Note on a paper by J. S. Frame and G. de B. Robinson

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NOTE ON A PAPER BY J. S. FRAME
AND G. DE B. ROBINSON

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1. Introduction. J. S. Frame and G. de B. Robinson have proved [1] the following

1.1 Let $p$ be a prime. The number of $p$-regular diagrams with $n$ nodes is equal to the number of $p$-regular classes of the symmetric group $S_n$ and hence to the number of modular irreducible representations of $S_n$.

A diagram is called $p$-regular if no $p$ of its rows are of equal length, otherwise $p$-singular.

Recently this result was refined by G. de B. Robinson [5] as follows:

1.2 The number of $p$-regular diagrams in a given block is equal to the number of modular irreducible representations in that block.

The author has also obtained 1.2 independently by a simple method. We shall also give an alternative proof of 1.1 by our method.

2. Remarks on diagrams. Let $[\alpha] = [\alpha_1, \alpha_2, \ldots, \alpha_n]$ be a diagram with $n$ nodes that contains $\alpha_i$ nodes in its $i$-th row. We denote the number of nodes in the $j$-th column of $[\alpha]$ by $a_j'$. We have evidently

$$\sum_{j=1}^{k} a_j' = n, \quad a_1' \geq a_2' \geq \ldots \geq a_k' > 0 \quad (k = \alpha_i).$$

We set

$$\rho_j = a_j' - a_{j+1}' \quad (j = 1, 2, \ldots, k - 1),$$

$$\rho_k = a_k'.$$

Then $[\alpha]$ is completely determined by a set of non-negative integers $\{\rho_j\}$ since

$$a_j' = \sum_{i=j}^{k} \rho_i \quad (j = 1, 2, \ldots, k).$$

It follows from our definition that $[\alpha]$ is $p$-regular if and only if every $\rho_j$ is less than $p$. We see also that $[\alpha]$ is $p$-regular if and only
if $|\alpha|$ does not contain a $p$-hook of leg length $p - 1$.

If $[\alpha]$ is $p$-singular, then there exists at least one $\rho_j$ greater than $p$. We set

$$
2.1 \quad \rho_j = \rho_j^{(1)} \cdot \rho_j^{(2)} \quad 0 \leqslant \rho_j^{(1)} < p.
$$

Then $[\alpha]$ is completely determined by $\{\rho_j^{(1)}\}$ and $\{\rho_j^{(2)}\}$. Let $[\alpha^{(1)}]$ and $[\alpha^{(2)}]$ be the diagrams determined by $\{\rho_j^{(1)}\}$ and $\{\rho_j^{(2)}\}$ in the above sense respectively. Since $\rho_j^{(1)} < p$, $[\alpha^{(1)}]$ is $p$-regular and $[\alpha^{(2)}]$ is not vacuous for a $p$-singular diagram $[\alpha]$. If $[\alpha^{(2)}]$ has $a$ nodes, then $[\alpha^{(1)}]$ has $m = n - ap$ nodes. Moreover we see easily that $[\alpha^{(1)}]$ is obtained by removing $a$ $p$-hooks of leg length $p - 1$ successively from $[\alpha]$. Since the $p$-regular diagram $[\alpha^{(1)}]$ is determined uniquely by $[\alpha]$, we shall call $[\alpha^{(1)}]$ the $p$-regular diagram corresponding to $[\alpha]$. We have the

**Lemma 1.** $[\alpha]$ and $[\alpha^{(1)}]$ have the same $p$-core.

**Example.** If $[\alpha] = [6, 4, 3^3, 1^4]$ for $p = 3$, then $[\alpha^{(1)}] = [6, 4, 1]$ and $[\alpha^{(2)}] = [3, 1^2]$. $[\alpha]$ and $[\alpha^{(1)}]$ have the same $p$-core $[\alpha_0] = [3, 1^2]$.

Let $[\beta]$ be a given $p$-regular diagram with $m$ nodes and let $[\gamma]$ be an arbitrary diagram with $a$ nodes. Then $[\beta]$ and $[\gamma]$ determine uniquely a diagram $[\alpha]$ with $n = m + ap$ nodes such that

$$
2.2 \quad [\beta] = [\alpha^{(1)}], \quad [\gamma] = [\alpha^{(2)}].
$$

Hence if we denote by $k(n)$ the number of diagrams with $n$ nodes, i.e. the number of classes of $S_n$, then for a given $p$-regular diagram $[\beta]$ with $m$ nodes there exist exactly $k(a)$ diagrams $[\alpha]$ with $n$ nodes such that $[\alpha^{(1)}] = [\beta]$. Therefore we obtain the

**Lemma 2.** Let $h(n)$ be the number of $p$-regular diagrams with $n$ nodes. Then

$$
2.3 \quad h(n) = k(n) - \sum_{a=1}^r h(n - ap)k(a),
$$

where $n = tp + r$, $0 \leqslant r < p$.

3. **Proof of 1.1.** Let us denote by $k'(n)$ the number of $p$-regular classes of $S_n$. We then have [2, Lemma 3]

$$
3.1 \quad k'(n) = k(n) - \sum_{a=1}^r k'(n - ap)k(a).
$$
Certainly the theorem is true for \( n = 1 \). We shall assume that \( 1.1 \) is true for \( m < n \). We then have

\[
h(n - ap) = k'(n - ap) \quad (a = 1, 2, \ldots, t).
\]

It follows immediately from 2.3 and 3.1 that \( h(n) = k'(n) \). This proves 1.1.

4. Proof of 1.2. Let \( B \) be a block of weight \( b \) having a given \( p \)-core \([a_0]\). The number \( l(b) \) of ordinary irreducible representations in \( B \) and the number \( l'(b) \) of modular irreducible representations in \( B \) are independent of the \( p \)-core and we have \([2; 3; 4]\)

\[
l'(b) = l(b) - \sum_{a=1}^{p} l'(b - a)k(a).
\]

If we denote by \( g(w) \) the number of \( p \)-regular diagrams in a block of weight \( w \) having a given \( p \)-core \([a_0]\), then we see by Lemma 1 that

\[
g(b) = l(b) - \sum_{a=1}^{b} g(b - a)k(a).
\]

Certainly 1.2 is true for \( b = 1 \). We shall assume that 1.2 is true for \( w < b \). Then 4.1 and 4.2 yield \( g(b) = l'(b) \). Since \( l'(b) \) is independent of the \( p \)-core, \( g(b) \) is also independent of the \( p \)-core. This completes the proof of 1.2.

References


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(Received September 3, 1956)