Mathematical Journal of Okayama University

Volume 6, *Issue* 2 1956 *Article* 5

MARCH 1957

Galois theory of simple rings II

Hisao Tominaga*

*Okayama University

GALOIS THEORY OF SIMPLE RINGS II

HISAO TOMINAGA

In his previous paper [15][1], the present author considered Galois theory of simple rings under the conditions $(*)$ and $(*_f)$, proved the existence of a one-to-one dual correspondence between closed regular subgroups of the total group and intermediate regular subrings, and showed the validity of the so-called extension theorem, being parallel to the theory for division rings constructed in [9] except the latter part of [9, Theorem 9].

The purpose of this paper is, giving some supplementary remarks to the previous theory and introducing some more natural notions instead of local simplicity and $(*)$, to make our previous one more desirable.

In § 1, as preliminaries, we shall sketch the outline of the previous theory, and in § 2 the notions of local finiteness and local finite-dimensionality which are equivalent to that of local simplicity and the condition $(*)$ respectively will be defined and results in [15] will be restated in these terminologies, some of which will be sharpened. § 3 is devoted to prove that our previous results are still true under the conditions similar to (α) — (δ) considered in [9, § 3], and at the same time, the realization of the analogy for the latter part of [9, Theorem 9] will be shown. At last in § 4, we shall try to weaken the notion of regular groups.

Here the author wishes to express his hearty thanks to the late Mrs. E. Isizuka who gave him continuous encouragement both openly and secretly.

1. Outline of the previous theory.

By a ring we mean, throughout this paper except § 4, a ring with an identity. By a subring we mean one which contains this identity element. By a *simple ring* we shall mean a simple ring (with an identity) with minimum condition. In this section, we assume R is a simple ring which is Galois over a simple subring S. And we consider the following conditions:

$(*)$ For any finite subset F in R, there exists a simple subring

1) Numbers in brackets refer to the references cited at the end of this paper.

N normal, finite over S and containing $S(F)$[1].

$(*_f)$ $[V_R(S): V_R(R)] < \infty$.

We shall state here our previous results without proofs in order:

(**a**) Let R be locally simple over S. If $\mathfrak{G} = \mathfrak{G}(R/S)$ is almost outer then it is locally finite ([15, Theorem 4]).

(**b**) If the condition $(*)$ is satisfied then either $\mathfrak{G}(R/S)$ is outer or $V_R(S)$ is finite over the center of S ([15, Theorem 6]).

(**c**) If the conditions $(*)$ and $(*_f)$ are satisfied then any closed regular subgroup of $\mathfrak{G}(R/S)$ is a regular total subgroup, and conversely ([15, Theorem 7)][2].

(**d**) Let the condition $(*)$ be satisfied. Then $\mathfrak{G}(R/S)$ is compact if and only if it is locally finite, or if it is almost outer. And it is discrete if and only if R is finite over S ([15, Theorem 8]).

(**e**) Under the condition $(*)$, there holds $J(\mathfrak{G}(R'), R) = R'$ for any intermediate regular subring R' finite over S ([15, Lemma 8]).

We consider here the following additional condition:[3]

$(*_1)$ For any finite set F in R, there exists a simple subring N s-normal, finite over S and containing $S(F)$.

(**f**) If the condition $(*_1)$ is satisfied then $H = V_R(V_R(S))$ is simple and $(*_1)$ is satisfied with respect to R/H ([15, Theorem 11]).

(**g**) If the condition $(*_1)$ is satisfied then $\mathfrak{J}$ is dense in $\mathfrak{G}(R/H)$, where $\mathfrak{J}$ is the totality of inner automorphisms contained in $\mathfrak{G}(R/S)$ and $H = V_R(V_R(S))$ ([15, Theorem 12]).

(**h**) Let the condition $(*_1)$ be satisfied. Then $\mathfrak{G}(R/S)$ is locally compact if and only if $[V_R(S) : V_R(R)] < \infty$ ([15, Theorem 14]).

(**i**) Under the conditions $(*)$ and $(*_f)$, the condition $(*)$ is fulfilled with respect to H/S and $\mathfrak{G}(H/S) = \mathfrak{G}(R/S)_H$ ([15, Lemmas 12, 13]).

(**j**) Under the conditions $(*)$ and $(*_f)$, there holds $J(\mathfrak{G}(R'), R) = R'$ for any intermediate regular subring R' ([15, Lemma 14]).

(**k**) Under the conditions $(*)$ and $(*_f)$, there exists a one-to-one dual correspondence between closed regular subgroups of the total group and intermediate regular subrings in the usual sense of Galois theory ([15, Theorem 15]).

(**l**) Under the conditions $(*)$ and $(*_f)$, for any intermediate regular subring R', each S-isomorphism ρ of R' into R can be extended to

1) Whenever the condition $(*)$ is satisfied, $\mathfrak{G}(R/S)$ should be considered as a topological group in the sense of [15].

2) This fact is required only to prove (k).

3) Soon one will see that this additional condition is superfluous (Corollary 2).

an automorphism in $\mathfrak{G}(R/S)$, where we assume that $V_R(R^{\prime\prime})$ is simple ([15, Theorem 16]).

(m) Under the conditions $(*)$ and $(*_f)$, any intermediate regular subring T with simple $V_T(S)$ is Galois over S if and only if $\tilde{\mathfrak{T}}$ is dense in $\mathfrak{G}(R/S)$, where $\tilde{\mathfrak{T}}$ is the composite of $\mathfrak{T}=\{\sigma\in\mathfrak{G}(R/S)\,;\,T^\sigma=T\}$ and the totality of $J(\mathfrak{T},\ R)$-inner automorphisms of R ([15, Remark 3])[1].

2. Local finite-dimensionality.

Throughout this section, we assume that $R=\sum^{n}De_{ij}$ is a simple ring, where e_{ij}'s are matric units and $D=V_R(\{e_{ij}\})$ is a division ring, and that S is a simple subring of R. Further we set $S_1=S(\{e_{ij}\})$[2] $=\sum^{n}D_1e_{ij}$, where $D_1=V_{S_1}(\{e_{ij}\})$.

We introduce here the following definitions:

Definition 1. R is said to be *locally finite over* S[3] if, for any finite set F in R, $S(F)$ is finite over S (as a left S-module), accordingly $[S(F):S]<\infty$ ([1, p. 68]).

Definition 2. S is said to be *twofold regular in* R if $V_R(V_R(S))$ is a simple subring.

Definition 3. Let $\mathfrak{G}$ be a group of S-automorphisms in R. We say that $(R/S,\ \mathfrak{G})$ is *locally finite-dimensional* (abbreviated, l. f. d.)[4] if for any finite set F in R, $S(F^{\mathfrak{G}})$ is finite over S. Particularly, in case R is Galois over S and $\mathfrak{G}$ coincides with the total group $\mathfrak{G}(R/S)$, we say simply $\mathfrak{G}$ is l. f. d.

