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ON PRIMITIVE ELEMENTS OF GALOIS EXTENSIONS OF DIVISION RINGS*

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It is a result of F. Kasch [3]¹⁾ that if a division ring K is Galois and finite over a division subring L and the center of $V_K(L)$, the *centralizer* of L in K , is separable over the center of K then $K = L(d, udu^{-1})$ with some d, u in K . In this note, under the same assumptions as above, we shall prove the following:

Principal Theorem. *Let D be an intermediate division subring of K/L and $\mathfrak{S}$ be the group of all L -inner automorphisms of K/L . If, for each element x of D , $\{x\}\mathfrak{S} \cap D$ is a finite set, then $D = L(d, udu^{-1})$ with some d, u in D where $\{x\}\mathfrak{S}$ means the set of all images of x by $\mathfrak{S}$ and $\{x\}\mathfrak{S} \cap D$ the complement of $\{x\}\mathfrak{S}$ in D .*

Clearly this theorem contains the result of F. Kasch as a special case. Further, our proof is completed without aid of Lagrange's interpolation formula used in the proof of F. Kasch.

§ 1. Preliminaries. Throughout this note, L will be a division ring, K be a division ring which is Galois and finite over L , and C be the center of K . If L is a finite ring then so is K , and hence, by the well-known Theorem of Wedderburn, our Principal Theorem is always true without special assumptions. Therefore, we shall assume in the sequel that L is an infinite division ring. For any division subring T of K , we denote by $\mathfrak{G}(K/T)$ the *total group* of K/T , that is, $\mathfrak{G}(K/T)$ is the group of all automorphisms of K which leave T element-wise invariant. Let now $V_K(L)$ be the *centralizer* of L in K . Then $V_K(V_K(L)) = H$ is *normal* over L , the total group $\mathfrak{G}(H/L)$ of H/L is outer and the *total group* $\mathfrak{G}(K/H)$ of K/H is $\mathfrak{S}$, the group of all inner automorphisms of K/L . For any subset S of K , we consider the subring $L(S)$ of K , the minimal subring of K containing S and L . Clearly, $L(S)$ is a division subring of K .

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1) Numbers in brackets refer to the references cited at the end of this note.

Lemma 1. *Let S be an infinite subset of L and let a, b be elements of K . If, for $\mathfrak{S} = \bigcup_{x \in S} \mathfrak{G}(K/L(a+xb))$, $\{a\}^{\mathfrak{S}}$ and $\{b\}^{\mathfrak{S}}$ are finite, then there exists an element s in S such that $L(a, b) = L(a+sb)$.*

Proof. We denote by $\{a_1 = a, a_2, \dots, a_r\}$ and $\{b_1 = b, b_2, \dots, b_t\}$ all different elements of $\{a\}^{\mathfrak{S}}$ and $\{b\}^{\mathfrak{S}}$ respectively. Since S is infinite there exists an element $s \in S$ such that $a+sb \neq a_i+sb_j$ for all pairs $(i, j) \neq (1, 1)$. For each automorphism $\sigma \in \mathfrak{G}(K/L(a+sb))$, we have $a^\sigma + sb^\sigma = (a+sb)^\sigma = a+sb$, which means that $a^\sigma = a$ and $b^\sigma = b$. Since, by Galois theory (see [2], [3] or [5]), $L(a+sb)$ is the fixed subring of $\mathfrak{G}(K/L(a+sb))$ in K , both a and b must be contained in $L(a+sb)$, whence $L(a, b) = L(a+sb)$.

By induction, one will readily prove the next:

Corollary 1. *If $\{a_1, a_2, \dots, a_n\}$ is a finite subset of K such that $\{a_i\}^{\mathfrak{G}(K/L)} (i=1, 2, \dots, n)$ are finite, then there exists a finite subset $\{x_1, x_2, \dots, x_n\}$ of L such that $L(a_1, a_2, \dots, a_n) = L(\sum_{i=1}^n x_i a_i)$.*

The next result of N. Nobusawa ([6]) follows readily from the above corollary.

