

Mathematical Journal of Okayama University

Volume 6, Issue 2

1956

Article 8

MARCH 1957

Compositions of linear topological spaces

Minoru Tomita*

*Okayama University

Copyright ©1956 by the authors. *Mathematical Journal of Okayama University* is produced by
The Berkeley Electronic Press (bepress). <http://escholarship.lib.okayama-u.ac.jp/mjou>

COMPOSITIONS OF LINEAR TOPOLOGICAL SPACES

MINORU TOMITA

Introduction

A topology of a linear space $\mathcal{L}$ is determined by a system of semi-norms on $\mathcal{L}$, then a conception of a composition of topological representations of $\mathcal{L}$ may be strictly and naturally formed by that of semi-norms of $\mathcal{L}$. In this point of view we shall develop a "composition theory of semi-norms" which lays its foundation on three elementary properties of semi-norms.

(1) A semi-norm is a functional on $\mathcal{L}$, and a space of semi-norms allows some rules of compositions of functionals. For instance, if $s_1, \dots, s_n$ are semi-norms, $q = \max(s_1, \dots, s_n)$ and $r = (s_1^p + s_2^p + \dots + s_n^p)^{1/p}$ ($p \geq 1$) are semi-norms. More generally, if a function of n real variables $k(x_1, \dots, x_n)$ satisfies a suitable condition (c. f. §4), the composite functional $k(s_1, \dots, s_n)$ is a semi-norm.

(2) A semi-norm s on $\mathcal{L}$ determines uniquely a representation of $\mathcal{L}$ on another normed space $\mathcal{L}(s)$, then a composition $p = k(s_1, \dots, s_n)$ of semi-norms $s_1, \dots, s_n$ defines a composition of the representative normed spaces $\mathcal{L}(p) = k(\mathcal{L}(s_1), \dots, \mathcal{L}(s_n))$.

(3) A space of semi-norms on $\mathcal{L}$ is a topological space, which has a weak topology as a space of functionals on $\mathcal{L}$. The general Heine-Borel principle (The equivalency between bounded-closedness and compactness) is valid even in the space of semi-norms. (Theorem 1 in §1). It enables us to extend the former algebraical compositions to transcendental compositions $\sup_{\lambda \in \Delta} \lambda, (\int_{\Delta} \lambda^p d\rho(\lambda))^{1/p}; \sup_{\lambda \in \Delta} \mathcal{L}(\lambda), (\int_{\Delta} \mathcal{L}(\lambda)^p d\rho(\lambda))^{1/p}$ etc. on bounded sets of semi-norms and corresponding sets of normed spaces, respectively.

§1 treats a topological space of semi-norms and the general Heine-Borel principle, §2 and 3 prepare the correspondence theory between semi-norms and representations. In §4 we introduce a concept of a composition of semi-norms in a wide sense. §5 and §6 treats some sets of semi-norms convex by some operations of compositions, and their extremal problems. §7 determines the composition law of representative normed spaces and their dual spaces which correspond to compositions of semi-norms.

§1. Topology and order of semi-norms.

We consider a fixed (real or complex) linear space $\mathcal{L}$. The *totality of complex functionals* on $\mathcal{L}$ is a topological space by its *weak topology*. A *weak neighbourhood* of a functional φ is a set of functionals $U(\varphi: f_1, \dots, f_n, \varepsilon) = \{\psi: |\varphi(f_i) - \psi(f_i)| < \varepsilon, 1 \leq i \leq n\}$. If φ and ψ are real functionals, the *order* $\varphi \geq \psi$ is defined by $\varphi(f) \geq \psi(f)$ ($f \in \mathcal{L}$). The *general Heine-Borel Theorem* is valid in a space of functionals.

Lemma 1.1. *A set A of functionals is compact if and only if it is closed and bounded.*

In fact, if A is absolutely bounded by a functional ψ , A is contained in the compact space $U(\psi) = \{\varphi: \varphi \leq \psi\}$. $U(\psi)$ is the Tychonoff product $\prod_{f \in \mathcal{L}} \{ \varphi(f) : |\varphi(f)| \leq \psi(f) \}$ of closed circulars in the complex domain defined for respective f in $\mathcal{L}$. Then a bounded closed set is compact.

Conversely, assume A be compact. Every f in $\mathcal{L}$ determines the (continuous) *coordinate function* $\varphi \rightarrow \varphi(f)$ of variable φ defined on the totality of functionals on $\mathcal{L}$. Then the image $\{\varphi(f): \varphi \in A\}$ of A is bounded closed, and A has an absolute bound $\psi(f) = \sup_{\varphi \in A} |\varphi(f)|$.

A *semi-norm* s of $\mathcal{L}$ is a functional on $\mathcal{L}$ so that (1). $s(f) \geq 0$, (2). $s(\alpha f) = |\alpha|s(f)$, and (3). $s(f+g) \leq s(f) + s(g)$. By the continuity of the coordinate functions $s \rightarrow s(f)$, the *totality P of semi-norms on $\mathcal{L}$* is a closed set of functionals, then the Heine-Borel Theorem is valid in P as well.

Theorem 1. *A set of semi-norms is compact if and only if it is bounded and closed.*

§ 2. Relations between semi-norms and representations.

If $f \rightarrow f_s$ is a representation (a linear mapping) of $\mathcal{L}$ on another normed linear space $\mathcal{L}(s)$, the functional s on $\mathcal{L}: s(f) = |f_s|$ (the norm of f_s) is a semi-norm on $\mathcal{L}$. The *kernel* $\mathcal{N}(s)$ of the representation, $\mathcal{N}(s) = \{f \in \mathcal{L}: f_s = 0\}$, is identical with the *zero-point-set* ($f \in \mathcal{L}: s(f) = 0$) of s .

Conversely, if s is a semi-norm on $\mathcal{L}$, its zero-point-set $\mathcal{N}(s)$ is a linear space, and the residue space $\mathcal{L}(s) = \mathcal{L} / \mathcal{N}(s)$ is a normed space. The natural mapping $f \in \mathcal{L} \rightarrow f_s = f / \mathcal{N}(s)$ is called the *canonical representation* of $\mathcal{L}$ by s , and the norm $|f_s|$ in $\mathcal{L}(s)$ is determined by $|f_s| = s(f)$.

The totality $\mathcal{L}^*$ of linear functionals on $\mathcal{L}$ is a linear space, $\mathcal{L}^*$ is clearly a closed set of functionals on $\mathcal{L}$, in which the Heine-Borel Theorem is preserved.

Lemma 2.1. *The totality $F(\mathcal{L}^*)$ of weakly closed sub-sets of $\mathcal{L}^*$ is a topological space. A complete system of neighbourhoods of a $W \in F(\mathcal{L}^*)$ consists of those sub-sets of $F(\mathcal{L}^*)$: $\mathfrak{B}(W: f_1, \dots, f_n, \varepsilon) = (X \in F(\mathcal{L}^*): X \subseteq U(W: f_1, \dots, f_n, \varepsilon) \text{ and } W \subseteq U(X: f_1, \dots, f_n, \varepsilon))$. $U(W: f_1, \dots, f_n, \varepsilon)$ denotes the set theoretical sum of the totality of $U(s: f_1, \dots, f_n, \varepsilon)$ neighbourhoods of $s \in W$.*

If s is a semi-norm, the dual Banach space of $\mathcal{L}(s)$ is denoted by $\mathcal{L}^*(s)$. $\mathcal{L}^*(s)$ is the totality of $\varphi \in \mathcal{L}^*$ so that $|\varphi(f)| \leq s(f)$ for every $f \in \mathcal{L}$. The norm $s^*(\varphi)$ of φ in $\mathcal{L}^*(s)$ is $s^*(\varphi) = \sup_{s(f) \leq 1} |\varphi(f)|$. The following fundamental relations between semi-norms s and unit spheres $U(s)$ of $\mathcal{L}^*(s)$ is well-known.

