqslant i > q$. Comparisons of the coefficients of t^{m+q} in (1) and that of t^{m+q+1} in (2) imply the following:

$$\sum_{h+k=m+q} \alpha_h^{\sigma^k} \gamma^{\sigma^k} \beta_k = \sum_{h+k=m+q} \alpha_h \gamma^{\sigma^h} \beta_k^{\sigma^h}, \tag{1'}$$

$$\sum_{h+k=m+q} \alpha_h^{\sigma^{k+1}} \beta_k = \sum_{h+k=m+q} \alpha_h \beta_k^{\sigma^{h+1}}. \tag{2'}$$

And our induction hypotheses will show that $\alpha_m\beta_q\gamma^{\sigma^q} + \alpha_q^{\sigma^m}\gamma^{\sigma^m} = \alpha_m\beta_q^{\sigma^m}\gamma^{\sigma^m} + \alpha_q\gamma^{\sigma^q}$ and $\alpha_m\beta_q + \alpha_q^{\sigma^{m+1}} = \alpha_m\beta_q^{\sigma^{m+1}} + \alpha_q$. Further, from the first, we have $(\alpha_m\beta_q - \alpha_q)^{\sigma^m} = \alpha_m\beta_q - \alpha_q$, and from the second, $(\alpha_m\beta_q - \alpha_q)^{\sigma^{m+1}} = \alpha_m\beta_q - \alpha_q$. Hence $\alpha_m\beta_q - \alpha_q$ is in $GF(p)$. Combining this fact with (2′), there holds that $(\alpha_m\beta_q - \alpha_q)(\gamma^{\sigma^q} - \gamma^{\sigma^m}) = 0$. Since $\gamma^{\sigma^q} \neq \sigma^{\sigma^m}$ for some $\gamma \in F$, we obtain $\alpha_m\beta_q = \alpha_q$, which completes our induction. Hence $x = \alpha_m y$.

For any $GF(p^n) \subset F$, if we set $L = V_K(GF(p^n))$, then K/L is finite and Galois, and its total group is locally finite. (The order of the group of all L-inner automorphisms of K is $(p^n - 1)/(p - 1)$.)

(c) *An infinite Galois extension whose total group is not locally finite but l. f. d.*

We consider the division ring $K = K_1 \times K_2 \times \ldots\ldots$ in Example (a). And let $H = H_1 \times H_2 \times \ldots\ldots$, where $K_i \supset H_i \supset Z$. Then $V_K(H) = V_K(H_1) \times V_K(H_2) \times \ldots\ldots$ and $V_K(V_K(H)) = H_1 \times H_2 \times \ldots\ldots$, whence K/H is Galois. Let a be an arbitrary element of K and σ be any automorphism in $\mathfrak{G}(K/H)$, then $\{a, a^\sigma\}$ is contained in some $A_p = K_1 \times {}_Z K_2 \times \ldots \times {}_Z K_p$. And σ induces an B_p-isomorphism σ^* of $B_p(a)$ into A_p, where $B_p = H_1 \times {}_Z H_2 \times \ldots \times {}_Z H_p$. Since A_p is finite over Z, there exists an inner automorphism $\tilde{b}$ in $\widetilde{V_{A_p}(B_p)}$ $(\subset \widetilde{V_K(H)})$ such that $\sigma^* = \tilde{b}_{B_p(a)}$. Hence $a^\sigma = a^{\tilde{b}}$ with some b in $V_K(H)$.

In particular, we consider the division subring $M = M_1 \times M_2 \times \ldots$, where M_i's are maximal subfields of K_i's mentioned in (a). Then $M = V_K(M)$ ($\supsetneqq Z$) is a maximal subfield of K, and so $\mathfrak{G}(K/M)$ is not locally finite (see Remark 1). And $M(a) = M(\{a\}^{\tilde{M}}) = M(\{a\}^{\mathfrak{G}(K/M)})$ implies that $\mathfrak{G}(K/M)$ is l. f. d.

(d) *An infinite Galois extension for which the conditions* (α)-(δ) *in* §3 *are satisfied but whose total group is not l. f. d.*

We consider again the division ring $K_0 = K_1 \times_Z C$ in Example (a), where $C = M_2 \times M_3 \times \ldots\ldots$. As is noted in Example (a), C is Galois over the rational number field Z. Now we shall verify that K_0 and Z can be taken as K and L in Theorem 9 respectively. At first, $V_{K_0}(K_0) = C$ and $[V_{K_0}(Z) : C] = [K_1 : Z] < \infty$, which is the condition (β). As we can take any Z-basis of K_1 for $\{d_1, \ldots, d_n\}$ in Theorem 9, and as K_0/K_1 is Galois by Example (a), the rest of the verification is clear.

If $a \in K_0 \backslash C$ then $Z(\{a\}^{\mathfrak{G}(K_0/Z)}) \supset Z(\{a\}^{\tilde{K_0}}) = K_0$ by Cartan's theorem ([1] or [2]), which shows that $\mathfrak{G}(K_0/Z)$ is not l. f. d.

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