Definition 4. Let $\mathfrak{G}$ be a group of automorphisms in R. Then S is said to be $\mathfrak{G}$-*normal* if $S^\sigma=S$ for all σ in $\mathfrak{G}$.

---

1) This is the so-called normality theorem in our case, however in [15], we did especially ignore it. Because, for the theorem, it seems that more general considerations as in [10] should be carried out. But such considerations will contain many difficulties, and will be liable to be mistaken. For instance, the proof of [13, Theorem 5] will be not yet complete without the supplementary assumption that $V_G(C_0)$ is regularly generated. (This is the case when the characteristic C is different from 2.)

2) For a subset F in R, $S(F)$ mean the subring of R generated by F over S.

3) Cf. [9].

4) Cf. [9].

The next two lemmas are clear, and the former has been noted in [15].

Lemma 1. *Let* T *be a ring with the identity element* 1. *If* D' *and* T' *are a division subring and a simple subring of* T *containing* 1 *respectively then* $D' \cap T'$ *is a division ring.*

Lemma 2. *If a division ring* K *is locally finite over a division subring* L *then any intermediate subring is a division subring.*

Corollary 1. *If* R *is locally finite over* S *then any intermediate subring containing* S_1 *is a regular subring.*

Proof. As S_1 is finite over S, it possesses minimum condition for left ideals, accordingly so does D_1. Hence D_1 is a division ring, and so S_1 is simple. And one will readily see that D is locally finite over D_1. The rest of the proof is clear from Lemma 2 and the fact that $V_R(T) = V_D(E)$, where $T = \sum^n Ee_{ij}$ is an arbitrary subring containing S_1.

By Corollary 1, we see that the local finiteness of R over S is equivalent to the local simplicity in [15], and [7, Satz 3] proves the following precision of (a).

(a*) $\mathfrak{G}(R/S)$ *is locally finite if and only if it is almost outer and* R *is locally finite over* S.

Proof. We shall prove here only that if $\mathfrak{G} = \mathfrak{G}(R/S)$ is locally finite then R is locally finite over S. For any finite subset F in R, consider $T = S(\{\{e_{ij}\}, F\}^{\mathfrak{G}})$ and $T = S(\{\{e_{ij}\}, F, V_R(S)\}^{\mathfrak{G}})$ according as $\mathfrak{G}$ is outer or not. Clearly $T = \sum^n Be_{ij}$ with $B = V_T(\{e_{ij}\})$. Taking the division subring B^* of D generated by B, $T^* = \sum^n B^* e_{ij}$ is simple, and is the least subring of R containing T such that if $t \in T^*$ is regular then t^{-1} is contained in T^*. (One can construct such T^* in the obvious way.) Then noting that, in case $\mathfrak{G}$ is non-outer, $V_R(S)$ is finite by [7, Satz 3], we can readily see $\mathfrak{G}_{T^*}$ is a finite regular group of T^*/S in Nakayama's sense [10]. Hence we have $[T^* : S] < \infty$ by [10, Theorem 1], and so R is locally finite over S.

Lemma 3. *Let* T *be a (simple) subring of* R *finite over* S *and containing* S_1, *and let* $\mathfrak{H}$ *be a set of* S*-automorphisms in* R. *Then* *there holds* $\mathfrak{H}_T \subset \sum^{u} \oplus \sigma_T^{(i)} (V_R(S))_r$ *with some* $\sigma^{(i)}$'s *in* $\mathfrak{H}$[1].

Proof. As $\mathfrak{H}_T R_r$ is contained in $\mathfrak{M}_R(T)$ with $[\mathfrak{M}_R(T) : R_r] < \infty$ ([15, §1]), $\mathfrak{H}_T R_r$ is finite over R_r. Thus, there exists the least integer u such that $\mathfrak{H}_T R_r = \sum^{u} \sigma_T^{(i)} R_r$. Then, by making use of the same method as in the proof of [15, Theorem 4], one will readily see that $\mathfrak{H}_T R_r = \sum^{u} \oplus \sigma_T^{(i)} R_r$ and that $\mathfrak{H}_T \subset \sum^{u} \oplus \sigma_T^{(i)} (V_R(S))_r$.

For a while, we assume that R is locally finite over a regular subring S, and denote by $\mathfrak{G}$, $\mathfrak{J}$ the groups of all S-automorphisms, of all S-inner automorphisms in R respectively. We consider here the following conditions:

(I) For any $r \in R$, $[S(\{r\}^{\mathfrak{G}}) : S] < \infty$.

(I′) For any $r \in R$, $[S(\{r\}^{\mathfrak{J}}) : S] < \infty$.

(II) $[V_R(S) : V_S(S)] < \infty$.

(II′) $[S(V_R(S)) : S] < \infty$.

(III) $V_R(S) = V_R(R)$.

Clearly, under the local finiteness of R over S, (I), (I′) mean that $(R/S, \mathfrak{G})$, $(R/S, \mathfrak{J})$ are l. f. d. respectively, and (III) is nothing but to say that $\mathfrak{G}$ is an outer group.

Lemma 4. (I) $\rightarrow$ (I′), (II) $\leftrightarrow$ (II′), (II′) $\rightarrow$ (I).

Proof. (I) $\rightarrow$ (I′) and (II) $\leftrightarrow$ (II′) are almost clear, because $S(V_R(S)) = S \times_{V_S(S)} V_R(S)$. (In particular, we have $[V_R(S) : V_S(S)] = [S(V_R(S)) : S]$.)

(II′) $\rightarrow$ (I). For any $r \in R$, we set $T = S_1(r)$, which is a regular subring of R finite over S by Corollary 1. From Lemma 3, we have $\mathfrak{G}_T \subset \sum^{u} \oplus \sigma_T^{(i)} (V_R(S))_r$ with some $\sigma^{(i)}$'s in $\mathfrak{G}$. Hence there holds $S(\{r\}^{\mathfrak{G}}) \subset S(r^{\sigma^{(1)}}, \dots, r^{\sigma^{(u)}}, V_R(S))$, which proves (I).

Lemma 5. *If* $\mathfrak{G}$ *is almost outer then it is locally finite, and conversely. In particular,* (III) $\rightarrow$ (I).

Proof. The converse part is [7, Satz 3]. Now, for any $r \in R$, we set $T = S_1(r)$, which is a regular subring finite over S. Then, by

---

1) Here $\mathfrak{H}_T$ be considered as a subset of the module of homomorphisms of the left S-module T into R. And as is easily seen, if $\sigma_T = \sum^{u} \sigma^{(i)}{}_T v^{(i)}{}_r (\sigma \in \mathfrak{H})$ then non-zero $v^{(i)}$'s are regular elements.