Lemma 2.2. *The unit sphere $U(s)$ of $\mathcal{L}^*(s)$ is a weakly compact convex and symmetric (i. e. $\varphi \in U(s)$ and $|\alpha| = 1$ imply $\alpha\varphi \in U(s)$) subset of $\mathcal{L}^*$. $s \leftrightarrow U(s)$ is a one-to-one correspondence between the totality $\mathbf{P}$ of semi-norms and the totality $\mathbf{U}$ of weakly compact convex and symmetric sub-sets of $\mathcal{L}^*$.*

$\mathbf{U}$ is a topological space, as a sub-space of $F(\mathcal{L}^*)$. Our interest is how the order and the topology of $\mathbf{P}$ are transposed to those of $\mathbf{U}$.

Theorem 2. *The correspondence $s \leftrightarrow U(s)$ between $\mathbf{P}$ and $\mathbf{U}$ is an order-preserving ($r \geq s$ if and only if $U(r) \supseteq U(s)$) homeomorphism. A neighbourhood $U(s: f_1, \dots, f_n, \varepsilon)$ of an $s \in \mathbf{P}$ corresponds to the neighbourhood $\mathfrak{B}(U(s): f_1, \dots, f_n, \varepsilon)$ of $U(s)$.*

Proof. We only need to prove that $|r(f) - s(f)| < \varepsilon$ is equivalent to $U(U(r): f, \varepsilon) \supseteq U(s)$ and $U(U(s): f, \varepsilon) \supseteq U(r)$. If $|r(f) - s(f)| < \varepsilon$, every $\varphi \in U(s)$ satisfies $|\varphi(f)| \leq s(f) \leq r(f) + \varepsilon$, while we can choose a $\psi \in U(r)$ with $r(f) = \psi(f)$. There is a suitable number α ($|\alpha| \leq 1$) so that $|\varphi(f) - \alpha\psi(f)| < \varepsilon$ and $\varphi \in U(\alpha\psi: f, \varepsilon) \subseteq U(U(r): f, \varepsilon)$. This means $U(s) \subseteq U(U(r): f, \varepsilon)$ and $U(r) \subseteq U(U(s): f, \varepsilon)$. Conversely if $U(s) \subseteq U(U(r): f, \varepsilon)$ and $U(r) \subseteq U(U(s): f, \varepsilon)$, $s(f) = \max_{\varphi \in U(s)} |\varphi(f)|$ and $r(f) = \max_{\psi \in U(r)} |\psi(f)|$ satisfies $|r(f) - s(f)| < \varepsilon$. Hence $s \rightarrow U(s)$ is an homeomorphism which maps the set $U(s: f_1, \dots, f_n, \varepsilon)$ to the set $\mathfrak{B}(U(s): f_1, \dots, f_n, \varepsilon)$.

§ 3. Orderly semi-norms on vector lattices.

We apply the result of the former § to semi-norms on vector lattices. Let $\mathcal{Q}$ denote a compact space and $C(\mathcal{Q})$ denote the totality of continuous functions on it. A semi-norm s on $C(\mathcal{Q})$ is said to be *orderly* if $s(f) \geq s(g)$ whenever $|f(\lambda)| \geq |g(\lambda)|$ on $\mathcal{Q}$. Generally, a normed lattice with a unit is represented as a uniformly dense sub-lattice of a suitable $C(\mathcal{Q})$ with an orderly norm, then we shall treat a normed lattice under such a concrete representation.

Among many orderly semi-norms of $C(\mathcal{Q})$, the following semi-norms are well-known.

(1) Let μ be a regular measure on $\mathcal{Q}$. Then $|\mu|_p(f) = \left(\int |f(\lambda)|^p d\mu(\lambda) \right)^{\frac{1}{p}}$ ($p \geq 1$) is an orderly semi-norm. We call it an L^p -semi-norm on $C(\mathcal{Q})$.

(2) If X is a closed sub-set of $\mathcal{Q}$, $|X|(f) = \sup_{\lambda \in X} |f(\lambda)|$ is an orderly semi-norm. We call it an L^∞ -semi-norm.

$C(\mathcal{Q})$ is always treated as a Banach space with the norm $\|f\| = |\mathcal{Q}|(f) = \sup |f(\lambda)|$. The dual Banach space $C^*(\mathcal{Q})$ of $C(\mathcal{Q})$ consists of the totality of completely additive set functions on the Borel field in $\mathcal{Q}$. Every $\varphi \in C^*(\mathcal{Q})$ has the Lebesgue decomposition $\varphi = \varphi^+ + \varphi^-$. The *total variation* of φ : $|\varphi| = \varphi^+ + \varphi^-$ is a regular measure on $\mathcal{Q}$.

A regular measure is called *normalized* if its total mass is 1. The totality $\mathcal{M}(\mathcal{Q})$ of normalized regular measures on $\mathcal{Q}$ is a bounded regularly convex sub-set of $C^*(\mathcal{Q})$.

We recall some elementary properties of orderly semi-norms.

(3.1) An orderly semi-norm s is smaller than the norm $s(1)|\mathcal{Q}|$. In fact, $|f(\lambda)| \leq \|f\|$ implies $s(f) \leq s(\|f\|1) = \|f\|s(1)$.

(3.2) An orderly semi-norm s is said to be *normalized* if $s(1) = 1$. The totality P_1 of normalized orderly semi-norms on $C(\mathcal{Q})$ is bounded, closed and compact; the totality P of orderly semi-norms s with $s(1) \leq 1$ is as well.

Lemma 3.1. The totality L^p of normalized L^p -semi-norms on $C(\mathcal{Q})$ is compact (for each $1 \leq p \leq \infty$).

Proof. In case $p < \infty$, the Lemma follows from that L^p is the range of the weakly continuous mapping $\mu \rightarrow |\mu|_p$ on the weakly compact space $\mathcal{M}(\mathcal{Q})$ of normalized regular measures.

In case $p = \infty$, the Lemma follows from the I. Gelfand's theorem on normed algebras that a necessary and sufficient condition for an (orderly)

semi-norm s to be an L^∞ -semi-norm is $s(fg) \leq s(f)s(g)$ and $s(f^2) = s(f)^2$ for f, g in $C(\Omega)$.

(3.3) We denote by $C(s)$ the normed space canonically represented by an orderly semi-norm s . The dual space $C^*(s)$ of $C(s)$ is contained in $C^*(\Omega)$. The total variation $|\varphi| = \varphi^+ - \varphi^-$ of every element φ in $C^*(s)$ belongs to $C^*(s)$ and has the same norm $s^*(|\varphi|) = s^*(\varphi)$ with φ .

In fact, $|\varphi|(|f|)$ is determined by $|\varphi|(|f|) = \sup_{|f| \geq 1, |f| \geq 0} |\varphi(f)|$. Thus $|\varphi|(|f|) \leq |\varphi|(|f|) = \sup_{|f| \geq 1, |f| \geq 0} |\varphi(f)| \leq \sup_{s(f) \geq s(0)} |\varphi(f)| = s^*(\varphi)s(f)$. Then $s^*(|\varphi|) \leq s^*(\varphi)$. On the other hand, $|\varphi(f)| \leq |\varphi|(|f|)$ implies $s^*(\varphi) \leq s^*(|\varphi|)$ and $s^*(\varphi) = s^*(|\varphi|)$.

