

Mathematical Journal of Okayama University

Volume 45, Issue 1

2003

Article 10

JANUARY 2003

On a Question of Furuta on Chaotic Order, II

Masatoshi Fuji*

Eizaburo Kamei†

Ritsuo Nakamoto‡

*Osaka Kyoiku University

†Maebashi Institute of Technology

‡Ibaraki University

On a Question of Furuta on Chaotic Order, II

Masatoshi Fuji, Eizaburo Kamei, and Ritsuo Nakamoto

Abstract

The chaotic order $A \gg B$ among positive invertible operators on a Hilbert space is introduced by $\log A \geq \log B$. Related to the Furuta inequality for the chaotic order, Furuta posed the following question: For $A, B > 0$, $A \gg B$ if and only if holds for all $p \geq 1$, $r \geq t$, $s \geq 1$ and $t \in [0, 1]$? Recently he gave a counterexample to the “only if” part. In our preceding note, we pointed out that the condition (Q) characterizes the operator order $A \geq B$. Moreover we showed that (Q) characterizes the chaotic order in some sense. The purpose of this note is to continue our preceding discussion on the operator inequality (Q) under the chaotic order. Among others, we prove that if $A \gg B$ for $A, B > 0$, then for $p \geq 1$, $s \geq 1$, $r \geq 0$ and $t \leq 0$, where $A_s B = A$ and particularly $A_s B = A$ for $s \in (0, 1)$. $\frac{1}{2}(A \# B) \geq A^{\frac{1}{2}} B A^{\frac{1}{2}}$ and particularly $A_s B = A$ for $s \in [0, 1]$.

KEYWORDS: Furuta inequality, grand Furuta inequality, chaotic order and chaotic Furuta inequality.

Math. J. Okayama Univ. **45** (2003), 123–131

ON A QUESTION OF FURUTA ON CHAOTIC ORDER, II

MASATOSHI FUJII, EIZABURO KAMEI AND RITSUO NAKAMOTO

ABSTRACT. The chaotic order $A \gg B$ among positive invertible operators on a Hilbert space is introduced by $\log A \geq \log B$. Related to the Furuta inequality for the chaotic order, Furuta posed the following question: For $A, B > 0$, $A \gg B$ if and only if

$$(Q) \quad A^{r-t} \geq \{A^{\frac{r}{2}}(A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}})^sA^{\frac{r}{2}}\}^{\frac{r-t}{(p-t)s+r}}$$

holds for all $p \geq 1$, $r \geq t$, $s \geq 1$ and $t \in [0, 1]$? Recently he gave a counterexample to the "only if" part. In our preceding note, we pointed out that the condition (Q) characterizes the operator order $A \geq B$. Moreover we showed that (Q) characterizes the chaotic order in some sense. The purpose of this note is to continue our preceding discussion on the operator inequality (Q) under the chaotic order. Among others, we prove that if $A \gg B$ for $A, B > 0$, then

$$A^{t-r} \sharp_{\frac{1+r-t}{(p-t)s+r}} (A^t \sharp_s B^p) \leq B$$

for $p \geq 1$, $s \geq 1$, $r \geq 0$ and $t \leq 0$, where $A \sharp_s B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^sA^{\frac{1}{2}}$ and particularly $\sharp_s = \sharp_s$ for $s \in [0, 1]$.

1. INTRODUCTION

In succession with the Furuta inequality [11], Furuta [14] proposed the interpolational inequality between the Ando-Hiai inequality [2] and the Furuta inequality, which is called the grand Furuta inequality in [8]. See [10], [15], [16], [23] and [25]. For convenience, we denote by $A > 0$ if A is a positive invertible operator on a Hilbert space and by $A \gg B$ for $A, B > 0$ if $\log A \geq \log B$, which is called the chaotic order in [7].

The grand Furuta inequality. *If $A \geq B > 0$, then for each $t \in [0, 1]$,*

$$(G) \quad A^{1-t+r} \geq \{A^{\frac{r}{2}}(A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}$$

holds for all $s \geq 1$, $p \geq 1$ and $r \geq t$.