Lemma 3, we have $\mathfrak{G}_T \subset \sum^{u} \oplus \sigma^{(i)}_T(V_R(S))_r$ with some $\sigma^{(i)}$'s in $\mathfrak{G}$. The rest of the proof is the latter half of the proof of [15, Theorem 4].

Lemma 6. (I′) *implies either* (II′) *or* (III).

Proof. We set $V_R(S) = \sum^{l} Kg_{pq}$, where g_{pq}'s are matric units and $K = V_{V_R(S)}(\{g_{pq}\})$ is a division ring. To prove our assertion, it suffices to show that, in case $\mathfrak{G}$ is non-outer, $V_R(S)$ is contained in some simple subring which is finite over S. To this end we shall distinguish two cases: (i) $V_R(S)$ *is not a division ring*, and (ii) $V_R(S)$ *is a division ring*. In these cases, considering $N = S(\{\{e_{ij}\}, \{g_{pq}\}\}^{\mathfrak{J}})$ and $N = S(\{\{e_{ij}\}, r, v\}^{\mathfrak{J}})$ for any $r \in R$, $v \in V_R(S)$ with $rv \neq vr$ respectively, our present proof will proceed as in the proof of [15, Theorem 6], and the details may be left to readers.

Now combining Lemmas 4, 5 and 6, we obtain the next which is evidently a generalization of [9, Theorem 3] as well as of **(b)** in §**1.**

Theorem 1. *Let a simple ring R be locally finite over a regular subring S, and $\mathfrak{G}$, $\mathfrak{J}$ be the groups of all S-automorphisms, of all S-inner automorphisms in R respectively. Then $(R/S, \mathfrak{G})$ is l. f. d. if and only if either $\mathfrak{G}$ is outer or $[V_R(S) : V_S(S)] < \infty$. Moreover, $(R/S, \mathfrak{G})$ is l. f. d. if and only if $(R/S, \mathfrak{J})$ is l. f. d.*

Lemma 7. *If $(R/S, \mathfrak{G})$ is l. f. d. then $(D/D_1, \mathfrak{G}(S_1)_D)$ is l. f. d. And if moreover R/S is Galois then so is D/D_1.*

Proof. Evidently S_1 is a regular subring and $(R/S_1, \mathfrak{G}(S_1))$ is l. f. d. Noting that, for any $\sigma \in \mathfrak{G}(S_1)$, there holds $(V_R(\{e_{ij}\}))^\sigma = V_R(\{e_{ij}\})$, our assertion will be readily seen.

Corollary 2. *Let $(R/S, \mathfrak{G})$ be l. f. d. Then, for any finite set F in R, there exists a regular subring T containing $S(F)$ which is $\mathfrak{G}$-normal, finite over S and twofold regular in R. If moreover R/S is Galois then the condition $(*_1)$ is satisfied.*

Proof. Taking the regular subring $N = S(\{\{e_{ij}\}, F\}^{\mathfrak{G}}) = \sum^{n'} Ee_{ij}$ which is $\mathfrak{G}$-normal and finite over S, we have $V_R(V_R(N)) = \sum^{n} V_D(V_D(E))e_{ij}$, where $E = V_N(\{e_{ij}\})$.

Remark 1. Let $S = \sum^{n^*} D^* e^*_{ij}$ be a simple subring of R, where e^*_{ij}'s are matric units and $D^* = V_S(\{e^*_{ij}\})$ is a division ring. Then there holds $R=\sum^{n^*}E^*e^*_{ij}$ with $E^*=V_R(\{e^*_{ij}\})$ and any generating elements of E^* over D^* are those of R over S at the same time. If moreover R is Galois over S then so is E^* over D^* with the total group $(\mathfrak{G}(R/S))_{E^*}$. Similarly R is locally finite over S when and only when E^* is so over D^*. Hence we may say that Galois theory of simple rings can be reduced to the case where the fixed subring is a division ring.

It is a pretty result of Nagahara that any division ring which is Galois and finite over a division subring L is generated over L by two elements which are conjugate to each other ([8, Theorem 4]). Making use of this fact and Lemma 7, we can prove the next corresponding to [6, Satz 14]:

Theorem 2. *If R is Galois and finite over S then $R=S(x, y, z)$ with some regular elements x, y, z*[1].

Proof. As is noted in Remark 1, it suffices to prove our assertion for the case where S is a division subring. We shall distinguish two cases: (i) $W=V_S(R)$ *is finite.* As evidently $V_R(R)$ is finite over $V_S(R)$, $V_R(S)$ is finite, that is, $\mathfrak{G}(R/S)$ is locally finite. $\mathfrak{G}(D/D_1)$ is therefore locally finite, and so $D=D_1(x)$ by [12, Theorem 6]. On the other hand, as is easily verified, there holds $e_{ij}=y'^{i-1}z'^{n-1}y'^{n-1}z'^{j-1}$, where $y'=\sum_2^n e_{ii-1}$ and $z'=\sum_2^n e_{i-1i}\left(=\left(\sum^n e_{i\,n-i+1}\right)y'\left(\sum^n e_{i\,n-i+1}\right)^{-1}\right)$. Hence we have $S_1=S(y', z')=\sum^n D_1e_{ij}$, and eventually $R=S(x, y, z)$, where $y=1-y'$ and $z=1-z'$. (ii) W *is infinite.* In this case, $W(\{e_{ii}\})=W(y')$ with some y'. In fact, $y'=\sum^n w_ie_{ii}$ is a required (regular) one, where w_i's are non-zero elements in W with $w_i\neq w_j$ $(i\neq j)$. And by Nagahara's result, $D=D_1(x, dxd^{-1})$ with some x, d in D. Now we set $t'=d(1-we_{nn-1})(1-e_{n-1n-2})\dots(1-e_{21})$ and $t''=(1+e_{n-1n})(1+e_{n-2,n-1})\dots(1+e_{12})$, where w is an element in W such that $w(w+1)\neq 0$. Then a similar computation as in [5, p. 98] will show that $(t'W(y')t'^{-1}, t''W(y')t''^{-1})$ contains all e_{ij}'s. And as $t'xt'^{-1}=dxd^{-1}$ and $t''xt''^{-1}=x$, we obtain $R=S(t'W(x, y')t'^{-1}, t''W(x, y')t''^{-1})=S(t'xt'^{-1}, t'y't'^{-1}, x, t''y't''^{-1})$. Noting that S is a division ring, we readily see that $S(t't''^{-1})$ contains the inverse of $t't''^{-1}$. We have therefore our assertion $R=S(x, y, z)$, where $y=t''y't''^{-1}$ and $z=t't''^{-1}$.

By the latter part of Corollary 2, we can replace both $(*)$ and $(*_1)$ in

1) As is easily seen from the proof, $R=S(d, d', r, r')$, where d, d' and r, r' are conjugate in D and R respectively.