(3.4) Let $U^+(s)$ denote the totality of regular measures contained in the unit sphere of $C^*(s)$ of an orderly semi-norm s . For each $f \in C(\Omega)$ we can choose $\mu \in U^+(s)$ so that $s(f) = \mu(|f|) = |\mu|_1(f)$.

In fact, let φ be an element of $C^*(s)$ so that $s^*(\varphi) = 1$ and $s(f) = \varphi(f)$, then its total variation $|\varphi|$ belongs to $U^+(s)$ and satisfies $|\varphi|(|f|) \geq |\varphi(f)| = s(f)$ and $|\varphi|(|f|) = s(f)$.

When μ, ν are regular measures, we say $\mu \leq \nu$ if $\nu - \mu$ is a measure. A set V of regular measures is said to be a *star* if $\mu \in V$ and $\mu \geq \nu$ implies $\nu \in V$. The $U^+(s)$ of an orderly semi-norm s is clearly a star.

Theorem 3. $s \leftrightarrow U^+(s)$ is a one-to-one correspondence between $\mathbf{P}^+$ and the totality U^+ of regularly convex stars of regular measures on Ω with masses ≤ 1 .

Proof. By 3.7, every $s \in \mathbf{P}^+$ satisfies $s(f) = \sup_{\mu \in U^+(s)} |\mu|_1(f)$ and $s = \sup_{\mu \in U^+(s)} |\mu|_1$, then $s \leftrightarrow U^+(s)$ is one-to-one. It is sufficient to say that every $V \in U^+$ is a $U^+(s)$ of an $s \in \mathbf{P}^+$. The set $U = \{\varphi \in C^*(\Omega) : |\varphi| \in V\}$ is regularly convex and symmetric in $C^*(\Omega)$, then it determines a semi-norm s on $C(\Omega)$ whose unit sphere $U(s)$ of the space $C^*(s)$ is U . s is orderly. In fact, $\varphi \in U$ implies $|\varphi| \in U$, where $|\varphi|(|f|) = \sup_{|\varphi| \geq 1, |\varphi| \geq 0} |\varphi(f)|$ and $s(f) = \sup_{\varphi \in U} |\varphi(f)| = \sup_{\varphi \in U} |\varphi|(|f|) = \sup_{\varphi \in U} \int |f(\lambda)| d|\varphi|(\lambda)$. Then V is identical with $U^+(s)$, the totality of measures in V .

Corollary. $s \leftrightarrow U^+(s)$ is an order-preserving homeomorphism between $\mathbf{P}^+$ and U^+ .

The topology of U^+ is in the sense of Lemma 2.1.

§ 4. Concept of a composition.

We turn once more to the linear space $\mathcal{L}$ in §1. Let k be an orderly semi-norm on the n -dimensional Euclidean space $\mathcal{E}^n$. k is a function of n variables $k(x_1, \dots, x_n)$ so that (1) $k(x_1, \dots, x_n) \leq k(y_1, \dots, y_n)$ whenever $|x_i| \leq |y_i|$ ($1 \leq i \leq n$). (2) $k(\alpha x_1, \dots, \alpha x_n) = |\alpha| k(x_1, \dots, x_n)$ and (3) $k(x_1 + y_1, \dots, x_n + y_n) \leq k(x_1, \dots, x_n) + k(y_1, \dots, y_n)$. If $s_1, \dots, s_n$ are n semi-norms on $\mathcal{L}$, the composite functional $q = k(s_1, \dots, s_n)$ (that is, $q(f) = k(s_1(f), \dots, s_n(f))$) is a semi-norm. We call it a *composite semi-norm* or *composition* of $s_1, \dots, s_n$. A composition is called *normalized* if $k(1, \dots, 1) = 1$. In particular, $q = (\sum \alpha_i s_i^p)^{\frac{1}{p}}$ and $r = \max(s_1, \dots, s_n)$ are called an L^p -sum and an L^∞ -sum of those s_i , respectively.

More generally a composition is defined in a bounded set of semi-norms. Let A be a bounded set of semi-norms. A function on A is uniformly continuous¹⁾ if and only if it is extended to a continuous function on the closure $\bar{A}$ of A . Then the space $C(\bar{A})$ is identical with the totality of uniformly continuous functions on A . Let k be an orderly semi-norm on $C(\bar{A})$. If $f \in \mathcal{L}$ is fixed, the function $|f_\lambda| = \lambda(f)$ of variable λ in A is uniformly continuous on A . A *composition* on A is a semi-norm on $\mathcal{L}$ determined by $k^\Lambda(f) = k(|f_\lambda|)$. A composition k is called *normalized* if the semi-norm k is normalized. A composition by an L^p -semi-norm on $C(\mathcal{Q})$ is called an L^p -sum of A . If A is compact, every L^p -sums are expressed as follows.

Lemma 4.1. *If A is a compact set of semi-norms, then*

(1) *An L^p -sum ($p < \infty$) $|\mu|_p^\Lambda$ on A is a weak integral by a regular measure μ on A ; $|\mu|_p^\Lambda = (\int \lambda^p d\mu(\lambda))^{\frac{1}{p}}$.*

(2) *An L^∞ -sum $|X|^\Lambda$ is the least upper bound $|X|^\Lambda = \sup_{\lambda \in X} \lambda$ (i.e. $|X|^\Lambda(f) = \sup_{\lambda \in X} \lambda(f)$) on a closed set X in A .*

Usually every L^p -sum on a bounded set A of semi-norms is treated as an L^p -sum on the closure $\bar{A}$. But, if necessary, we represent an L^p -sum ($p < \infty$) by a *weak Radon integral* on A .

A finitely additive set-function defined on the totality of sub-sets of a topological space $\mathcal{Q}$ is called a *Radon measure*.

The Radon integral exists for every bounded function f on $\mathcal{Q}$.

Lemma 4.2. *If s is an L^p -sum ($p < \infty$) on a bounded set A of semi-norms, s is a weak Radon integral $s = (\int_A \lambda^p d\rho(\lambda))^{\frac{1}{p}}$ by a suitable Radon measure ρ on A .*

1) Uniformly continuous by the uniform structure of the weak topology.

Proof. The space $C(\bar{A})$, regarded as the totality of uniformly continuous functions on A , is contained in the totality $B(A)$ of bounded functions on A . If ρ is a Radon measure, the Radon integral $\rho(f) = \int f d\rho$ is a positive linear functional on $B(A)$. Its restriction on $C(\bar{A})$ is a positive linear functional on $C(\bar{A})$, and represents a regular measure μ on $\bar{A}$. Then the weak Radon integral $(\int \lambda(f)^p d\rho(\lambda))^{\frac{1}{p}}$ is equal to an L^p -sum $(\int \lambda(f)^p d\mu(\lambda))^{\frac{1}{p}} = |\mu|_p^{\frac{1}{p}}(f)$.

The "Extension theorem of the measure" mentions that (See the next Remark) every positive linear functional on $C(\bar{A})$ is extended to a positive linear functional (a Radon integral) on $B(A)$. Then every L^p -sum is a weak Radon integral on A .

Remark. The following extension theorem is often useful.