Recently Furuta solved the following question posed by himself [17]:

Furuta's question. *For $A, B > 0$, $A \gg B$ if and only if*

$$(Q) \quad A^{r-t} \geq \{A^{\frac{r}{2}}(A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}})^sA^{\frac{r}{2}}\}^{\frac{r-t}{(p-t)s+r}}$$

holds for all $p \geq 1$, $r \geq t$, $s \geq 1$ and $t \in [0, 1]$?

Mathematics Subject Classification. 47A63.

Key words and phrases. Furuta inequality, grand Furuta inequality, chaotic order and chaotic Furuta inequality.

The principal part of this question is "only if", for which he gave a negative answer by finding the conterexample $A = e^X$ and $B = e^Y$, where

$$(1) \quad X = \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 1 & 3 \\ 3 & -2 \end{pmatrix}.$$

Then $\log A = X \geq Y = \log B$ and (Q) does not hold for $r = 2$, $t = 1$, $s = 2$ and $p = 2$.

In our preceding note [9], we pointed out that "(Q) holds for all $p \geq 1$, $r \geq t$, $s \geq 1$ and $t \in [0, 1]$ " characterizes the operator order $A \geq B$ for $A, B > 0$ and consequently it is strictly stronger than the chaotic order $A \gg B$. Moreover we discussed a characterization of chaotic order in the frame of operator inequalities of grand Furuta type. Very recently, Furuta improved our characterization of the chaotic order which is a typical application of the Furuta inequality, Theorems 2.1 and 2.2 in [18].

Inspired by Furuta's improvement, we consider operator inequalities of grand Furuta type under the chaotic order in this note. As a matter of fact, we propose the following inequality as a satellite of the chaotic grand Furuta inequality: If $A \gg B$ for $A, B > 0$, then

$$A^{t-r} \sharp_{\frac{1+r-t}{(p-t)s+r}} (A^t \natural_s B^p) \leq B$$

for $p \geq 1$, $s \geq 1$, $r \geq 0$ and $t \leq 0$, where

$$A \natural_s B = A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^s A^{\frac{1}{2}}$$

and particularly $\sharp_s = \natural_s$ for $s \in [0, 1]$. Furthermore we discuss some extensions of the above inequality.

Concluding this section, we have to say that our viewpoint depends on a satellite inequality [19] to the Furuta inequality; if $A \geq B > 0$, then

$$(2) \quad A^{-r} \sharp_{\frac{1+r}{p+r}} B^p \leq B \leq A \leq B^{-r} \sharp_{\frac{1+r}{p+r}} A^p$$

for $p \geq 1$ and $r \geq 0$.

2. FURUTA'S IMPROVEMENT

The origin of the question (Q) may come from the chaotic Furuta inequality (FC), i.e., for $A, B > 0$, $A \gg B$ if and only if

$$A^r \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{r}{p+r}}$$

holds for all $p, r \geq 0$, see [4], [13], [6], [26] and Uchiyama [24]. The meaning of (FC) becomes clear if one compares with the Furuta inequality (FI), [11] and see [3], [12], [19] and [22]: If $A \geq B \geq 0$, then

$$A^{1+r} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1+r}{p+r}}$$

holds for all $p \geq 1$, $r \geq 0$.

For convenience, we cite the original form of the Furuta inequality:

Furuta inequality: If $A \geq B \geq 0$,
then for each $r \geq 0$,

$$(A^{\frac{r}{2}} A^p A^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$$

and

$$(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}} B^p B^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for p and q such that $p \geq 0$
and $q \geq 1$ with