Mathematical Journal of Okayama University, Vol. 6 [1956], Iss. 2, Art. 5

(**b**) — (**m**) of §**1** by the local finite-dimensionality of $\mathfrak{G}=\mathfrak{G}(R/S)$, however before the restatement of these propositions, we shall insert one more supplementary remark about [15, Lemma 11] which played an important rôle in the previous theory.

Lemma 8. *Let S be a regular subring of a simple ring R, N be twofold regular in R and finite over S. If S is twofold regular in N and N contains $V_R(S)$ then S is twofold regular in R, $T=V_R(V_N(N))$ is simple, and there holds that $H=V_T(V_T(S))$ and $[T:H]=[V_T(S):V_T(T)]<\infty$, where H signifies $V_R(V_R(S))$.*

Proof. As $V_N(V_N(S))$ and $V_N(S)$ are simple by assumption, N is Galois and finite over $V_N(V_N(S))$. We see therefore $[V_N(S):V_N(N)]<\infty$, and the rest of the proof is the same with that of [15, Lemma 11].

In particular, if R is a division ring then [9, Lemma 6] takes the following simple form:

Corollary 3. *Let S be a division subring of a division ring R, N be a subring of R finite over S. If N contains $V_R(S)$ then, setting $T=V_R(V_N(N))$, there holds $H=V_T(V_T(S))$ and $[T:H]=[V_T(S):V_T(T)]<\infty$, where $H=V_R(V_R(S))$.*

Another easy consequence of Lemma 8 is (**f**), because if $\mathfrak{G}=\mathfrak{G}(R/S)$ is l. f. d. and non-outer, for any finite subset F in R, $N=S(\{\{e_{ij}\},F\}^{\mathfrak{G}}, V_R(S))$ is Galois and finite over S, whence $[V_N(S):V_N(N)]<\infty$, and so $V_N(V_N(S))$ is simple.

We shall restate here (**b**) — (**m**) of §**1** in our present terminologies:

(**b***) *Let R be locally finite over S. Then $\mathfrak{G}$ is l. f. d. if and only if either $\mathfrak{G}$ is outer or $[V_R(S):V_S(S)]<\infty$*[1].

(**d***) *Let $\mathfrak{G}$ be l. f. d. Then it is compact if and only if it is locally finite, or if it is almost outer. And it is discrete if and only if R is finite over S.*

(**e***) *If $\mathfrak{G}$ is l. f. d. then, for any intermediate regular subring R' finite over S, $J(\mathfrak{G}(R'),R)=R'$.*

(**f***) *If $\mathfrak{G}$ is l. f. d. then $H=V_R(V_R(S))$ is simple and $\mathfrak{G}(R/H)$ is l. f. d.*

(**g***) *If $\mathfrak{G}$ is l. f. d. then $\mathfrak{J}$ is dense in $\mathfrak{G}(R/H)$.*

(**h***) *Let $\mathfrak{G}$ be l. f. d. Then $\mathfrak{G}$ is locally compact if and only if $[V_R(S):V_R(R)]<\infty$.*

1) In (**b***)—(**m***), $\mathfrak{G}$ means $\mathfrak{G}(R/S)$, and for other notations, see the corresponding propositions in §**1**.

Accordingly we have:

(c*) *If* $\mathfrak{G}$ *is l. f. d. and locally compact then any closed regular subgroup of* $\mathfrak{G}$ *is a regular total subgroup, and conversely.*

(i*) *If* $\mathfrak{G}$ *is l. f. d. and locally compact then the outer group* $\mathfrak{G}(H/S)$ *coincides with* $\mathfrak{G}_H$.

(j*) *If* $\mathfrak{G}$ *is l. f. d. and locally compact then, for any intermediate regular subring* R', $J(\mathfrak{G}(R'), R) = R'$.

(k*) *If* $\mathfrak{G}$ *is l. f. d. and locally compact, there exists a one-to-one dual correspondence between closed regular subgroups of* $\mathfrak{G}$ *and intermediate regular subrings, in the usual sense of Galois theory.*

(l*) *If* $\mathfrak{G}$ *is l. f. d. and locally compact then, for any intermediate regular subring* R', *each S-isomorphism* ρ *of* R' *into* R *can be extended to an automorphism in* $\mathfrak{G}$, *where we assume that* $V_R(R'^{\rho})$ *is simple.*

Corollary 4. *If* $\mathfrak{G}$ *is l. f. d. and locally compact then, for any intermediate regular subring* R' *normal and Galois over* S, *the topological group* $\mathfrak{G}(R'/S)$ *is (topologically) isomorphic to* $\mathfrak{G}/\mathfrak{G}(R/R')$.

(m*) *If* $\mathfrak{G}$ *is l. f. d. and locally compact then an intermediate regular subring* T *with simple* $V_T(S)$ *is Galois over* S *if and only if* $\tilde{\mathfrak{T}}$ *is dense in* $\mathfrak{G}$.

In the sequel, we shall continue our consideration under the additional assumption that S is finite over the center $V_S(S)$. Our first lemma is the next:

Lemma 9. *Let* U *be a simple subring of a ring* T *(with an identity). If* $[U : V_U(U)] < \infty$ *then* $H = U \times_{V_U(U)} V_H(H)$, *where* $H = V_T(V_T(U))$. *In particular,* $[H : U] < \infty$ *if and only if* $[V_H(H) : V_U(U)] < \infty$.

Proof. As U is a central simple algebra (of finite rank) and H is an algebra over $V_U(U)$ $(\subset V_H(H))$, we have $H = U \times_{V_U(U)} V_H(U) = U \times_{V_U(U)} V_H(H)$ by a well-known theorem of Wedderburn[1]. The rest of the proof is clear.

As an easy consequence of the preceding lemma, we obtain the next ([15, Theorem 13]):

1) Wedderburn's theorem [1, Theorem 7.3F] asserts the following: If a central simple finite dimensional algebra A, over a field, is a subalgebra of an algebra B and if the identity element of A is also that of B, then $B = A \times_{V_A(A)} V_B(A)$.

Corollary 5. *Let R/S be Galois and $\mathfrak{G}(R/S)$ be l. f. d. and non-outer. If $[S: V_S(S)] < \infty$ then $H = V_R(V_R(S))$ is finite over S.*

Proof. $[V_R(S): V_S(S)] < \infty$ by (**b***), and so $[V_H(H): V_S(S)] < \infty$, whence it follows our assertion.

And we have the next which will contain [9, Theorem 8].

