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One parameter groups of unitary operators in a Hilbert space are well considered and many properties are known. In this note we consider families of unitary operators having somewhat weakened group-property — in article 1 families of unitary operators with one real parameter, and in article 2 families on Lie groups. These considerations will be applied to the theory of unitary representations of solvable Lie groups.

1. Let $\mathfrak{H}$ be a Hilbert space, and $U(t)$'s be unitary oporators on $\mathfrak{H}$ with one real parameter t (t moves from $-\infty$ to ∞).

If $U(t)$'s depend on t continuously with respect to strong (weak) topology, and moreover satisfy

$$U(t+s) = \text{const. times } U(t)U(s)$$

for any t, s $(-\infty < t,\ s < \infty)$, then we can reduce this family to a one parameter group of unitary operators $\{V(t)\,;\ -\infty < t < \infty\}$ by multiplying each $U(t)$ by constant.

In the first place we note that $U(0)$ is a constant operator, because of the identity

$$U(0) = U(0+0) = \text{const. times } U(0)U(0).$$

So we may multiply $U(0)^{-1}$ to all of $U(t)$'s which reduces our family to another one satisfying $U(0) = 1$. In like manner we may assume that $U(t)$'s satisfy

$$U(-t) = U(t)^* \qquad (-\infty < t < \infty)$$

where $*$ means the adjoint operator. Under these normalizations if we put

$$U(t)U(s) = \alpha(t, s)U(t+s), \qquad \alpha(t, s): \text{ const.}$$

for any t, s $(-\infty < t,\ s < \infty)$, $\alpha(t, s)$ must satisfy the followings.

(i) $\alpha(t, s)$ is a continuous function of t, s $(-\infty < t,\ s < \infty)$ with absolute value 1.

(ii) $\alpha(t, s)\alpha(t+s, u) = \alpha(t, s+u)\alpha(s, u)$.
(iii) $\alpha(0, t) = \alpha(t, 0) = 1$.
(iv) $\alpha(t, -t) = 1$.

Using this function we construct a group G. For that sake let G be the set of all the pairs

$$(t, \alpha),$$

where t runs through all the real numbers, and α runs through all the complex numbers of absolute value 1. In G we define the multiplication operation by

$$(t, \alpha)(s, \beta) = (t+s, \alpha(t, s)\alpha\beta).$$

Then as one sees easily G becomes a Lie group. It contains the subgroup C of all the elements of the form $(0, \alpha)$ as a central subgroup. The factor group G/C is isomorphic to the additive group of real numbers, and so G contains a one parameter subgroup R consisting of the elements of the form $(t, \alpha(t))$ $(-\infty < t < \infty)$, where $\alpha(t)$ depends continuously on t. From

$$\begin{aligned}(t+s, \alpha(t+s)) &= (t, \alpha(t))(s, \alpha(s)) \\ &= (t+s, \alpha(t, s)\alpha(t)\alpha(s)),\end{aligned}$$

$\alpha(t)$'s satisfy

$$\alpha(t)\alpha(s)\alpha(t, s)\alpha(t+s)^{-1} = 1 \qquad (-\infty < t, s < \infty).$$

Set

$$V(t) = \alpha(t)U(t) \qquad (-\infty < t < \infty),$$

then clearly $V(t)$'s depend on t continuously with respect to strong topology, and moreover satisfy

$$\begin{aligned}V(t)V(s) &= \alpha(t)\alpha(s)U(t)U(s) \\ &= \alpha(t)\alpha(s)\alpha(t, s)U(t+s) \\ &= \alpha(t)\alpha(s)\alpha(t, s)\alpha(t+s)^{-1}V(t+s) \\ &= V(t+s).\end{aligned}$$

So we arrive at last at the desired reduction.

2. Now we shall slightly generalize our proposition. In stead of assuming $\{U\}$ be the family on the additive group of real numbers, we now assume that it be the family on an arbitrary connected Lie group G. And we show that, *if $U(g)$'s depend on $g \in G$ continuously with respect to strong (weak) topology, and mereover satisfy*

$$U(gh) = \text{const. times } U(g)U(h)$$

for any $g, h \in G$, then these can be regarded as defining a representation of a Lie group G_0 which has one dimensional central subgroup C such that $G_0/C = G$.

The procedure is precisely the same. We may assume $U(e) = 1$, e being the identity of G, and on putting

$$U(g)U(h) = \alpha(g, h)U(gh),$$

$\alpha(g, h)$ satisfies the followings.

(i) $\alpha(g, h)$ is a continuous function on $G \times G$ with absolute value 1.