$$(1+r)q \geq p+r.$$

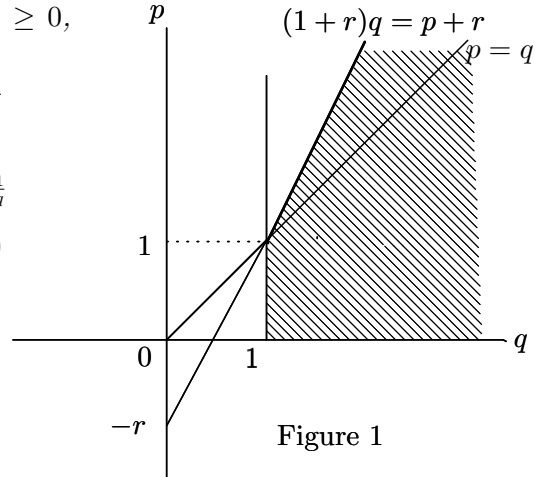


Figure 1

We remark that Furuta's question can be rephrased as the existence of a parallelism between (FI) - (FC) and (G) - (Q), where (FI) and (G) are considered under the operator order, and (FC) and (Q) are done under the chaotic order. Unfortunately our result says that it is incomplete for the case $t \in [0, 1]$. So we paid our attention to the bounds of t in Furuta's question (Q) and showed characterizations of the chaotic order and the operator order under replacing to $t \leq 0$. Furuta [18] improved them, in which $1 \leq s \leq 2$ is improved to $s \geq 1$. That is,

Theorem A. For $A, B > 0$, $A \gg B$ if and only if (Q) holds for $p \geq 0$, $r \geq 0$, $s \geq 1$ and $t \leq 0$.

Theorem B. For $A, B > 0$, $A \geq B$ if and only if (G) holds for $p \geq 1$, $r \geq 0$, $s \geq 1$ and $t \leq 0$.

Incidentally Theorem B is shown in Cor. 5. Theorem A is proved by a combination of (FC) and (FI) as follows: If $A \gg B$, then

$$(3) \quad A_1 = A^{-t} \geq (A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^{\frac{-t}{p-t}} = B_1$$

for $p \geq 0$ and $t \leq 0$. Applying (FI) to $A_1 \geq B_1 > 0$, it follows that

$$A_1^{1+r_1} \geq (A_1^{\frac{r_1}{2}} B_1^{p_1} A_1^{\frac{r_1}{2}})^{\frac{1+r_1}{p_1+r_1}}$$

for $p_1 \geq 1$ and $r_1 \geq 0$. Since we may assume $t \neq 0$ by (FC), we put

$$p_1 = \frac{(p-t)s}{-t} \geq 1 \quad \text{and} \quad r_1 = \frac{r}{-t} \geq 0,$$

so that we have the conclusion (Q) for $p \geq 0$, $r \geq 0$, $s \geq 1$ and $t \leq 0$.

3. MEAN THEORETIC APPROACH

We begin with rephrasing (Q) and (G) in terms of operator means respectively;

$$(QM) \quad A^{t-r} \sharp_{\frac{r-t}{(p-t)s+r}} (A^t \natural_s B^p) \leq 1$$

and

$$(GM) \quad A^{t-r} \sharp_{\frac{1+r-t}{(p-t)s+r}} (A^t \natural_s B^p) \leq A.$$

Thus Theorems A and B are represented as follows:

Theorem AM. For $A, B > 0$, $A \gg B$ if and only if (QM) holds for $p \geq 0$, $r \geq 0$, $s \geq 1$ and $t \leq 0$.

Theorem BM. For $A, B > 0$, $A \geq B$ if and only if (GM) holds for $p \geq 1$, $r \geq 0$, $s \geq 1$ and $t \leq 0$.

On the other hand, (FC) is rephrased, too: For $A, B > 0$, $A \gg B$ if and only if

$$(FCM) \quad A^{-r} \sharp_{\frac{r}{p+r}} B^p \leq 1 \quad \text{for } p, r \geq 0.$$

We here note that (FCM) is the special case $t = 0$ and $s = 1$ in (QM). It is pointed out in [5, 13, 21] that (FC) implies the following Furuta type operator inequality which is nothing but the first inequality in (2).

Theorem C. If $A \gg B$ for $A, B > 0$, then

$$(SM) \quad A^{-r} \sharp_{\frac{1+r}{p+r}} B^p \leq B \quad \text{for } p \geq 1 \text{ and } r \geq 0.$$

As mentioned in [21], Theorem C automatically connects to the Furuta inequality because (FI) is expressed as follows: If $A \geq B \geq 0$, then

$$(FIM) \quad A^{-r} \sharp_{\frac{1+r}{p+r}} B^p \leq A \quad \text{for } p \geq 1 \text{ and } r \geq 0.$$

Under such circumstances among (QM), (GM), (FCM), (FIM) and (SM), we propose the following inequality under the chaotic order, which implies Theorem B as well as Theorem C does (FIM) under the assumption $A \geq B \geq 0$.