Corollary 2. *If $\mathfrak{G}(K/L)$ is locally finite¹⁾ then any intermediate subring D possesses a single primitive element over L : $D = L(d)$ for some d in D .*

Since the total group $\mathfrak{G}(H/L)$ of $H = V_K(V_K(L))$ is outer, $\mathfrak{G}(H/L)$ is locally finite²⁾. Hence we obtain by Corollary 2 the next:

Corollary 3. *Any intermediate division subring T of H possesses a single primitive element over L .*

Lemma 2. *Let R be a proper division subring of K containing C and $\{c_1, c_2, \dots, c_n\}$ be a subset of C consisting of n different elements. If a is an element in R such that $ab \neq ba$ for some b in $K \setminus R$ then $\{(b+c_i)a(b+c_i)^{-1}; i=1, 2, \dots, n\}$ is a subset of K consisting of n different elements and there exists at most one element c in C such that $(b+c)a(b+c)^{-1}$ is contained in R .*

Proof. If c_1, c_2 are different elements in C then $(b+c_1)a(b+c_1)^{-1} \neq (b+c_2)a(b+c_2)^{-1}$. For, if not, $(b+b_1)a(b+c_1)^{-1} = (b+c_2)a(b+c_2)^{-1}$

1) See [4, § 1].

2) See [4, Theorem 1].

$= a'$ imply that $(c_1 - c_2)a = a'(c_1 - c_2)$, whence $a = a'$. But $(b + c_1)a(b + c_1)^{-1} = a$ leads to a contradiction $ba = ab$. Now we suppose that $(b + c_1)a(b + c_1)^{-1} = a_1 \in R$ and $(b + c_2)a(b + c_2)^{-1} = a_2 \in R$. Then $ba - a_1b = a_1c_1 - c_1a$, $ba - a_2b = a_2c_2 - c_2a$, whence $(a_2 - a_1)b = (a_1c_1 - c_1a) - (a_2c_2 - c_2a)$. Since $a_1 \neq a_2$, we obtain the contradiction that $b = (a_2 - a_1)^{-1} \{(a_1c_1 - c_1a) - (a_2c_2 - c_2a)\} \in R$.

§ 2. Primitive elements of Galois extensions. At first we shall state the following generalization of H. Cartan's theorem¹⁾.

Lemma 3. *Let R, S be division subrings of a division ring K . If all inner automorphisms induced by non-zero elements of S leave R set-wise invariant, then $R \supset S$ or $R \subset V_K(S)$.*

Lemma 4. *Let $\mathfrak{G}(K/L)$ be not locally finite and D be an intermediate subring of K/L such that $D^{\mathfrak{G}} = D$. If the center Z of $V_K(L)$ is separable over C then $D = L(d, udu^{-1})$ with some d, u in D .*

Proof. By Galois theory, there holds that $[K:L] = [\mathfrak{G}(K/L) : \mathfrak{F}] [V_K(L) : C]$. Clearly, each automorphism in $\mathfrak{F}$ is induced by some non-zero element of $V_K(L)$. Since $\mathfrak{G}(K/L)$ is not locally finite $\mathfrak{F}$ must be an infinite group, whence C is infinite by the relation $[V_K(L) : C] \leq [K:L]$. Further, K is non-commutative because $\mathfrak{F}$ is not the identity group.