(ii) $\alpha(g, h)\alpha(gh, k) = \alpha(g, hk)\alpha(h, k)$.

(iii) $\alpha(e, g) = \alpha(g, e) = 1$.

(iv) $\alpha(g, g^{-1}) = \alpha(g^{-1}, g)$.

Let G_0 be the set of all the pairs

$$(g, \alpha)$$

where g runs through all the elements of G, and α runs through all the complex numbers of absolute value 1. The multiplication operation in G_0 is defined by

$$(g, \alpha)\ (h, \beta) = (gh, \ \alpha(g, h)\alpha\beta),$$

and G_0 thus turns into a connected lie group. G_0 has the subgroup C of all the elements of the form (e, α) as a central subgroup and $G_0/C = G$. On setting

$$W(g, \alpha) = \alpha U(g) \qquad ((g, \alpha) \in G_0),$$

it is easily seen that we get a representation of G_0 by unitary operators.

3. Before establishing the applications of the results obtained in articles 1, 2, we write here of the 3-dimensional nilpotent Lie group which shows peculiar behavier when represented by unitary operators.

The only non-trivial (non-abelian) 3-dimensional connected nilpotent Lie groups are the group H consisting of all the matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \quad (x,\ y,\ z \text{ run through all real numbers})$$

and its factor groups. So we only need to consider H. Its center Z consists of matrices of the form

$$\begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (z \text{ runs through all real numbers}).$$

When H is represented by unitary operators on a Hilbert space, and if in that time the center of H is represented non-trivially and by constant operators, then this representation is the direct product of an irreducible representation and a constant (*trivial*) *representation of some dimension.*

First of all we shall determine all the irreducible unitary representations by the method of induced representation due to G. W. Mackey[1]. Set K the abelian normal subgroup of H composed of all the elements of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (-\infty < x,\ z < \infty).$$

Take a character of this abelian group K. It has the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \exp i(x\xi + z\zeta)$$

for some real numbers ξ, ζ. The transformation induced on the character space of H from adjoint transformation by an element

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

moves this character to a new one which has the same form, ξ replaced

1) Cf. G. W. Mackey, *Imprimitivity for representations of locally compact groups I*, Proc. Nat. Acad. Sci., Vol. 35 (1945), pp. 539—545

by $\xi - b\zeta$. So the orbits in the character space of K under the group of transformations thus described are either points (when $\zeta = 0$), or straight lines (when $\zeta \neq 0$). The orbits are thus regular in Mackey's term, and his theory applies. The normalized form of irreducible representations are as follows.

1. $$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \exp\ i(x\xi + y\eta) \qquad (-\infty < x, y, z < \infty)$$

for some fixed real numbers ξ, η which is not other than the representation of abelian factor group H/Z.

2. Let $\mathfrak{H}_0 = L^2(-\infty, \infty)$, and on $\mathfrak{H}_0$ define $U_\zeta(x, y, z)$ $(\zeta \neq 0)$ by

$$f(X) \rightarrow \exp\ i(z + yX)\zeta \cdot f(X + x).$$

Clearly this is a unitary operator on $\mathfrak{H}_0$ and the mapping

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow U_\zeta(x, y, z) \qquad (-\infty < x, y, z < \infty)$$

defines an irreducible unitary representation of H on $\mathfrak{H}_0$.

According to Mackey's theory any irreducible unitary representation must be unitarily equivalent to one of these representations. The second type of representations has a remarkable property. It is completely determined by the representation of center. Indeed the central element

$$\begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is mapped into the operator which multiplies each element of $\mathfrak{H}_0$ by $e^{iz\zeta}$, and if ζ differs then this representation of Z differs.

Return now to our proposition afore-mentioned. As we assumed that under the representation now considering Z is represented non-trivially by constant operators, i. e. the operators multiplying by constants, e. g.

$$\begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow e^{iz\zeta} \times,\ ^{1)}$$

1) Under $\alpha\times$ we understand the operator which multiplies all the elements of the spaec by the constant α.

this representation, when decomposed into the direct integral sum of irreducible representations as was done by Mautner[1], has only the equivalent irreducible component representations, for under each component representation the center Z is likewise represented by

$$\begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow e^{iz\xi}\times .$$

Now a theorem of Mautner[2] applies, and our representation must have the form mentioned in the proposition.

4. Here I briefly speak about the theory of unitary representations of solvable Lie groups, and in connection with this we shall note the application of the results of the preceeding articles.

