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## On a holomorphically projective correspondence in an almost complex space

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## ON A HOLOMORPHICALLY PROJECTIVE CORRESPONDENCE IN AN ALMOST COMPLEX SPACE

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In a previous paper in collaboration with Prof. Ōtsuki [2], the present author has introduced and investigated holomorphically flat curves in Kählerian spaces and the correspondence between Kählerian spaces, preserving such curves, which is called holomorphically projective (h. p.) correspondence. He has thereby shown that a space h. p. to a Euclidean space is of constant holomorphic curvature.

On the other hand, Prof. K. Yano and I. Mogi [5] has characterized a Kählerian space of constant holomorphic curvature by the axiom of holomorphic planes and also by the holomorphic free mobility. The method of real representation used by them is valid also in a pseudo-Kählerian space.

In the present paper we shall generalize the notions of holomorphically flat curves and h. p. correspondence to the case of almost complex spaces with affine connection. Next, a tensor invariant under such a correspondence will be obtained. Finally, in a metric case we shall obtain the tensor of constant holomorphic curvature found by Prof. K. Yano and I. Mogi [5].

As to the notations and conventions, we follow J. A. Schouten [4] and K. Yano [7].

§ 1. Let  $X_{2n}$  be a space with an almost complex structure defined by  $F_i^h$ :

$$(1.1) \quad F_i^j F_j^h = -A_i^h,$$

and let  $X_{2n}$  be endowed with a symmetric affine connection  $\Gamma_{ji}^h$ . Denoting by  $\nabla$  the covariant differentiation with respect to  $\Gamma_{ji}^h$ , we assume that

$$(1.2) \quad \nabla_j F_i^h = 0$$

which means geometrically that the fields of proper planes  $\gamma_n$  and  $\bar{\gamma}_n$  of the almost complex structure  $F_i^h$  are separately parallel with respect to the connection [6].

The geodesics are defined by differential equations of the form

$$(1.3) \quad \frac{d^2 x^h}{dt^2} + \Gamma_{ji}^h \frac{dx^j}{dt} \frac{dx^i}{dt} = \alpha(t) \frac{dx^h}{dt}$$

which mean that the tangent displaced parallelly along the curve remains tangent to the curve.

We now introduce the curves satisfying the differential equations

$$(1.4) \quad \frac{d^2 x^h}{dt^2} + \Gamma_{ji}^h \frac{dx^j}{dt} \frac{dx^i}{dt} = \alpha(t) \frac{dx^h}{dt} + \beta(t) F_i^h \frac{dx^i}{dt}.$$

Such a curve is a plane curve and has the property that the tangent holomorphic plane displaced parallelly along it remains holomorphically tangent to the curve. We call such a curve a holmorphically flat curve.

If, in an almost complex space, there are two connections  $\Gamma_{ji}^h$  and  $'\Gamma_{ji}^h$ , and if any curve which is holomorphically flat with respect to one of the connections is always holomorphically flat with respect to the other, than they have to be related such as

$$(1.5) \quad '\Gamma_{ji}^h = \Gamma_{ji}^h + 2P_{(j}A_{i)}^h + 2Q_{(j}F_{i)}^h.$$

Under the restriction (1.2) on both of the connections, we have

$$P_k F_i^k A_j^h - P_i F_j^k A_k^h + Q_k F_i^k F_j^h + Q_i A_j^h = 0,$$

from which, contracting the indices  $h$  and  $j$ , and taking account of  $F_h^h = 0$ ,

$$(1.6) \quad Q_i = -P_h F_i^h.$$

Accordingly the relation (1.5) can be written as

$$(1.7) \quad '\Gamma_{ji}^h = \Gamma_{ji}^h + 2P_{(j}A_{i)}^h - 2P_k F_{(j}^k F_{i)}^h.$$

This correspondence is called a holomorphically projective one (cf. [2]).