Theorem 3. *Let a simple ring R be locally finite over a simple subring S. If $V_R(S) = V_R(R)$ and S is finite over $V_S(S)$ then any intermediate subring is simple, and there exists a one-to-one correspondence between intermediate (simple) subrings R' and subfields Z' of $V_R(R)$ containing $V_S(S)$ under the relations $R' = S \times_{V_S(S)} Z'$ and $Z' = V_{R'}(R')$.*

Proof. At first we note $V_S(S) \subset V_{R'}(R') \subset V_R(R)$, and then we have $R' = S \times_{V_S(S)} V_{R'}(R')$ $(= S \times_{V_S(S)} V_{R'}(S))$ by Wedderburn's theorem. As R is locally finite over S, $V_R(R)$ is locally finite (algebraic) over $V_S(S)$, whence $V_{R'}(R')$ is a subfield of $V_R(R)$. This fact means also that R' is a simple ring. Conversely, if Z' is a subfield of $V_R(R)$ containing $V_S(S)$ then $S(Z') = S \times_{V_S(S)} Z'$ and $Z' = V_{S(Z')}(S(Z'))$.

In particular, if R/S is Galois (and $\mathfrak{G} = \mathfrak{G}(R/S)$ is outer) then the preceding theorem permits us to say that Galois theory of R/S can be reduced to that of $V_R(R)/V_S(S)$[1]. In case R is a division ring, the local finiteness of $\mathfrak{G}(R/S)$ and $[S: V_S(S)] < \infty$ imply that $\mathfrak{G}(R/S)$ is outer, however in the present case, it follows only that if $\mathfrak{G}(R/S)$ is non-outer then R is a complete matrix ring over a finite field. Excluding this finite case, we may regard Theorem 3 as a generalization of [9, Theorem 8].

Remark 2. As is easily seen from Theorem 1 (or (**b***)), if $\mathfrak{G}(R/V_R(R))$ is l. f. d. then R is finite over its center. This fact will suggest that Galois theory of $R/V_R(R)$ with really infinite $[R: V_R(R)]$ is different from our present one.

3. A generalization of the previous theory.

In this section, our consideration will proceed under the assumptions rather general which should be fulfilled when $\mathfrak{G}(R/S)$ is l. f. d. and locally compact. The author hopes that such a treating will be not mea-

1) In case $[R: S] < \infty$, $[S: V_S(S)] < \infty$ is equivalent with $[R: V_R(R)] < \infty$ ([14, Lemma]). Accordingly this is a well-known fact for algebras of finite rank.

ningless from the view-point to make clear the essential part of our theory.

Throughout this section we assume again $R=\sum^{n} De_{ij}$ is a simple ring which is Galois over a simple subring S. Now we consider the following condition:

(1) $$[V_R(S): V_R(R)] < \infty.$$

In case (1) is satisfied, as is well-known, $H=V_R(V_R(S))$ is simple and there holds $[R:H]=[V_R(S):V_R(R)]$, accordingly there exists a finite independent H-basis $\{r_1,\ldots,r_t\}$ of R. We set here $H=\sum^{m} Cd_{hk}$, where d_{hk}'s are matric units and $C=V_H(\{d_{hk}\})$ is a division ring, and consider the following condition:

(2) $S_0=S(\{e_{ij}\}, \{d_{hk}\}, \{r_v\})$ *is finite over* S.

Under these assumptions, we can easily see that S_0 is simple. (See the proof of Corollary 1.) In the rest of this section, unless otherwise specified, we assume the following conditions besides (1) and (2):

(3) $H=V_R(V_R(S))$ *is locally finite over* S.

(4) R *is Galois over* S_0.

And $\{r_1,\ldots,r_t\}$, H and S_0 will mean always these mentioned above.

Clearly by (**e***) and (**h***), the conditions (1), (2), (3) and (4) are satisfied if $\mathfrak{G}(R/S)$ is l. f. d. and locally compact, and which, in case R is a division ring, correspond to (β), (γ), (α) and (δ) in [9] respectively.

Lemma 10. *R is locally finite over S.*

Proof. Evidently $\mathfrak{G}^{(0)}=\mathfrak{G}(R/S_0)$ is outer and $H_0=H\cap S_0=J(\mathfrak{G}^{(0)}_H, H)$. Hence $\mathfrak{G}^{(0)}_H$ is a regular (outer) group of H/H_0. (Note here $V_H(H)=V_H(S)$.) Now let $F=\{a_u=\sum^{t} h_{uv}r_v\,;\ u=1,\ldots,w\,;\ h_{uv}\in H\}$ be an arbitrary finite set in R, and set $H^*=H_0(\{h_{uv}\})$, $\mathfrak{G}^*=\mathfrak{G}^{(0)}(H^*)$. Then there holds $J(\mathfrak{G}^*_H, H)=H^*$, because H^* is (simple and) finite over H_0. (Recall here that (3) and $V_H(S)=V_H(H)$ imply the local finite-dimensionality of $\mathfrak{G}(H/S)$, in virtue of (**a***).) From this fact, one can readily see $J(\mathfrak{G}^*, R)=\sum^{t}\oplus H^*r_v$, which is a subring containing $S(F)$ and finite over H^*, and so finite over S. We have proved therefore our lemma.

In the sequel, H_0 always signifies the simple subring $H\cap S_0$, and $\mathfrak{G}$ means $\mathfrak{G}(R/S)$.

The next propositions correspond to (ii), (iii) of [9, Lemma 9] respectively. The proofs of these can be similarly obtained, and may be left to readers.

Lemma 11. *There exists a one-to-one correspondence between subrings H' of H with $[H' : H_0] < \infty$ and subrings S' of R with $[S' : S_0] < \infty$ in the relations $H' = S' \cap H$ and $S' = \sum^{t} \oplus H' r_v$. In this correspondence, H' is $\mathfrak{G}(H/H_0)$-normal if and only if S' is $\mathfrak{G}(R/S_0)$-normal.*

Lemma 12. $\mathfrak{G}(H/H_0) = \mathfrak{G}(R/S_0)_H$ *and* $\mathfrak{G}(H/S) = \mathfrak{G}_H$.

Proof. We shall sketch here the outline of the proof. Consider the homomorphism $\varphi : \sigma \to \sigma_H$ of the compact Hausdorff group $\mathfrak{G}^{(0)} = \mathfrak{G}(R/S_0)$ into the compact Hausdorff group $\mathfrak{G}(H/H_0)$. Then Lemma 11 and the fact that $\mathfrak{G}_H^{(0)}$ is a regular group of H/H_0 show that φ is continuous and that $\mathfrak{G}_H^{(0)}$ is dense in $\mathfrak{G}(H/H_0)$ respectively, and so we have $\mathfrak{G}_H^{(0)} = \mathfrak{G}(H/H_0)$. Now, set $H' = S(H_0 \mathfrak{G})$, which is a simple subring of H that is normal and finite over S. If the finite group $\mathfrak{G}(H'/S)$ is induced by $\{\sigma^{(1)}, \ldots, \sigma^{(u)}\} \subset \mathfrak{G}$, then $\mathfrak{G}(H/S)$ is done by $\{\sigma^{(1)} \mathfrak{G}(\sum^{t} \oplus H' r_v), \ldots, \sigma^{(u)} \mathfrak{G}(\sum^{t} \oplus H' r_v)\}$, whence $\mathfrak{G}(H/S) = \mathfrak{G}_H$.