Theorem 4. *Let A be a topological space, $\mathcal{Q}$ be a bounded set in the dual space of a Banach space and $Co(\mathcal{Q})$ be the smallest regularly convex set which contains $\mathcal{Q}$. If $\lambda \rightarrow T\lambda$ is a mapping of A on $\mathcal{Q}$, every element φ of $Co(\mathcal{Q})$ is a weak Radon integral $\varphi = \int T\lambda d\rho(\lambda)$ by a suitable Radon measure with the total mass 1.*

If $\mathcal{Q}$ is compact and if $\lambda \rightarrow T\lambda$ is continuous, the measure ρ can be chosen as a normalized regular measure on $\mathcal{Q}$.

Proof. The totality $\mathbf{R}$ of Radon measures on $\mathcal{Q}$ with the total mass 1 is the smallest regularly convex sub-set of the dual space $B^*(\mathcal{Q})$ of $B(\mathcal{Q})$, which contains the totality of point measures $\delta_\lambda: \int f d\delta_\lambda = f(\lambda)$ ($\lambda \in \mathcal{Q}$). Since $\rho \in \mathbf{R} \rightarrow \int T\lambda d\rho(\lambda)$ is weakly continuous on $\mathbf{R}$, its Range is weakly compact, convex and contains every $T\lambda = \int T\lambda d\delta_\lambda(\lambda)$ ($\lambda \in \mathcal{Q}$). Then it is identical with the set $Co(A)$.

If $\mathcal{Q}$ is compact and if the mapping is continuous, a weak Radon integral $\int T\lambda d\rho(\lambda)$ is equal to a weak integral $\int T\lambda d\mu(\lambda)$ by a regular measure μ . μ is the restriction of the positive linear functional ρ on $B(\mathcal{Q})$ within the space $C(\mathcal{Q})$.

Corollary. (Extension theorem of the measure). *Let $\mathcal{Q}$ be a topological space, and A be a compact space. If $\lambda \rightarrow T\lambda$ is a mapping of $\mathcal{Q}$ in a dense sub-space of A , every regular measure μ on A , as a positive linear functional on $C(A)$, is extended to a Radon integral ρ*

on $B(\Omega)$ so that

$$\int_{\Omega} f(T\lambda) d\rho(\lambda) = \int_{\Lambda} f(\nu) d\mu(\nu).$$

If Ω is compact, and if T is continuous, ρ is chosen as a regular measure on Ω .

Proof. We apply Theorem 4 to the mapping $\lambda \rightarrow \delta_{T\lambda}$. The point mass $\delta_{T\lambda}$ at $T\lambda$ in A is a positive linear functional on $C(\Omega)$. Then every positive linear functional μ on $C(A)$ in the set $X = \text{Co}(\delta_{T\lambda} : \lambda \in \Omega)$ is a weak Radon integral $\int \delta_{T\lambda} d\rho(\lambda)$. X is the totality of normalized regular measures on A , and every normalized regular measure μ is a weak integral $\mu = \int \delta_{T\lambda} d\rho(\lambda)$. Then for every $f \in C(A)$,

$$\int f d\mu = \int \delta_{T\lambda}(f) d\rho(\lambda) = \int f(T\lambda) d\rho(\lambda).$$

§ 5. Convex sets of semi-norms.

A set A of semi-norms is called L^p -sum convex ($1 \leq p \leq \infty$) if every normalized L^p -sum of every pair of elements in A belongs to A . A is called *universally convex* if it is L^1 -sum convex, and $s, t \in A$ imply $\max(s, \alpha t) \in A$ for every $0 \leq \alpha \leq 1$.

Theorem 5. *The totality $\text{Co}^p(A)$ of normalized L^p -sums on a bounded set A of semi-norms on $\mathcal{L}$ is the smallest closed L^p -sum convex set which contains A .*

Theorem 6. *The totality $\text{Co}^u(A)$ of normalized compositions on a bounded set A of semi-norms on $\mathcal{L}$ is the smallest closed universally convex set which contains A .*

Proof of Theorem 5. In case $p < \infty$, $\text{Co}^p(A)$ is the range of the weakly continuous mapping $\mu \rightarrow |\mu|_p^{\frac{1}{p}} = (\int \lambda^p d\mu(\lambda))^{\frac{1}{p}}$ defined on the totality $\mathcal{M}(A)$ of normalized regular measures on A . This mapping transposes the addition $\alpha\mu + \beta\nu$ ($\alpha + \beta = 1$) in $\mathcal{M}(A)$ to the L^p -sum operation $\{\alpha(|\mu|_p^{\frac{1}{p}})^p + \beta(|\nu|_p^{\frac{1}{p}})^p\}^{\frac{1}{p}}$ in $\text{Co}^p(A)$. $\mathcal{M}(A)$ is the smallest weakly compact convex set which contains those point measures δ_λ at λ in A , then $\text{Co}^p(A)$ is the smallest compact L^p -sum convex set which contains all elements $\nu = (\int \lambda^p d\delta_\nu(\lambda))^{\frac{1}{p}}$ in A .

In case $p = \infty$, $Co^\infty(A)$ is compact as the image of the continuous mapping $k \rightarrow |k|^\Delta$ of the compact set of the totality of L^∞ -semi-norms on $C(\bar{A})$. (c. f. Lemma 3.2), then it is sufficient to see that $Co^\infty(A)$ contains everywhere densely those L^∞ -sums $\max(\lambda_1, \dots, \lambda_n)$ of finite elements in A , that is, every neighbourhood $U(|X|^\Delta: f_1, \dots, f_n, \varepsilon)$ of an L^∞ -sum $|X|^\Delta = \sup_{\lambda \in X} \lambda$ (the least upper bound on a closed set X in A) contains such an L^∞ -sum $\max(\lambda_1, \dots, \lambda_n)$. Let ν_i be a point in X so that $\nu_i(f_i) = \sup_{\lambda \in X} \lambda(f_i) = |X|^\Delta(f_i)$, then we can choose $\lambda_1, \dots, \lambda_n$ in A so that $|\nu_i(f_j) - \lambda_i(f_j)| < \varepsilon$ ($i, j = 1, 2, \dots, n$). The $\max(\lambda_1, \dots, \lambda_n)$ is a required one. Thus the Theorem is valid in every case.

Proof of Theorem 6. By (3.4) in §3 an orderly seminorm k satisfies $k(f) = \sup_{\mu \in U^+(k)} \int |f| d\mu = \sup_{\mu \in U^+(k)} |\mu|_1(f)$. Then the composition k^Δ is a L^∞ -sum of L^1 -sums on A :

$$k^\Delta(f) = \sup_{\mu \in U^+(k)} |\mu|_1(|f|) = \sup_{\mu \in U^+(k)} |\mu|_1^\Delta(f).$$

Lemma 5.2. *Every composition k^Δ on A is a L^∞ -sum of a system of L^1 -sums on A .*

Let A be a bounded set and let W denote the smallest universally convex set which contains A . Then $W = Co^p(W)$ for $p = 1$ and $p = \infty$. Let k^Δ be an arbitrary normalized composition. Then k is a L^∞ -sum of a suitable set Ω of L^1 -sums $|\mu|_1^\Delta$ on A by measures μ with masses $\mu(A) \leq 1$. On the other hand k^Δ is normalized, and there is at least one measure ν in $U^+(k)$ so that $k(1) = |\nu|_1(1) = 1$. Then k is a L^∞ -sum $k = \sup_{\mu \in \Omega} (\max(|\nu|_1^\Delta, |\mu|_1^\Delta))$. Every $|\mu|_1^\Delta$ is a multiplication $|\mu|_1^\Delta = \alpha \mu_0$ of $0 \leq \alpha \leq 1$ and a normalized L^1 -sum μ_0 in A . Since $|\nu|_1^\Delta$ and μ_0 belong to $Co^1(A) \subseteq W$, $\max(|\nu|_1^\Delta, |\mu|_1^\Delta) = \max(|\nu|_1^\Delta, \alpha \mu_0)$ belongs to W (by the definition of the universal convexity), then $k = \sup_{\mu \in \Omega} \max(|\nu|_1^\Delta, |\mu|_1^\Delta)$ belongs to $Co^\infty(W) = W$.