If we denote by  $R_{kji}^h$  the curvature tensor with respect to  $\Gamma_{ji}^h$ :

$$(1.8) \quad R_{kji}^h = 2\partial_{[k}\Gamma_{j]i}^h + 2\Gamma_{[k|l}^h \Gamma_{j]}^l,$$

then, by a straightforward and rather complicated computation, we obtain

$$(1.9) \quad 'R_{kji}^h = R_{kji}^h + 2A_{[j}^h P_{k]i} + 2P_{[k]j} A_i^h - 2P_{[k|l} F_{j]}^l F_i^h - 2P_{[k|l} F_{j]}^h F_i^l,$$

where we have put

$$(1.10) \quad P_{ji} = \nabla_j P_i - P_j P_i + P_i P_k F_j^i F_k^i.$$

By contraction over  $h$  and  $k$  in (1.9), we have

$$(1.11) \quad {}'R_{ji} = R_{ji} - 2(n+1)P_{ji} + 2P_{(ji)} - 2P_{(ba)}F_j^b F_i^a$$

and, multiplying (1.11) by  $F_j^j F_i^i$  and adding the result to (1.11),

$$(1.12) \quad {}'R_{ji} + {}'R_{ba}F_j^b F_i^a = R_{ji} + R_{ba}F_j^b F_i^a - 2(n+1)P_{ji} - 2(n+1)P_{ba}F_j^b F_i^a,$$

and, eliminating the last terms from (1.11) and (1.12), and solving the resulting equations in  $P_{ji}$ ,

$$(1.13) \quad 4(n^2-1)P_{ji} = M_{ji} - {}'M_{ji},$$

where we have put

$$(1.14) \quad M_{ji} = (2n-1)R_{ji} + R_{ij} - 2R_{(ba)}F_j^b F_i^a.$$

Substituting (1.13) into (1.9), we see that the tensor

$$(1.15) \quad P_{kji}^h = R_{kji}^h + \frac{1}{2(n^2-1)} [M_{[k|i|} A_{j]}^h + M_{[k|j]} A_i^h - M_{[k|i|} F_j^i F_i^h - P_{(k|i|} F_{j]}^h F_i^i]$$

is invariant under the h. p. correspondence. We call it the h. p. curvature tensor. It is written down explicitly as follows :

$$(1.16) \quad \begin{aligned} P_{kji}^h = R_{kji}^h &+ \frac{1}{2(n^2-1)} [ \{ (2n-1)R_{[k|i|} + R_{i(k} - 2R_{(ba)}F_{[k}^b F_{|i|}^a \} A_{j]}^h \\ &- \{ (2n-1)F_i^i R_{[k|i|} + F_i^i R_{i(k} + 2R_{(ba)}F_{[k}^b F_{|i|}^a \} F_{j]}^h ] \\ &+ \frac{1}{n+1} [R_{(kj)} A_i^h + F_{[k}^i R_{j]i} F_i^h]. \end{aligned}$$

We can verify that

$$(1.17) \quad P_{kji}^k = 0.$$

§ 2. An almost complex space with an affine connection is said to be h. p. flat if it can be related to a Euclidean space by a h. p. correspondence (1.7). The necessary condition to be h. p. flat is clearly  $P_{kji}^h = 0$ . Conversely, if  $P_{kji}^h = 0$ , then putting

$$P'_{ji} = \frac{1}{4(n^2-1)} \{ (2n-1)R_{ji} + R_{ij} - 2R_{(ba)}F_j^b F_i^a \},$$

the curvature tensor  $R_{kji}^h$  satisfies the equation

$$R_{kji}^h = -2A_{(j}P'_{k)i} - 2P'_{(kj)}A_i^h + 2P'_{(k|l|}F_{j)}^l F_i^h + 2P'_{(k|l|}F_{j)}^h F_i^l.$$

On the other hand, if the space is h. p. flat under (1.7), then  $P_{ji}$  satisfies the equation (1.9) in which the left hand side vanishes. Hence  $P_{ji}$  should be equal to the above  $P'_{ji}$ . Therefore, in order to prove that the space with  $P_{kji}^h = 0$  is h. p. flat, it is sufficient that there exists a vector field  $P_i$  such that

$$(2.1) \quad \nabla_j P_i = P_{ji} + P_j P_i - P_b P_a F_j^b F_i^a,$$

in the space having the curvature

$$(2.2) \quad R_{kji}^h = -2\{P_{(k|i|}A_{j)}^h + P_{(kj)}A_i^h - F_{(j}^a P_{k)a} F_i^h - F_{(j}^h P_{k)a} F_i^a\}.$$