Then by [15, Lemma 9], we have the following:

Corollary 6. *If R' is a simple subring of H containing S then there holds $J(\mathfrak{G}(R'), R) = R'$.*

Now we shall prove three more lemmas.

Lemma 13. *If $\mathfrak{G}$ is l. f. d. and locally compact then, for any intermediate regular subring T containing $S_1 = S(\{e_{ij}\}) = \sum^{n} D_1 e_{ij}$, $\mathfrak{G}(R/T)$ is l. f. d.*

Proof. Since $\mathfrak{G}(D/D_1)$ is l. f. d. and $[V_D(D_1) : V_D(D)] = [V_R(S_1) : V_R(R)] < \infty$, D is totally locally finite over D_1 ([9, Theorem 9]). If we set $T = \sum^{n} E e_{ij}$ then, for any subring $T' = \sum^{n} E' e_{ij}$ containing T, T' is normal over T when and only when E' is normal over E. Our assertion is clear from these facts.

Lemma 14. *If R is locally finite over S* and *$\mathfrak{G}$ is outer then, for any intermediate simple subring T, $\mathfrak{G}(R/T)$ is l. f. d.*

Proof. Let $N = \sum^{n} E e_{ij} = \sum^{s} S f_k$ be a simple subring normal, finite over S and containing $S_1 = S(\{e_{ij}\})$. Then $T^* = T(N)$ is simple and normal over T by Corollary 1. Since $\mathfrak{G}$ is locally finite, the set $\{f_1, \ldots, f_s\}^{\mathfrak{G}}$ is finite, accordingly $\mathfrak{G}(R/T)_{T^*}$ is a finite outer group of T^*/T.

Hence $[T^*: T] < \infty$ by [10, Theorem 1]. Now, for any finite set F in R, we set $M = N(F^{\mathfrak{G}})$. Then $T^{**} = T^*(M)$ $(\supset S_1)$ is finite over T^* by Lemma 13, and so finite over T. As evidently T^{**} contains $T(F^{\mathfrak{G}(R/T)})$, our proof is complete.

Lemma 15. *For any intermediate regular subring S' finite over S, there holds $J(\mathfrak{G}(S'), R) = S'$.*

Proof. As any simple ring is regularly generated, we can choose such an independent $V_R(R)$-basis $\{v_1, \dots, v_t\}$ of $V_R(S)$ that all v_i's are regular elements. Then, by Lemma 11, $S_0(v_1, \dots, v_t) = \sum^t \oplus H' r_v$ with some H' finite over H_0. Now let H'' be a simple subring of H containing H' and normal, finite over S, and we set $\mathfrak{G}'' = \mathfrak{G}(\sum^t \oplus H'' r_v)$. If the finite group $\mathfrak{G}(H''/S) = \{\sigma^{(1)}, \dots, \sigma^{(u)}\}_{H''}$ with some $\sigma^{(i)}$'s in $\mathfrak{G}$ then $\mathfrak{G}(H/S) = \{\sigma^{(1)}\mathfrak{G}'', \dots, \sigma^{(u)}\mathfrak{G}''\}_H$. We set here $r_j^{\sigma^{(i)}} = \sum^t h_{ijv} r_v$ $(i = 1, \dots, u; j = 1, \dots, t)$ and let N be a simple subring of H containing $H''(\{h_{ijv}\})$ and normal, finite over S. If we set $M = \sum^t \oplus N r_v$ then $\hat{\mathfrak{G}} = \{\sigma \in \mathfrak{G}; M^\sigma = M\}$ contains $\{\sigma^{(1)}\mathfrak{G}'', \dots, \sigma^{(u)}\mathfrak{G}''\}$ as well as $\widetilde{V_H(S)}$[1] $(\supset \{\tilde{v}_1, \dots, \tilde{v}_t\})$, and so we have $J(\hat{\mathfrak{G}}, R) \subset J(\{\sigma^{(1)}\mathfrak{G}'', \dots, \sigma^{(u)}\mathfrak{G}''\}, R) \cap J(\{\tilde{v}_1, \dots, \tilde{v}_t\}, R) = J(\{\sigma^{(1)}\mathfrak{G}'', \dots, \sigma^{(u)}\mathfrak{G}''\}_H, H) = S$, that is, $J(\hat{\mathfrak{G}}, R) = S$.

Next we shall prove $(R/S, \hat{\mathfrak{G}})$ is l. f. d. Let $F = \{a_{u'} = \sum^t h_{u'v} r_v; u' = 1, \dots, w'; h_{u'v} \in H\}$ be an arbitrary finite set in R, and we set $E = \{h_{u'v}^\sigma; u' = 1, \dots, w'; v = 1, \dots, t; \sigma \in \mathfrak{G}\}$, which is evidently a finite set. Then $S(F^{\hat{\mathfrak{G}}}) \subset M(E)$, and $M(E)$ is finite over S. Hence $S(F^{\hat{\mathfrak{G}}})$ is finite over S.

Now we set $V_R(S) = \sum^t K g_{pq}$, $V_R(S') = \sum^{t'} K' g'_{p'q'}$, where g_{pq}'s, $g'_{p'q'}$'s are matric units and $K = V_{V_R(S)}(\{g_{pq}\})$, $K' = V_{V_R(S')}(\{g'_{p'q'}\})$ are division rings. Then we consider $S^* = S(\{\{e_{ij}\}, \{g_{pq}\}, \{g'_{p'q'}\}, b, S'\}^{\hat{\mathfrak{G}}})$ for any $b \in R \backslash S'$. Evidently S^* is a simple ring (Corollary 1), and is finite over S by the above remark. And, as is easily verified, $V_{S^*}(S)$ and $V_{S^*}(S')$ are simple rings by Lemma 1. Clearly $S^{*\hat{\mathfrak{G}}} = S^*$, and so S^* is Galois and finite over S. Hence by [10, Theorem 5], there exists some

1) In general, for a subset F in R, $\tilde{F}$ means the totality of inner automorsphism induced by regular elements constained in F.

Mathematical Journal of Okayama University, Vol. 6 [1956], Iss. 2, Art. 5

$\tau \in \mathfrak{G}(S^*/S)$ such that $b^\tau \neq b$ and $x^\tau = x$ for all $x \in S'$. As $\mathfrak{G}(S^*/S) = \hat{\mathfrak{G}}_{S^*} \cdot \widetilde{V_{S^*}(S)}$ by [15, Theorem 1], we can extend τ to an automorphism in $\mathfrak{G}(S')$, which proves $J(\mathfrak{G}(S'), R) = S'$.