§ 6. An extremal theorem of semi-norms.

An element of an L^p -sum convex compact set A of semi-norms is called *L^p -sum indecomposable* if, whenever s be a normalized L^p -sum of two elements q, r in A , we have either $s = q$ or $s = r$. The next Theorem will be applied to decomposing operator algebras in §8.

Theorem 7. *If A is a compact L^p -sum convex set of semi-norms,*

A contains sufficiently many indecomposable elements in it. Then every element of A is an L^p -sum on the totality E^p of L^p -sum indecomposable elements in A (i. e. $A = Co^p(E^p)$).

Proof. Let q denote an upper bound of A . In case $p < \infty$, the Theorem follows the Krein-Milman's theorem. Replace each λ in A by a bounded functional $(\lambda(f)/q(f))^p$ on $\mathcal{L}$ provided $(\lambda(f)/q(f))^p = 0$ whenever $q(f) = 0$, then the set A and its L^p -sum operation are transposed to a set A^p of bounded functionals and the usual addition of functionals, respectively. $A^p = ((\lambda/q)^p : \lambda \in A)$ is a bounded regularly convex sub-set of the Banach space B^1 of the totality of bounded functionals on $\mathcal{L}$, and $\lambda \rightarrow (\lambda/q)^p$ is a weak homeomorphism between A and A^p . A^p contains sufficiently many extremal elements in it, while $(\lambda/q)^p$ in A^p is extremal if and only if λ is L^p -sum indecomposable in A . A contains therefore sufficiently many indecomposable elements in it.

We next consider in the case of $p = \infty$. Given each s in A and each f in $\mathcal{L}$, we can choose an L^∞ -indecomposable element t_f in A so that $t_f = s$ and $t_f(f) = s(f)$. In fact the totality of elements t in A so that $t \leq s$ and $t(f) = s(f)$ is inductive by the inverse order of semi-norms, and contains a minimal element t_f by the Zorn's lemma. t_f has no semi-norm r so that $t_f > r$ and $t_f(f) = r(f) = s(f)$. This t_f is indecomposable since $t_f = \max(q, r)$ ($q, r \in A$) implies either $t_f(f) = q(f) = s(f)$ or $t_f(f) = r(f) = s(f)$, and either $t_f = q$ or $t_f = r$. The given semi-norm s is an L^∞ -sum $s = \sup_{f \in \mathcal{L}} t_f$ of L^∞ -indecomposable elements t_f in A , hence A contains sufficiently many L^∞ -indecomposable elements.

Example. The totality P^+ of normalized orderly semi-norms on the space $C(\mathcal{Q})$ of a compact space $\mathcal{Q}$ is L^∞ -sum convex. It is shown that

Theorem 8. *A necessary and sufficient condition for an orderly semi-norm s to be indecomposable to any non-trivial L^∞ -sum of orderly semi-norms is that s be an L^1 -semi-norm.*

Proof. If an L^1 -semi-norm $|\mu|_1$ is an L^∞ -sum $|\mu|_1 = \max(q, r)$ of two orderly semi-norms q and r , then either $|\mu|_1(1) = q(1)$ or $|\mu|_1(1) = r(1)$. We can assume $|\mu|_1(1) = q(1)$ without loss of generality, then every f in $C(\mathcal{Q})$ with $0 \leq f \leq 1$ satisfies $|\mu|_1(1) = |\mu|_1(f) + |\mu|_1(1-f)$,

1) B is considered as a dual Banach space of a suitable normed space $l(\mathcal{L})$, the totality of functionals $\varphi(f)$ which vanishes except for finite elements in $\mathcal{L}$, and whose norm is $\sum_{f \in \mathcal{L}} \varphi(f)$.

$q(1) \leq q(f) + q(1-f)$, $q(f) \leq |\mu|_1(f)$ and $q(1-f) \leq |\mu|_1(1-f)$. This means $q(f) = |\mu|_1(f)$ and $q = |\mu|_1$, the indecomposability of $|\mu|_1$ to non-trivial L^∞ -sums.

Conversely let s be an L^∞ -indecomposable orderly semi-norm in $\mathbf{P}^+$. By (3.4) in §3 s is an L^∞ -sum $s = \sup_{\mu \in U^+(s)} |\mu|_1 = \sup_{\lambda \in \Lambda} \lambda$ on a suitable set A of L^1 -semi-norms. Every neighbourhood $U = U(s; f_1, \dots, f_n, \varepsilon)$ contains an element of A . In fact, let s_0 denote the least upper bound of semi-norms in the common-part of U and A , and let s_i denote the least upper bound of semi-norms r in A so that $r(f_i) \leq s(f_i) - \varepsilon$. Then s is an L^∞ -sum $s = \max(s_0, s_1, \dots, s_n)$. By the indecomposability of s , s is coincident with s_0 , and $U \cap A$ is non-empty. Then s is an L^1 -semi-norm as a limit element of L^1 -semi-norms (Lemma 3.1).

§ 7. Decompositions of normed spaces and its dual spaces.

Let k be a composite semi-norm on a set A of semi-norms on $\mathcal{L}$.

Then the representation $\mathcal{L} \rightarrow \mathcal{L}(k^\Delta)$ is said to be *composed* of the system of those representations $(\mathcal{L} \rightarrow \mathcal{L}(\lambda) : \lambda \in A)$. The consistency of such a definition is asserted by the one-to-oneness of the correspondence between semi-norms and representations. To study the relation between $\mathcal{L}(k^\Delta)$ and $\mathcal{L}(\lambda)$ ($\lambda \in A$), we define the carrier of a composition.

The *carrier* of an orderly semi-norm k on the space $C(\mathcal{Q})$ of a compact space $\mathcal{Q}$ is the smallest closed sub-set X of $\mathcal{Q}$ so that every f in $C(\mathcal{Q})$ which vanishes on X belongs to $\mathcal{N}(k)$ (i. e. $k(f) = 0$). The existence of the carrier is asserted by the next Lemma.

Lemma 7.1. *The carrier $D(k)$ of an orderly semi-norm k on $C(\mathcal{Q})$ is the closure of the set-theoretical sum of carriers of all regular measures in the space $U^+(k)$ ($U^+(k)$ is defined in 3.4, §3).*

In fact, if a continuous function f vanishes on the carrier of every regular measure μ in $U^+(k)$, then by 3.4 $|\mu|_1(f) = \int |f| d\mu = 0$ and $k(f) = \sup_{\mu \in U^+(k)} \int |f| d\mu = 0$. Conversely, if a continuous function f does not vanish on the carrier of a measure μ in $U^+(k)$, then $k(f) \geq \int |f| d\mu > 0$. Q. E. D.

If two functions f and g in $C(\mathcal{Q})$ has the same values on the carrier $D(k)$ of an orderly semi-norm k , then $k(f) = k(g)$. k is thus determined as an orderly norm on $C(D(k))$.