By Lemma 3, either $D \subset V_K(V_K(L)) = H$ or $D \supset V_K(L)$. In the first case, D has a single primitive element over L by Corollary 2 because H is Galois, finite over L , and $\mathfrak{G}(H/L)$ is locally finite. Hence we may, and shall, assume $D \not\subset H$ so that $D \supset V_K(L) (= V_D(L))$ and D is non-commutative. Now we set $V_D(V_K(L)) = V_D(V_D(L)) = H_0$, $V_D(D) = C_0$, and denote by W a separable, maximal subfield of $V_K(L)$ over Z . Noting that $V_K(D) \subset V_K(L) \cap V_K(V_K(L)) = V_{V_K(L)}(V_K(L)) = Z$, we have $C \subset C_0 \subset Z$. As Z is separable over C , so is W over C , whence $W = C(b) = C_0(b)$ with some b in W . Clearly $L \subset H_0 \subset H$, and so, by Corollary 3, $H_0 = L(a)$ for some $a \in H_0$. As is easily verified, $V_D(V_D(H_0)) = H_0$, which means that D is Galois over H_0 and $\mathfrak{G}(D/H_0)$ is inner. We set here $M = L(a, b) = H_0(b)$. As $H_0 \subset M \subset D$ and $W \subset V_D(M) \subset$

1) See [4, Lemma 2].

$V_D(L) \cap V_D(W) \subset V_K(L) \cap V_D(W) = V_{V_K(L)}(W) = W$, D is Galois, finite over M and $\mathfrak{G}(D/M) = \widetilde{W}$ by Galois theory, where $\widetilde{W}$ denote the group of inner automorphisms determined by all non-zero elements in W .

We shall prove next that $M = L(a, b) = L(a + lb)$ with some l in L . To show this, in virtue of Lemma 1, it suffices to show that $\{a\}^{\mathfrak{H}}$ and $\{b\}^{\mathfrak{H}}$ are finite, where $\mathfrak{H} = \bigcup_{x \in L/\langle 0 \rangle} \mathfrak{G}(K/L(a + xb)) (\subset \mathfrak{G}(K/L))$. Since $a \in H$, $\{a\}^{\mathfrak{G}(K/L)} = \{a\}^{\mathfrak{G}(H/L)}$ is finite. If y is in $V_K(L(a + xb))$ then $y \in V_K(L)$ and $y(a + xb) = (a + xb)y$, whence $yb = by$, and so y belongs to the centralizer of W in $V_K(L)$, that is, $y \in W$. On the other hand, one can easily see that $W \subset V_K(L(a + xb))$, that is, $V_K(L(a + xb)) = W$. For each $\sigma \in \mathfrak{G}(K/L(a + xb))$, and for any $y \in V_K(L(a + xb)) = W$, we have $y(a^\sigma + xb^\sigma) = y(a + xb)^\sigma = y(a + xb) = (a + xb)y = (a^\sigma + xb^\sigma)y$. Since $a^\sigma \in H^\sigma = H$ and $y \in V_K(L(a + xb)) = W \subset V_K(L)$, it follows that $yb^\sigma = b^\sigma y$ from the above equations, that is, $b^\sigma \in V_K(W)$. As σ leaves $V_K(L)$ setwise invariant, $b^\sigma \in W^\sigma \subset V_K(L)$ which shows $b^\sigma \in V_K(W) \cap V_K(L) = V_{V_K(L)}(W) = W$. Hence, $\{b\}^{\mathfrak{H}} \subset W$. Obviously, $\mathfrak{G}^*(K/L)$, the restriction of $\mathfrak{G}(K/L)$ on $V_K(L)$, has $V_L(L)$ as the fixed subring and $[V_K(L) : V_L(L)] < \infty^1$.

Since $V_L(L) \subset Z \subset W$, b satisfies some equation $f(x) = 0$ in $V_L(L)$. Therefore, for any $\sigma \in \mathfrak{H}$, we have $f(b^\sigma) = 0$; this means that $\{b\}^{\mathfrak{H}} (\subset W)$ is a finite subset of K . Hence, there exists some $l \in L$ such that $M = L(d)$ for $d = a + lb$.

If $L \subset C_0$ then $d \notin C_0$, for $M = L(d) \subset C_0$ implies $W = V_D(M) \supset V_D(C_0) = D$ but this gives a contradiction because D is non-commutative. On the other hand, if $L \not\subset C_0$ and $d \in C_0$ then, for any $l' \in L \setminus C_0$, $l'd$ is also a primitive element of M/L . Therefore, without loss of generality, we may assume that $M = L(d)$ for some $d \notin C_0$.