Now Lemmas 10, 11, 12, 14 and 15 enable us to apply the same method as in the proof of [9, Theorem 9] to obtaint the next:

Theorem 4. *Under the conditions* (1) — (4), *for any intermediate regular subring* R', *there holds* $J(\mathfrak{G}(R'), R) = R'$, *and* R *is locally finite over* R'.

Proof. The proof is similar to that of [9, Theorem 9]. However, we state the proof briefly, for the sake of completeness.

In virtue of (1), there exists a simple subring R'' of R' containing S with $[R'' : S] < \infty$ and $V_R(R'') = V_R(R')$. By Lemma 10 and Lemma 15, the conditions (1) — (4) are satisfied with respect to R/R'', hence we can apply Corollary 6 for R'', $H'' = V_R(V_R(R''))$ instead of S, H respectively to obtain $J(\mathfrak{G}(R'), R) = R'$.

Next to prove the second part, it suffices to show that our assertion is true in the case where R' is contained in H. Set $H' = R'(H_0)$, which is a simple subring of H finite over R' by Lemma 14. Then, by Lemma 12, one will readily see $\sum^{t} \oplus H' r_v = J(\mathfrak{G}^{(0)}(H'), R)$, which shows that $\sum^{t} \oplus H' r_v$ is a simple subring containing R' as well as S_0 (Lemma 11), where $\mathfrak{G}^{(0)} = \mathfrak{G}(R/S_0)$. We obtain therefore $[\sum^{t} \oplus H' r_v : R'] = t \cdot [H' : R'] < \infty$. On the other hand, as $\mathfrak{G}(R/S_0)$ is outer also, R is locally finite over $\sum^{t} \oplus H' r_v$ again by Lemma 14, and hence so is over R'.

Clearly (j*) is a direct consequence of this theorem, and moreover we have the next:

Corollary 7. *If* $\mathfrak{G}$ *is l. f. d. and locally compact then, for any intermediate regular subring* R', $\mathfrak{G}(R/R')$ *is l. f. d., and the topology of* $\mathfrak{G}(R/R')$ *considered as the total group is equivalent to the topology induced in it as a subgroup of* $\mathfrak{G}$.

Remark 3. Applying the same method as in the proof of [11, Theorem 7], we obtain (c*) as an easy consequence of Theorem 4, however it seems to the present author that the proof given in [15] is more elementary.

In this connection, we shall show that [15, Theorem 16] is still valid under the conditions (1) — (4). For this end, the following lemmas will be required.

Lemma 16. *For any intermediate regular subring S' finite over S, each S-isomorphtsm ρ of S' into R can be extended to an automorphism in $\mathfrak{G}$, where we assume that $V_R(S'^{\rho})$ is simple.*

Proof. Here we make use of the same notations as in the proof of Lemma 15. Set $V_R(S'^{\rho})=\sum^{t''} K''g''_{p''q''}$, where $g''_{p''q''}$'s are matric units and $K''=V_{V_R(S'^{\rho})}(\{g''_{p''q''}\})$ is a division ring, and consider $S^{\S}=S(\{\{e_{ij}\}, \{g_{pq}\}, \{g'_{p'q'}\}, \{g''_{p''q''}\}, S', S'^{\rho}\}^{\hat{\mathfrak{G}}})$. Then we can easily verify as for S^* in the proof of Lemma 15 that $S^{\S}$ is a simple ring which is Galois and finite over S such that $\mathfrak{G}(S^{\S}/S)=\hat{\mathfrak{G}}_{S^{\S}}\cdot\widetilde{V_{S^{\S}}(S)}$, and that $V_{S^{\S}}(S')$ and $V_{S^{\S}}(S'^{\rho})$ are simple. The rest of the proof is therefore clear from [10, Theorem 6].

The proof of the next is, in virtue of Lemma 12 and Lemma 16, similarly obtained as in that of [15, Lemma 15].

Lemma 17. *If R' is a simple subring of H containing S, for any S-isomorphism ρ of R' into R, there holds $R'^{\rho}\subset H$, where we assume that $V_R(R'^{\rho})$ is simple.*

Now we are going to prove our last theorem of this section.

Theorem 5. *Under the conditions* (1) — (4), *for any intermediate regular subring R', each S-isomorphism ρ of R' into R can be extended to an automorphism in $\mathfrak{G}$, where we assume $V_R(R'^{\rho})$ is simple.*

Proof. By (1), we can find a simple subring R'' of R' finite over S with $V_R(R')=V_R(R'')$ and $V_R(R'^{\rho})=V_R(R''^{\rho})$. Since (1) — (4) are satisfied with respect to R/R'' by Lemmas 10 and 15, we see $\rho_{R''}=\sigma_{R''}$ for some $\sigma\in\mathfrak{G}$ by Lemma 16. Clearly $\rho\sigma^{-1}$ is an R''-isomorphism of R' into R and $V_R(R'^{\rho\sigma^{-1}})=(V_R(R'^{\rho}))^{\sigma^{-1}}$ is simple. And as $R''\subset R'\subset H''=V_R(V_R(R''))$, there holds $R'^{\rho\sigma^{-1}}\subset H''$ by Lemma 17. Hence, applying [15, Theorem 10] for R'', H'' instead of S, R respectively, we see $\rho\sigma^{-1}=\tau_{R'}$ for some $\tau\in\mathfrak{G}(H''/R'')=\mathfrak{G}(R/R'')_{H''}$ (Lemma 12), accordingly $\rho\sigma^{-1}=\tau'_{R'}$ with some $\tau'\in\mathfrak{G}(R/R'')$. Clearly $\tau'\sigma$ is a required extension of ρ.

Remark 4. In [9] and [15], we were open to the charge of unnecessary use of the assumption that a simple ring considered is Galois over a simple subring, however in the present paper, we try hard to consider our theory without the assumption as possible as one can. Moreover, as one will readily see, some lemmas (and definitions) in [9] and [15] become needless and some previous results will be shown somewhat briefly. And the validity of Theorems 1, 4, 5 and (h*) will permit us to come to the conclusion that Galois theory of division rings constructed in [9] has been thoroughly extended to simple rings.

4. A generalization of the notion of regular groups.

In his paper [2], G. Azumaya introduced the notion of strong regularity of ring elements: Let a be an element of a ring A. a is *strongly regular* if there exist $x, y \in A$ such that $a^2x = a$, $ya^2 = a$. And, making use of this notion, he proved several interesting properties of (strongly) π-regular rings[1]. Now we shall begin this section by setting the next lemma concerning strongly π-regular rings.