The *carrier* of a composition k^Δ on a compact space A of semi-norms

is defined as the carrier of the orderly semi-norm k on $C(A)$.

Lemma 7.2. *If k^Λ is a composition on a compact set A of semi-norms on $\mathcal{L}$, the zero-point-set $\mathcal{N}(k^\Lambda)$ of k^Λ is contained in each zero-point set $\mathcal{N}(\lambda)$ of semi-norms in the carrier of k^Λ .*

In fact, if $f \in \mathcal{N}(k^\Lambda)$ and $k(|f_\lambda|) = 0$, then $|f_\lambda| = \lambda(f) = 0$ and f belongs to $\mathcal{N}(\lambda)$ for each λ in $D(k)$.

A composition of normed spaces is now defined.

Definition. 1. Let $\mathfrak{N}(\lambda)$ be a system of normed spaces defined at each point λ in a compact space A . A normed space $\mathfrak{N}$ is said to be composed of $(\mathfrak{N}(\lambda) : \lambda \in A)$ by an orderly semi-norm k on $C(A)$ if

- (1). For each $\mathfrak{N}(\lambda)$ a representation $f \rightarrow f_\lambda$ of $\mathfrak{N}$ in $\mathfrak{N}(\lambda)$ exists.
- (2). If an f in $\mathfrak{N}$ is fixed, a numerical function $|f_\lambda|$ of the variable λ is continuous on A .
- (3). Every two elements λ, μ in A are distinguished ($|f_\lambda| \neq |f_\mu|$) by the norm-function $|f_\lambda|$ of a suitable f in $\mathfrak{N}$.
- (4). The norm $|f|$ of f in $\mathfrak{N}$ is determined by $|f| = k(|f_\lambda|)$.

Definition 2. If the space $\mathfrak{N}$ in Definition 1 is composed by an L^p -semi-norm ($p < \infty$) by a regular measure μ on A , then $\mathfrak{N}$ is said to be an L^p -sum and denoted by $\mathfrak{N} = (\int \mathfrak{N}(\lambda)^p d\mu(\lambda))^{\frac{1}{p}}$.

If the space $\mathfrak{N}$ is composed by the L^∞ -semi-norm $|A|$, then $\mathfrak{N}$ is said to be the L^∞ -sum (or the least upper bound) of those spaces $(\mathfrak{N}(\lambda) : \lambda \in A)$, and denoted by $\mathfrak{N} = \sup_{\lambda \in A} \mathfrak{N}(\lambda)$.

Theorem 9. *Let k^Λ be a composite semi-norm on a compact set A of semi-norms on a linear space $\mathcal{L}$. Then the representative space $\mathcal{L}(k^\Lambda)$ is a composition of those normed spaces $(\mathcal{L}(\lambda) : \lambda \in D(k))$.*

Proof. If $\lambda \in D(k)$, $\mathcal{N}(\lambda) \supseteq \mathcal{N}(k^\Lambda)$ implies that $f_k \rightarrow f_\lambda = (f_k)_\lambda$ defines a representation of $\mathcal{L}(k^\Lambda) = \mathcal{L} / \mathcal{N}(k^\Lambda)$ on $\mathcal{L}(\lambda) = \mathcal{L} / \mathcal{N}(\lambda)$. k is a semi-norm on $C(D(k))$, and the representation $f_k \rightarrow (f_k)_\lambda$ satisfies clearly (1), (2), (3) and

$$(4); \quad |f_k| = k^\Lambda(f) = k(|f_k|_\lambda).$$

We now treat our final problem, the composition rules of the dual space of a composite normed space.

Let A be a compact set of semi-norms on $\mathcal{L}$. A spherical system $U(A)$ on A is the totality of those pairs (λ, φ) so that $\lambda \in A$ and $\varphi \in$

$U(\lambda)$ ($U(\lambda)$ is the unit sphere of $\mathcal{L}^*(\lambda)$). $U(A)$ is a topological space, as a sub-space of the product space $A \times \mathcal{L}^*$.

Theorem 10. *The spherical system $U(A)$ of a compact set A of semi-norms on $\mathcal{L}$ is weakly compact.*

In fact, let q denote the least upper bound of A . The $U(\lambda)$ of every λ in A is contained in $U(q)$, then $U(A)$ is contained in the compact space $A \times U(q)$. If f is a fixed element in $\mathcal{L}$, the functions $\lambda(f) - |\varphi(f)|$ is continuous on the product space $A \times U(q)$ with respect to the variables $\lambda \in A$ and $\varphi \in U(q)$. $U(A)$ is the common part of all those closed sets $((\lambda, \varphi) \in A \times U(q) : \lambda(f) - |\varphi(f)| \geq 0)$, then it is closed and compact in $U(q) \times A$ as well. Q. E. D.

If μ is a regular measure on $U(A)$, μ determines a regular measure μ_Δ on A so that $\int f(\lambda) d\mu_\Delta(\lambda) = \int f(\lambda) d\mu(\lambda, \varphi)$. μ_Δ is the *restriction* of the measure μ on A by the mapping $(\lambda, \varphi) \rightarrow \lambda$.

Theorem 11. *Let k^Δ be a composite semi-norm on a compact set A of semi-norms on $\mathcal{L}$, and let $U(A)$ denote the spherical system on A . Then every ψ in $U(k_\Delta)$ is a weak integral $\psi = \int \varphi d\mu(\lambda, \varphi)$ by a suitable regular measure μ on $U(A)$ whose restriction μ_Δ on A belongs to $U^+(k)$.*

Proof. If μ varies on a set of regular measures on $U(A)$, the mapping $\mu \rightarrow \mu_\Delta$ is weakly continuous. Then the totality W of those regular measures μ on $U(A)$ whose restriction μ_Δ belongs to $U^+(k)$ is regularly convex and bounded (since $\int 1 d\mu = \int 1 d\mu_\Delta \leq 1$). If $\mu \in W$, then the weak integral $\int \varphi d\mu(\lambda, \varphi)$ belongs to $U(k^\Delta)$. In fact by the relation $\lambda(f) \geq |\varphi(f)|$ for $(\lambda, \varphi) \in U(A)$, we have

$$\begin{aligned} |\varphi(f)| &= \left| \int \varphi(f) d\mu(\lambda, \varphi) \right| \leq \int \lambda(f) d\mu(\lambda, \varphi) \\ &= \int |f_\lambda| d\mu_\Delta(\lambda) \leq k(|f_\lambda|) = k^\Delta(f). \end{aligned}$$

The weakly continuous mapping $\varphi \rightarrow \int \varphi d\mu(\lambda, \varphi)$ maps therefore the set W in a regularly convex symmetric sub-set U of $U(k^\Delta)$.

Now suffice it to say $U = U(k^\Delta)$. This follows the fact that for each f in $\mathcal{L}$ we can choose a weak integral $\psi = \int \varphi d\mu(\lambda, \varphi)$ in U with

$k^\Lambda(f) = \psi(f)$, or $k(|f_\Lambda|) = \int \varphi(f) d\mu(\lambda, \varphi)$. The existence of such a measure μ is shown in the more stronger condition that

(1). μ vanishes out-side of a compact set $V = ((\lambda, \varphi) \in U(A) : \lambda(f) = \varphi(f))$.

(2). The restriction μ_Λ of μ within the space A belongs to $U^+(k)$ and satisfies $k(|f_\Lambda|) = \int |f_\Lambda| d\mu_\Lambda(\lambda) (= \int \varphi(f) d\mu(\lambda, \varphi))$.