To construct the general representation theory of solvable Lie groups is not so easy. The reason why so will undoubtedly be that the structure of solvable groups is not so simple, and is not even well known for one to speak of compared with that of semi-simple groups. It will be desirable that, when a group is given concretely, one can construct all the irreducible representations by some standard method, and for that purpose the most powerful tool will be the method of induced representation. It enables us in some cases to reduce the argument of a group of some dimension to that of groups of lower dimension, but other cases may happen which cause difficulties in so arguing. Another case in which we cannot use the method of induced representation so effectively and yet the whole argument goes successfully is what we want to mention below.

Let G be a simply connected solvable Lie group which contains as a normal subgroup the group H mentioned in article 3. Moreover suppose that the center Z of H is contained in the center of G.

Consider a unitary representation $g \rightarrow U(g)$ of G on a Hilbert space $\mathfrak{H}$ under which the center of G is mapped into constant operators — for example irreducible representation, factor representation. Z is then represented either trivially or non-trivially. The former case is without interest because in this case it turns out to argue the representation of a group of lower dimension. In the latter case we shall show that

1) Cf. F. I. Mautner, *Unitary representations of locally compact groups* I, Ann, Math., Vol. 51 (1950), pp. 1—25; II, Ann. Math., Vol. 52 (1950), pp. 528—556.

2) Cf. F. I. Mautner, loc. cit. II, Theorem 3.1, p. 539.

$\mathfrak{H}$ *is decomposed into the direct product of two Hilbert spaces* $\mathfrak{H}_1$ *and* $\mathfrak{H}_2$, *and in that time* $U(g)$*'s are likewise decomposed into the direct products of unitary operators on* $\mathfrak{H}_1$ *and* $\mathfrak{H}_2$, *and the whole argument is ultimately reduced to that of groups of lower dimension.* The situation is very likely in the case of finite dimensional representation[1)] which will not be expected generally in the case of unitary representation.

Z being isomorphic to the additive group of real numbers, we regard its element as real numbers, and under the representation considered let

$$z \to e^{iz\zeta}\times \qquad (z \in Z,\ \zeta \neq 0).$$

Then as we saw in article 3, $\mathfrak{H}$ is decomposed into the direct product of two Hilbert spaces $\mathfrak{H}_1$, $\mathfrak{H}_2$ and in the same time the operator $U(h)$ is decomposed into the form $U_1(h)\otimes 1$, where $U_1(h)$ is a unitary operator on $\mathfrak{H}_1$. $h \to U_1(h)$ defines an irreducible unitary representation under which $z \in Z$ is represented by $e^{iz\zeta}\times$.

Take a $g \in G$. As $h \to U_1(ghg^{-1})$ defines also an irreducible unitary representation of H, and under which $z \in Z$ is represented likewise by $e^{iz\zeta}\times$, this representation must be unitarily equivalent to the representation $h \to U_1(h)$. So there exists a unitary operator $A_1(g)$ on $\mathfrak{H}_1$ so that we have

$$U_1(ghg^{-1}) = A_1(g)U_1(h)A_1(g)^{-1} \qquad \text{for any } h \in H.$$

$A_1(g)$ is determined up to constant multipliers, and satisfies

$$A_1(g)A_1(g') = \text{const. times } A_1(gg').$$

Set $A(g) = A_1(g)\otimes 1$. Then clearly $B(g) = A(g)^{-1}U(g)$ commutes with all of $U(h)$'s ($h \in H$), and thus it has the form $1\otimes B_2(g)$, where $B_2(g)$ is a unitary operator on $\mathfrak{H}_2$. We have then

$$U(g) = A_1(g)\otimes B_2(g).$$

We here note that $A_1(g)$'s and thus also $B_2(g)$'s can be determined so as to depend continuously on g. For that purpose take first a one parameter subgroup $g(t)$ $(-\infty < t < \infty)$ in G, and elements $\varphi_0 \neq 0$ in $\mathfrak{H}_1$, ψ_0 ($\|\psi_0\| = 1$) in $\mathfrak{H}_2$. Choosing suitably $\delta > 0$, we have

$$\begin{aligned}&(U(g(t))\,(\varphi_0\otimes\psi_0),\ \varphi_0\otimes\psi_0)\\ &\quad = (A_1(g(t))\varphi_0,\ \varphi_0)\,(B_2(g(t))\psi_0,\ \psi_0) \neq 0 \qquad (0 \leqq t \leqq \delta),\end{aligned}$$

1) Cf. e.g. A. H. Clifford, *Representations induced in an invariant subgroup*, Ann. Math., Vol. 38 (1937), pp. 533—550.