Since W is finite and separable over C_0 , there exists only a finite number of subfields $\{W_1, W_2, \dots, W_n\}$ of W which properly contains C_0 . Then, as $W = V_D(M)$, all proper division subrings of D containing M are exhausted by $\{M_i = V_D(W_i) : i = 1, 2, \dots, n\}$. Let $\{t_1, t_2, \dots, t_n\}$ be chosen such as $t_i \in W_i \setminus C_0$. Then there exists a subset $\{f_i : i = 0, 1, \dots, n\}$ of D such that $df_0d^{-1} \neq f_0$, and $t_i f_i t_i^{-1} \neq f_i$ ($i = 1, 2, \dots, n$) by Hilfsatz 1 of [3], there exists an element $f \in D$ so that $dfd^{-1} \neq f$ and $t_i f t_i^{-1} \neq f$ ($i = 1, 2, \dots, n$). It is clear that $f \in D \setminus M_i$, $d \in M \subset M_i$ and $M_i \supset C_0$. Since C is infinite, by using Lemma 2 repeatedly, we can select an element $c \in C_0$ such that $d' = (f + c)d(f + c)^{-1} \notin M_i$ ($i = 1, 2,$

1) See [4, Lemma 4].

..., n) Suppose that $L(d, d') \subsetneq D$. Then $W = V_D(M) \supset V_D(M(d')) = V_D(L(d, d')) \supseteq C_0$, and so $V_D(L(d, d'))$ must coincide with some W_i ($1 \leq i \leq n$). However, this shows that $M_i = V_D(W_i) \supset L(d, d') \ni d'$, being contrary to the property of d' . Therefore, $L(d, d') = D$, q. e. d.

§ 3. Proof of Principal Theorem. Combining Corollary 2 with Lemma 4, we can now prove our principal theorem. In case $\mathfrak{G}(K/L)$ is *locally finite*, by Corollary 2, $D = L(d)$ with some element $d \in D$, and all the restrictions in our theorem is superfluous. On the other hand, in case $\mathfrak{G}(K/L)$ is not locally finite, C is infinite. Suppose that $D^{\mathfrak{S}} \neq D$, then there exists an element $g \in D$ such that, for some $v \in V_K(L)$, $vgv^{-1} \notin D$. Since C is infinite, accordingly $C \cap D$ is infinite, by making use of the same method as in the proof of Lemma 2, we see that the set $\{(v+x)g(v+x)^{-1}; x \in C \cap D\} \setminus D$ is infinite, which means that $\{g\}^{\mathfrak{S}} \setminus D$ is infinite. Therefore, there must hold $D^{\mathfrak{S}} = D$ and hence, $D = L(d, d')$ by Lemma 4, where $d' = udu^{-1}$ for some $u \in D$.

As an easy consequence of the principal theorem, we obtain the following:

Corollary 4. *Let K/L be Galois, $\mathfrak{G}(K/L)$ be locally finite-dimensional and locally compact¹⁾ and the center of $V_K(L)$ be separable over C . If D is an intermediate subring of K finite over L such that, for each $x \in D$, the set $\{x\}^{\mathfrak{S}} \setminus D$ is finite, then $D = L(d, udu^{-1})$ with some d, u in D .*

Proof. In case $\mathfrak{G}(K/L)$ is *locally finite*, our assertion follows from Corollary 2. On the other hand, if $\mathfrak{G}(K/L)$ is *not locally finite*, then, by assumption, $[V_K(L) : V_L(L)] < \infty$ [4, Theorem 6]. Since $\mathfrak{G}(K/L)$ is locally finite-dimensional, there exists a subring K' of K which is normal, finite over L and contains $D(V_K(L))$. Clearly the center of K' contains C and $V_K(L) = V_{K'}(L)$, and so our assertion is a direct consequence of the principal theorem.

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