In fact, by (3.4) we can choose a measure ν in $U^+(k)$ so that $k(|f_\Lambda|) = \int |f_\Lambda| d\nu(\lambda)$. It is sufficient to see the extensibility of the measure ν to a measure μ in the space V . This is done by the extension theorem of measures (Theorem 4 Remark in §4), and by the fact that the mapping $(\lambda, \varphi) \rightarrow \lambda$ maps the set $V = ((\lambda, \varphi) \in U(A) : \lambda(f) = \varphi(f))$ onto the set A , that is, for each λ in A , we can choose a φ in $U(\lambda)$ so that $\varphi(f) = \lambda(f)$. Thus such a measure μ exists, and U coincides with $U(k^\Lambda)$.

Theorem 12.¹⁾ *Let k^Λ be a composition on a separable compact set A of semi-norms, then every element ψ in the dual space $L^*(k^\Lambda)$ is a weak integral $\psi = \int \psi_\lambda d\nu(\lambda)$ by a suitable regular measure ν in A , where ψ_λ is a weakly measurable function on A so that each ψ_λ is an element in $L^*(\lambda)$ of norm $\lambda^*(\psi_\lambda) \leq 1$, and ν is a measure in $U^+(k)$ of norm $k^*(\nu) = k^{\Lambda*}(\psi)$.*

Proof. J. Dieudonné [2] extended the Doob's theorem to the following result. "If $\lambda \rightarrow T\lambda$ is a continuous mapping of a compact space Ω to another separable compact space A , every regular measure μ on Ω is a weak integral $\mu = \int \mu_\lambda d\nu(\lambda)$, where each μ_λ is a normalized regular measure on the space $\Omega_\lambda = (x : Tx = \lambda)$, and the measure ν is the restriction of the measure μ on the space A ". We now assume the space A in Theorem 11 be separable.

If ψ is an element of $\mathcal{L}^*(k^\Lambda)$ with the norm $k^{\Lambda*}(\psi) = 1$, ψ is a weak integral $\psi = \int \varphi d\mu(\lambda, \varphi)$ so that the restriction ν of μ within the space A belongs to $U^+(k)$. The Dieudonné's theorem is applicable to this measure μ with respect to the mapping $(\lambda, \varphi) \rightarrow \lambda$, and μ is a weak integral $\mu = \int_A \mu_\lambda d\nu(\lambda)$ of a weakly measurable function μ_λ . Each μ_λ is a normalized regular measure on the sphere $U(\lambda)$ (= the complete

¹⁾ Analogous cares to those of [2] must be exercised with the measurability of the norm functions $\lambda^*(\psi_\lambda)$, as well as the difference between weak equivalency and usual equivalency, of those functions ψ_λ in Theorem 12 and its Corollary.

inverse image of the point λ). Then

$$\begin{aligned}\psi(f) &= \int_{U(\lambda)} \varphi(f) d(\lambda, \varphi) = \int_{\Lambda} \left(\int_{U(\lambda)} \varphi(f) d\mu_{\lambda}(\varphi) \right) d\mu(\lambda) \\ &= \int \psi_{\lambda}(f) d\nu(\lambda).\end{aligned}$$

Each ψ_{λ} is a weak integral $\psi_{\lambda} = \int \varphi d\mu_{\lambda}(\varphi)$ on the unit sphere $U(\lambda)$ by a normalized regular measure μ_{λ} and belongs to $U(\lambda)$ as well, then ψ is a weak integral $\int_{\Lambda} \psi_{\lambda} d\nu(\lambda)$. The norm $k^*(\nu)$ of ν is 1 because it is not smaller than the norm $k^*(\psi) = 1$ of ψ . Q. E. D.

Corollary. Let s be a L^p -sum ($p < \infty$) on a compact set Λ of semi-norms; $s = (\int_{\Lambda} \lambda^p d\nu(\lambda))^{\frac{1}{p}}$. Then every ψ in $\mathcal{L}^*(s)$ is a weak integral $\psi = \int \psi_{\lambda} d\nu(\lambda)$ so that

(1). In case $p = 1$, $s^*(\psi) = \text{ess. max } \lambda^*(\psi_{\lambda})$.

(2). In case $p > 1$, $s^*(\psi) = (\int \lambda^*(\psi_{\lambda})^q d\nu(\lambda))^{\frac{1}{q}}$. $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$.

If s is the least upper bound on Λ , every ψ in $\mathcal{L}^*(s)$ is a weak integral $\psi = \int \psi_{\lambda} d\mu(\lambda)$ by a suitable regular measure μ on Λ so that

$$s^*(\psi) = \int \lambda^*(\psi_{\lambda}) d\mu(\lambda).$$

§ 8. Applications to decomposing operator algebras.

Let $\mathfrak{N}$ be a normed space, and $\mathcal{A}$ be a linear algebra of bounded operators on $\mathfrak{N}$ which contains the identity I . A semi-norm r on $\mathcal{A}$ so that $r(I) = 1$ and $r(AB) \leq r(A)r(B)$ is said to be *algebraic*. The norm $n(A) = |A|$ of A as an operator is clearly algebraic. The totality of algebraic semi-norms on $\mathcal{A}$ is L^∞ -sum convex.

If x is an element of $\mathfrak{N}$, the semi-norm $p(A) = |Ax|$ on $\mathcal{A}$ satisfies $p(AB) \leq n(A)p(B)$. In general, a semi-norm p on $\mathcal{A}$ so that $\sup p(B) \leq 1$, $p(AB) < \infty$ ($A \in \mathcal{A}$), is said to be *operative*. And if r is an algebraic semi-norm so that $p(AB) \leq r(A)p(B)$, r is said to *operate on* p , and p is said to be *operated by* r .

The canonical representation of $\mathcal{A}$ by an algebraic semi-norm r

is an algebraic representation of $\mathcal{A}$ to another normed algebra $\mathcal{A}(r)$. If p is an operative semi-norm, then $\mathcal{A}$ operates bounded linearly on the normed space $\mathcal{A}(p)$. That is, there is an algebraic representation $A \in \mathcal{A} \rightarrow A(p)$ on a linear algebra of bounded operators on $\mathcal{A}(p)$ so that $p(A) = |A(p)x|$ for a suitable cyclic element x in $\mathcal{A}(p)$. If p is operated by an algebraic semi-norm r moreover, then $A_r \rightarrow A(p)$ determines a topologico-algebraic representation of the normed algebra $\mathcal{A}(r)$ on an operator algebra on $\mathcal{A}(p)$. The totality of semi-norms operated by a fixed algebraic semi-norm r is universally convex.

An operative semi-norm p on $\mathcal{A}$ is said to be *normalized* if $p(I) = 1$. If p is normalized and operated by an algebraic semi-norm r , then $p \leq r$. (In fact, $p(A) = p(AI) \leq r(A)p(I) = r(A)$). Thus

Lemma 8.1. *The totality of semi-norms on a linear algebra $\mathcal{A}$ operated by a fixed algebraic semi-norm r on $\mathcal{A}$ is compact, universally convex and bounded by r .*

The present problem is to decompose a given (algebraic or operative) semi-norm p into simpler pieces.

The past works for operator-algebras were almost restricted to that of C^* -algebras, but for the G. E. Šilov's several early works. Then we shall reconstruct once more the decomposition theory of C^* -algebras in a new point of view. It may be perhaps useful to extend the result to general Banach algebras in a future.