Theorem 1. If $A \gg B$ for $A, B > 0$, then

$$A^{t-r} \sharp_{\frac{1+r-t}{(p-t)s+r}} (A^t \natural_s B^p) \leq A^t \sharp_{\frac{1-t}{p-t}} B^p \leq B$$

holds for $p \geq 1$, $s \geq 1$, $r \geq 0$ and $t \leq 0$. Consequently, if $A \geq B > 0$, then

$$A^{t-r} \sharp_{\frac{1+r-t}{(p-t)s+r}} (A^t \natural_s B^p) \leq A$$

holds for $p \geq 1$, $s \geq 1$, $r \geq 0$ and $t \leq 0$.

Proof. Fix arbitrary $p \geq 1$, $s \geq 1$, $r \geq 0$ and $t \leq 0$. Now (FC) implies that

$$(4) \quad A^{-t} \geq (A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^{\frac{-t}{p-t}} \quad \text{and so} \quad A^{-t} \gg (A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^{\frac{-t}{p-t}}.$$

Hence we have

$$(5) \quad A_1 = A^{1-t} \gg (A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^{\frac{1-t}{p-t}} = B_1.$$

Since $p_1 = \frac{(p-t)s}{1-t} \geq 1$ and $t_1 = -\frac{r}{1-t} \leq 0$, we can apply Theorem C to $p_1 \geq 1$, $r_1 = -t_1 \geq 0$ and $A_1 \gg B_1$; that is,

$$(6) \quad A_1^{t_1} \#_{\frac{1-t_1}{p_1-t_1}} B_1^{p_1} \leq B_1.$$

Applying this, we have

$$A^{-r} \#_{\frac{1+r-t}{(p-t)s+r}} (A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^s \leq (A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^{\frac{1-t}{p-t}},$$

so that

$$A^{t-r} \#_{\frac{1+r-t}{(p-t)s+r}} (A^t \natural_s B^p) \leq A^t \#_{\frac{1-t}{p-t}} B^p \leq B,$$

where the final inequality is just assured by Theorem C. $\square$

Corollary 2. *If $A \gg B$ for A , $B > 0$ and $\alpha > 0$, then*

$$A^{t-r} \#_{\frac{\alpha+r-t}{(p-t)s+r}} (A^t \natural_s B^p) \leq A^t \#_{\frac{\alpha-t}{p-t}} B^p \leq B^\alpha$$

holds for $p \geq \alpha$, $r \geq 0$, $s \geq 1$ and $t \leq 0$.

Proof. Put $A_1 = A^\alpha$, $B_1 = B^\alpha$, $p_1 = \frac{p}{\alpha} \geq 1$, $r_1 = \frac{r}{\alpha} \geq 0$ and $t_1 = \frac{t}{\alpha} \leq 0$. Then it follows from Theorem 1 that

$$A_1^{t_1-r_1} \#_{\frac{1+r_1-t_1}{(p_1-t_1)s+r_1}} (A_1^{t_1} \natural_s B_1^{p_1}) \leq A_1^{t_1} \#_{\frac{1-t_1}{p_1-t_1}} B_1^{p_1} \leq B_1,$$

so that we have the conclusion. $\square$

In Corollary 2, we put $s = \frac{\beta-t}{p-t}$ and $u = t-r$. Then we have the following inequalities, which are equivalent. The following reformulation in Corollary 3 will interpretate the meaning of Corollary 2 well, as in the Figure of the next section.

Corollary 3. *If $A \gg B$ for A , $B > 0$ and $\beta > \alpha > 0$, then*

$$A^u \#_{\frac{\alpha-u}{\beta-u}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \leq A^t \#_{\frac{\alpha-t}{p-t}} B^p \leq B^\alpha$$

and

$$B^u \#_{\frac{\alpha-u}{\beta-u}} (B^t \natural_{\frac{\beta-t}{p-t}} A^p) \geq B^t \#_{\frac{\alpha-t}{p-t}} A^p \geq A^\alpha$$

hold for $\alpha \leq p \leq \beta$ and $u \leq t \leq 0$.

4. A COMPLEMENTARY OPERATOR INEQUALITY

In this section, we consider the case $\alpha < 0$ in Corollary 3. For this, we prepare an operator inequality complementary to Theorem C.