Taking account of (1.2), the integrability condition of (2.1) is

$$-R_{kji}^h P_h = 2\{\nabla_{[k} P_{j)i} + \nabla_{(k} P_{j)} P_i + P_{(j} \nabla_{k)} P_i - F_{(j}^b \nabla_{k)} P_b P_a F_i^h - P_b F_{(j}^b \nabla_{k)} P_a F_i^a\}$$

or, substituting (2.1) and (2.2),

$$(2.3) \quad \nabla_{(k} P_{j)i} = 0.$$

Now, if the identity of Bianchi [4, p.147] is applied to (2.2), we have

$$(2.4) \quad 0 = \nabla_{[l} P_{k|i|} A_{j]}^h + \nabla_{[l} P_{kj)} A_i^h - \nabla_{[l} P_{k|a|} F_{j]}^a F_i^h - \nabla_{[l} P_{k|a|} F_{j]}^h F_i^a.$$

By contraction over  $h$  and  $i$ , we have

$$(2.5) \quad \nabla_{[l} P_{kj]} = 0$$

and, by contraction over  $h$  and  $j$ ,

$$(2.6) \quad (2n-1) \nabla_{[l} P_{k)i} + 2 \nabla_{[b} P_{a)(l} F_{k]}^b F_i^a = 0.$$

Alternating indices  $i, k, l$  in (2.6) and considering (2.5), we have

$$(2.7) \quad 2 \nabla_{[b} P_{a)(l} F_{k]}^b F_i^a + \nabla_{[b} P_{a)i} F_l^b F_k^a = 0,$$

substituting (2.7) into (2.6),

$$(2.8) \quad (2n-1) \nabla_{[l} P_{k)i} - \nabla_{[b} P_{a)i} F_l^b F_k^a = 0,$$

and finally, solving this equation with respect to  $\nabla_{(i} P_{k)l}$ ,

$$(2.9) \quad (2n-3)(2n-1) \nabla_{(i} P_{k)j} = 0.$$

Hence, the integrability condition (2.3) of (2.1) is a consequence of (2.2). This proves

**Theorem 1.** *An almost complex space with an affine connection is holomorphically projectively flat if and only if the h. p. curvature  $P_{kji}^h$  vanishes.*

§ 3. In Hermitian metric case, our restriction (1.2) implies that the space is pseudo-Kählerian [7]. Consequently, Ricci tensor is symmetric and satisfies [3]

$$(3.1) \quad R_{ji} = R_{ba} F_j^b F_i^a,$$

and the h. p. curvature tensor may be reduced to

$$(3.2) \quad \begin{aligned} P_{kjih} &\equiv P_{kji}^l g_{lh} \\ &= R_{kjih} + \frac{1}{2(n^2-1)} [2(n-1)R_{(k+i)j} g_{j)h} - (2n-1)R_{a(k} F_{j)h} F_i^a] \\ &\quad - \frac{1}{n-1} R_{a(k} F_{j)h} F_i^a. \end{aligned}$$

If the space is h. p. flat, i. e.,  $P_{kjih} = 0$ , then contracting by  $g^{ji}$ , we have

$$(3.3) \quad 2nR_{kh} = Rg_{kh}.$$

Hence  $R$  is a constant, and we put

$$(3.4) \quad R = n(n+1)k$$

or

$$(3.5) \quad R_{ji} = \frac{n+1}{2} kg_{ji}.$$

Then we obtain

$$(3.6) \quad R_{kjih} = \frac{k}{4} (g_{ji}g_{kh} - g_{ki}g_{jh} + F_{ji}F_{kh} - F_{ki}F_{jh} - 2F_{kj}F_{ih}),$$

which is the expression of constant holomorphic curvature, found by Prof.

K. Yano and I. Mogi [5], and corresponding to one introduced by S. Bochner [1], see also [2]. Thus we have

**Theorem 2.** *If a Kählerian space is h. p. flat, then it is of constant holomorphic curvature.*

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