so we can determine uniquely $B_2(g(t))$'s $(0 \leqq t \leqq \delta)$ by

$$(B_2(g(t))\psi_0, \psi_0) > 0 \qquad \text{for } 0 \leqq t \leqq \delta.$$

Let E be the projection operator on $\mathfrak{H}$ defined by

$$E(\varphi \otimes \psi) = \varphi \otimes (\psi, \psi_0)\psi_0 \qquad (\varphi \in \mathfrak{H}_1, \ \psi \in \mathfrak{H}_2).$$

Then

$$\begin{aligned} \| EU(g(t))\varphi \otimes \psi_0 \| &= (B_2(g(t))\psi_0, \psi_0) \| A_1(g(t))\varphi \| \ \| \psi_0 \| \\ &= (B_2(g(t))\psi_0, \psi_0) \| \varphi \| \qquad (0 \leqq t \leqq \delta). \end{aligned}$$

The left-hand side of this equation being continuous with respect to t, the right-hand side is also continuous with respect to t for $0 \leqq t \leqq \delta$. Thus from

$$\begin{aligned} &(U(g(t))\varphi \otimes \psi_0, \ \varphi' \otimes \psi_0) \\ &= (A_1(g(t)) \ \varphi, \varphi') \ (B_2(g(t))\psi_0, \psi_0) \qquad \text{for } \varphi, \ \varphi' \in \mathfrak{H}_1 \end{aligned}$$

we have that $A_1(g(t))$'s depend continuously on t $(0 \leqq t \leqq \delta)$, and from

$$\begin{aligned} &(U(g(t))\varphi_0 \otimes \psi, \varphi_0 \otimes \psi') \\ &= (A_1(g(t))\varphi_0, \varphi_0) \ (B_2(g(t))\psi, \ \psi') \qquad \text{for } \psi, \ \psi' \in \mathfrak{H}_2 \end{aligned}$$

we have that $B_2(g(t))$'s depend continuously on t $(0 \leqq t \leqq \delta)$. For arbitrary $t > 0$, let $t = n\delta + t'$ $(0 \leqq t' < \delta)$ and set

$$A_1(g(t)) = A_1(g(\delta))^n A_1(g(t')),$$

$$B_2(g(t)) = B_2(g(\delta))^n B_2(g(t')).$$

For $t < 0$, set

$$\begin{aligned} A_1(g(t)) &= A_1(g(-t))^*, \\ B_2(g(t)) &= B_2(g(-t))^*. \end{aligned}$$

Then $A_1(g(t))$'s, $B_2(g(t))$'s thus determined depend continuously on t and satisfy the other conditions imposed on.

As we assumed that G is a simply connected solvable Lie group, by choosing suitably one-parameter subgroup $g_1(t)$, $g_2(t), \dots,$ $g_r(t)$ in G, an arbitrary element g in G can be uniquely written down in the form

$$g = g_1(t_1)g_2(t_2) \cdots g_r(t_r)h \qquad (h \in H)$$

and here $t_1, t_2, \ldots, t_r, h$ depend continuously on g. Determine now $A_1(g_k(t))$, $B_2(g_k(t))$ $(k=1, 2, \ldots, r)$ as we showed before so as to depend continuously on t, and set

$$A_1(g)=A_1(g_1(t_1))A_1(g_2(t_2))\cdots A_1(g_r(t_r))U_1(h),$$
$$B_2(g)=B_2(g_1(t_1))B_2(g_2(t_2))\cdots B_2(g_r(t_r)).$$

Then these $A_1(g)$'s, $B_2(g)$'s are what we wanted to obtain.

$A_1(g)$'s, $B_2(g)$'s thus determined have the following properties.

(i) $A_1(g)$'s, $B_2(g)$'s depend continuously on $g \in G$.

(ii) $A_1(g)\, A_1(g')=\alpha(g, g')A_1(gg')$,

$B_2(g)B_2(g')=\alpha(g, g')^{-1}B_2(gg')$,

where $\alpha(g, g')$ denotes a constant.

(iii) $U(g)=A_1(g)\otimes B_2(g)$.

The function $\alpha(g, g')$ on $G\times G$ or rather on $G/H\times G/H$ is in essence unique, and proper to the group G. It satisfies the conditions mentioned in article 2. Thus we can say as follows. *To determine all the representations of the mentioned type is reduced to determine the representaions of the group G_0, constructed from G/H and the above-mentioned function $\alpha(g, g')$ in the manner as in article 2, of such type under which the central element $(\bar{e}, \alpha)$ ($\bar{e}$ being the identity of G/H) is represented by α.*

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