Lemma 18. *Let $\mathfrak{H}$ be a set of automorphisms in a strongly π-regular ring A. Then $T = J(\mathfrak{H}, A)$ is strongly π-regular too.*

Proof. Let t be an arbitrary element in T. Then t^n is strongly regular for some positive integer n. And so, by [2, Lemma 1], there exists a unique element $a \in A$ such that $t^n a = at^n$, $t^{2n}a = t^n$ and $t^n a^2 = a$. Clearly, for any $\sigma \in \mathfrak{H}$, we have $t^n a^\sigma = a^\sigma t^n$, $t^{2n} a^\sigma = t^n$ and $t^n (a^\sigma)^2 = a^\sigma$. Hence the uniqueness of a shows $a^\sigma = a$, whence a is contained in T. This proves that T is strongly π-regular.

It has been desired, throughout our consideration, to give some more intrinsic characterization of regular total subgroups. In relation to this request, we shall introduce here the next which is evidently a generalization of the notion of regular groups:

Definition 5. An automorphism group $\mathfrak{H}$ of a simple ring R is said to be $(*)$-*regular* if $V_{\mathfrak{H}} = V_R(J(\mathfrak{H}, R))$ is simple and $\mathfrak{H}$ contains $\tilde{V}_{\mathfrak{H}}$.

In the rest of this section, we assume R is a simple ring which is Galois over a simple subring S and $\mathfrak{G}$ signifies the total group $\mathfrak{G}(R/S)$.

Lemma 19. *Let R be finite over S. Then any $(*)$-regular subgroup $\mathfrak{H}$ of $\mathfrak{G}$ is regular.*

Proof. Denote by $\mathfrak{J}$ the totality of inner automorphisms in $\mathfrak{G}$,

1) A ring A is called a *π-regular ring* [*strongly π-regular ring*] if for any element $a \in A$ there exist an element $x \in A$ and a positive integer n such that $a^n x a^n = a^n$ [if there exist elements $x, y \in A$ and positive integers m, n such that $a^{m+1}x = a^m$, $ya^{n+1} = a^n$]. Of course a strongly π-regular ring is π-regular ([2], [4]).

then there holds $[\mathfrak{G}:\mathfrak{J}]\cdot[V_R(S):V_R(R)]=[R:S]$ by [15, Theorem 1]. Noting that $\mathfrak{H}/\mathfrak{H}\cap\mathfrak{J}\cong\mathfrak{H}\cdot\mathfrak{J}/\mathfrak{J}\subset\mathfrak{G}/\mathfrak{J}$ and that $V_{\mathfrak{H}}\subset V_R(S)$, we see that $\mathfrak{H}$ is regular in the sense of Nakayama [10], whence $J(\mathfrak{H}, R)$ is simple by [10, Theorem 1].

In case $\mathfrak{G}$ is l. f. d. and locally compact, the next will enable us to characterize regular total subgroups of $\mathfrak{G}$ in somewhat autonomous manner.

Theorem 6. *Let R be a simple ring which is Galois over a simple subring S. If $\mathfrak{G}=\mathfrak{G}(R/S)$ is l. f. d. and locally compact then any* (*)*-regular subgroup $\mathfrak{H}$ of $\mathfrak{G}$ is regular.*

Proof. By [3, Theorem13], a two-sided simple ring with an identity is primitive. And any primitive π-regular ring of bounded index is simple by [4, Theorem 2.3]. Hence, by the light of Lemma 18, it suffices to prove that $T=J(\mathfrak{H}, R)$ is two-sided simple, for a simple ring is (strongly) π-regular and of bounded index. To this end, we shall distinguish two cases: (i) $\mathfrak{G}$ *is outer*. In this case, we have $T=\bigcup_N(N\cap T)$, where N runs over all simple subrings which are normal and finite over S. As $V_N(N)=V_N(S)$, $\mathfrak{H}_N$ is a finite outer group, and so $N\cap T=J(\mathfrak{H}_N, N)$ is simple. Now the two-sided simplicity of T is an easy consequence of this fact. (ii) $\mathfrak{G}$ *is non-outer*. In this case, $[V_R(S): V_S(S)]<\infty$ by (**b***), so that there exists a simple subring S^* containing $V_R(S)$ and finite over S such that $V_R(T)=V_R(T\cap S^*)$ (by the local compactness of $\mathfrak{G}$). As is easily seen, we have $T=\bigcup_N(N\cap T)$, where N runs over all simple subrings containing S^* and normal, finite over S. Further, as $\mathfrak{H}_N$ is evidently (*)-regular, by Lemma 19, it follows that $T\cap N=J(\mathfrak{H}_N, N)$ is simple, accordingly T is two-sided simple.

Combining the above with (**c***), we obtain:

(**c****) *If $\mathfrak{G}$ is l. f. d and locally compact then any closed* (*)*-regular subgroup of $\mathfrak{G}$ is a regular total subgroup, and conversely.*

REFERENCES

[1] E. ARTIN, C. NESBITT and R. THRALL: Rings with minimum condition, Ann Arbor (1944).

[2] G. AZUMAYA: Strongly π-regular rings, J. Fac. Sci. Hokkaido Univ. Ser. I, 13 (1954), 34—39.

[3] N. JACOBSON: Structure theory of simple rings without finiteness assumptions, Trans. Amer. Math. Soc., 57 (1945), 228—245.

[4] I. KAPLANSKY: Topological representation of algebras, Trans. Amer. Math. Soc., 68 (1950), 62—75.

Mathematical Journal of Okayama University, Vol. 6 [1956], Iss. 2, Art. 5

[5] F. KASCH: Zu Erzeugug separabler Algebren, Norsk Math. Tidsskr., 34 (1952), 97—99.
[6] ————: Über den Endomorphismenring eines Vektorraumes und den Satz von Normalbasis, Math. Ann., 126 (1953), 447—463.
[7] ————: Eine Bemerkung über innere Automorphismen, Math. J. Okayama Univ., to appear.
[8] T. NAGAHARA: On Generating elements of Galois extensions of division rings, Math. J. Okayama Univ., to appear.
[9] T. NAGAHARA and H. TOMINAGA: On Galois theory of division rings, Math. J. Okayama Univ., 6 (1956), 1—21.
[10] T. NAKAYAMA: Galois theory of simple rings, Trans. Amer. Math. Soc., 73 (1952), 276—292.
[11] N. NOBUSAWA: An extension of Krull's Galois theory to division rings, Osaka Math. J., 7 (1955), 1—6.
[12] ————: On compact Galois groups of division rings. Osaka Math. J., 8 (1956), 43—50.
[13] A. ROSENBERG and D. ZELINSKY: Galois theory of continuous transformation rings, Trans. Amer. Math. Soc., 79 (1955), 429—452.
[14] H. TOMINAGA: On a theorem of N. Jacobson, Proc. Jap. Acad., 31 (1955), 653—654.
[15] ————: Galois theory of simple rings, Math. J. Okayama Univ., 6 (1956), 29—48.

DEPARTMENT OF MATHEMATICS,
OKAYAMA UNIVERSITY

(*Received January 31, 1957*)