Let $\mathcal{A}$ be a C^* -algebra (i. e. a uniformly closed self-adjoint algebra of bounded linear operators on a Hilbert space.) with the identity I . An algebraic semi-norm p on $\mathcal{A}$ so that $p(A^*A) = p(A)^2$ and $p(A^*) = p(A)$ is said to be a B^* -semi-norm. If $\mathcal{N}$ is a two-sided closed ideal of $\mathcal{A}$, by the theorem of Gelfand-Kaplansky ([4] and [6]) the residue algebra $\mathcal{A}/\mathcal{N}$ with the norm $|A/\mathcal{N}| = \inf_{X \in \mathcal{N}} |A - X|$ is a C^* -algebra. Those natural representations $A \rightarrow A/\mathcal{N}$ by closed ideals $\mathcal{N}$ exhaust the totality of representations of $\mathcal{A}$ to C^* -algebras. Then the zero-point-set $\mathcal{N}(p)$ of a B^* -semi-norm p is a two-sided ideal, and $A \rightarrow A/\mathcal{N}(p)$ is the canonical representation of $\mathcal{A}$ on the C^* -algebra $\mathcal{A}/\mathcal{N}(p)$ so as to be $p(A) = \inf_{X \in \mathcal{N}(p)} |A - X| = |A/\mathcal{N}(p)|$.

In such a way, B^* -semi-norms and two-sided closed ideals of $\mathcal{A}$ correspond one-to-one with each other, however, the correspondence reverses their respective orders. *The totality B^* of B^* -semi-norms on $\mathcal{A}$ is L^∞ -sum convex, closed and bounded by the norm $n(A)$ of $\mathcal{A}$.* If p is a least upper bound $p = \sup_{\lambda \in \Lambda} \lambda$ on a set Λ of B^* -semi-norms, then the

ideal $\mathcal{N}(p)$ should be the greatest lower bound (the common ideal) of the system of corresponding ideals $(\mathcal{N}(\lambda): \lambda \in A)$. In particular, $p \in B^*$ is L^∞ -sum indecomposable if and only if the ideal $\mathcal{N}(p)$ (and the algebra) $\mathcal{A}/\mathcal{N}(p)$ is irreducible (c. f. [7]), that is, $\mathcal{N}(p)$ has no expression as the common ideal of any pair of ideals containing $\mathcal{N}(p)$ properly. Thus

Lemma 8.2. *Every C^* -algebra with the identity I is a L^∞ -sum of (ideal theoretically) irreducible algebras.*

If $\mathcal{A}$ is a projective C^* -algebra (A C^* -algebra $\mathcal{A}$ is said to be *projective* if it is spanned uniformly by its projection elements.), then the result is more strengthened.

Theorem 13. *Every projective C^* -algebra $\mathcal{A}$ with the identity I is a L^∞ -sum of several sub-direct sum irreducible C^* -algebras.*

Proof. $\mathcal{A}$ is said to be *sub-direct sum irreducible* (c. f. [1], [7]) if it contains the smallest closed two-sided ideal. For each projection E , we consider a minimal B^* -semi-norm p_E so that $p_E|_E = 1$. If q is a B^* -semi-norm properly smaller than p_E , then $q(E) < p(E)$ and $q(E) = q(E^2) = q(E)^2$ imply $q(E) = 0$. The least upper bound r of all those B^* -semi-norms properly smaller than p_E vanishes at E as well. r is then the largest B^* -semi-norm properly smaller than p_E , and $\mathcal{N}(p_E)$ is the smallest closed ideal which contains $\mathcal{N}(p_E)$ properly. Thus $\mathcal{N}(p_E)$ and $\mathcal{A}/\mathcal{N}(p_E)$ are totally irreducible. Since $\mathcal{A}$ is projective, the least upper bound s of those B^* -semi-norms p with the totally irreducible zero-point-sets (ideals) $\mathcal{N}(p)$ does not vanish at any projections in $\mathcal{A}$. This means $\mathcal{N}(s) = 0 = \mathcal{N}(n)$, the equality of the semi-norm s and the norm n of $\mathcal{A}$. Q. E. D.

If $\mathcal{A}$ is a C^* -algebra with the identity on a Hilbert space $\mathfrak{H}$, every element x in $\mathfrak{H}$ determines a semi-norm p on $\mathcal{A}$ operated by the norm $n(A)$ of $\mathcal{A}$. An H -semi-norm on $\mathcal{A}$ operated by the norm n is defined in the following condition. The space $\mathcal{A}(p)$ of the canonical representation $A \rightarrow A_p$ is a Hilbert space, $\mathcal{A}$ operates on which as a self-adjoint algebra, that is, $((AB)_p, C_p) = (B_p, (A^*C)_p)$ for every A, B, C in $\mathcal{A}$.

Notice that $p(A) = (A_p, A_p)^{\frac{1}{2}}$, then an H -semi-norm p determines the inner-product $(A_p, B_p) = \frac{1}{4} \{p(A+B)^2 - p(A-B)^2 + ip(A+iB)^2 - ip(A-iB)^2\}$ of $\mathcal{A}_p$, and the totality of normalized H -semi-norms on $\mathcal{A}$ operated by the norm of $\mathcal{A}$ is compact and L^2 -sum convex. An normalized H -semi-norm p is L^2 -sum indecomposable if and only if the algebra $\mathcal{A}$ operates irreducibly on the Hilbert space $\mathcal{A}(p)$.

The above-stated is of cause merely a reproduction of the I. Gelfand's theory. His theory of the canonical representation of a positive definite function consists of two relation-theories. One treats the relation between positive definite functions and H -semi-norms. Another treats the relation between H -semi-norms and the canonical representations. We can eliminate its somewhat superfluous former half.

Coincidence between indecomposability of (operative and algebraic) semi-norms and algebraic irreducibility of canonical representations observed in B^* - and H -semi-norms may not be expectable for general operator-algebras. However, to investigate the properties of general algebraic and operative semi-norms may be an important remained problem.

REFERENCES

- [1] G. BIRKOFF: Sub-directunions in universal algebra, Bull. Amer. Math. Soc. vol. 50 (1954) pp. 764—768.
- [2] J. DIEUDONNÉ: Sur le théorème de Lebesgue-Nykodym IV. Jour. Indian Math. Soc. N. S. 15 (1951).
- [3] ———: La dualité dans les espaces vectoriels topologiques Ann. École Norm. Sup. vol. 54 (1941) pp. 107—139.
- [4] I. GELFAND and NAIMARK: On the imbedding of normed rings into the ring of operators in Hilbert space. Rec. Math. (Mat Sbornik) N. S. 12 (1943) 197—213.
- [5] R. GODEMENT: A theory of spherical functions I. Trans. Amer. Math. Soc. vol. 73 (1952) pp. 555—596.
- [6] I. KAPLANSKY: Normed algebras, Duke Math. Jour. vol. 16 (1949) 339—418.
- [7] N. H. MACCOY: Sub-direct sums of rings. Bull. Amer. Math. J. vol. 53 (1947) 856—877.
- [8] G. W. MACKEY: On infinite-dimensional linear spaces Trans. Amer. Math. Soc. vol. 57 (1945) 155—207.
- [9] I. E. SEGAL: Irreducible representations of operator algebras. Bull. Amer. Math. Soc. 53 (1947) 73—88.
- [10] G. E. ŠILOV: Rings of type C. Rings of type C on the line and on the circumference. Doklady Akad. Nauk SSSR (N. S.) 66 (1949) pp. 813—816 and pp. 1063—1066.

DEPARTMENT OF MATHEMATICS,
OKAYAMA UNIVERSITY

(Received March 17, 1957)