Lemma 4. *If $A \gg B$ for $A, B > 0$ and $\alpha \leq 0$, then*

$$A^t \sharp_{\frac{\alpha-t}{p-t}} B^p \leq A^\alpha$$

holds for $p \geq 0$ and $t \leq \alpha$.

Proof. Since we may assume $t < \alpha < 0$ by (FC), it follows from (FC) that

$$A^t \sharp_{\frac{\alpha-t}{p-t}} B^p = A^t \sharp_{\frac{\alpha-t}{-t}} (A^t \sharp_{\frac{-t}{p-t}} B^p) \leq A^t \sharp_{\frac{\alpha-t}{-t}} 1 = A^\alpha,$$

where we used the multiplicativity of $\sharp_\gamma$, i.e., $A \sharp_{st} B = A \sharp_s (A \sharp_t B)$. $\square$

Theorem 5. *If $A \gg B$ for $A, B > 0$ and $\alpha \leq 0 \leq \beta$, then*

$$A^u \sharp_{\frac{\alpha-u}{\beta-u}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \leq A^t \sharp_{\frac{\alpha-t}{p-t}} B^p \leq A^\alpha$$

and

$$B^u \sharp_{\frac{\alpha-u}{\beta-u}} (B^t \natural_{\frac{\beta-t}{p-t}} A^p) \geq B^t \sharp_{\frac{\alpha-t}{p-t}} A^p \geq B^\alpha$$

hold for $0 \leq p \leq \beta$ and $u \leq t \leq \alpha$.

Proof. The proof is quite similar to that of Theorem 1. Since

$$(A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^{\frac{1}{p-t}} \ll A$$

by (FC), we have

$$B_1 = (A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^{\frac{\alpha-t}{p-t}} \ll A^{\alpha-t} = A_1.$$

Therefore it follows from Theorem C that

$$A_1^{t_1} \sharp_{\frac{1-t_1}{p_1-t_1}} B_1^{p_1} \leq B_1$$

for $p_1 \geq 1$ and $t_1 \leq 0$. So we put $p_1 = \frac{\beta-t}{\alpha-t}$ and $t_1 = \frac{u-t}{\alpha-t}$, that is,

$$A^{u-t} \sharp_{\frac{\alpha-u}{\beta-u}} (A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^{\frac{\beta-t}{p-t}} \leq (A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^{\frac{\alpha-t}{p-t}}.$$

Arranging it and applying Lemma 4, we have

$$A^u \sharp_{\frac{\alpha-u}{\beta-u}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \leq A^t \sharp_{\frac{\alpha-t}{p-t}} B^p \leq A^\alpha,$$

which is the required inequality. The latter is obtained by $B^{-1} \gg A^{-1}$. $\square$

Remark 6. Finally we note the remarkable contrast between the second inequalities in Corollary 3 and Theorem 5. That is, suppose that $A \gg B$ for $A, B > 0$. Then the following (1) and (2) hold.

(1) If $\beta > \alpha > 0$, then

$$B^u \#_{\frac{\alpha-u}{\beta-u}} (B^t \#_{\frac{\beta-t}{p-t}} A^p) \geq B^t \#_{\frac{\alpha-t}{p-t}} A^p \geq A^\alpha$$

for $\alpha \leq p \leq \beta$ and $u \leq t \leq 0$.

(2) If $\alpha \leq 0 \leq \beta$, then

$$B^u \#_{\frac{\alpha-u}{\beta-u}} (B^t \natural_{\frac{\beta-t}{p-t}} A^p) \geq B^t \#_{\frac{\alpha-t}{p-t}} A^p \geq B^\alpha$$

for $0 \leq p \leq \beta$ and $u \leq t \leq \alpha$.

We here draw the path $\{X \#_c Y; c \in [0, 1]\}$ as the curve combining X and Y in the following figure. Then it may express this contrast in (1) and (2) by setting the line $x = \alpha$. In each case $\alpha < 0$ or $\alpha > 0$, the order among the upper three points of intersection with $x = \alpha$ is preserved for positive operators. Incidentally, since the assumption $A \gg B$ does not imply $A^\alpha \geq B^\alpha$ for $\alpha > 0$ and $B^\alpha \geq A^\alpha$ for $\alpha < 0$, the order among the four points of intersection with $x = \alpha$ does not hold, contrary to the numerical case.

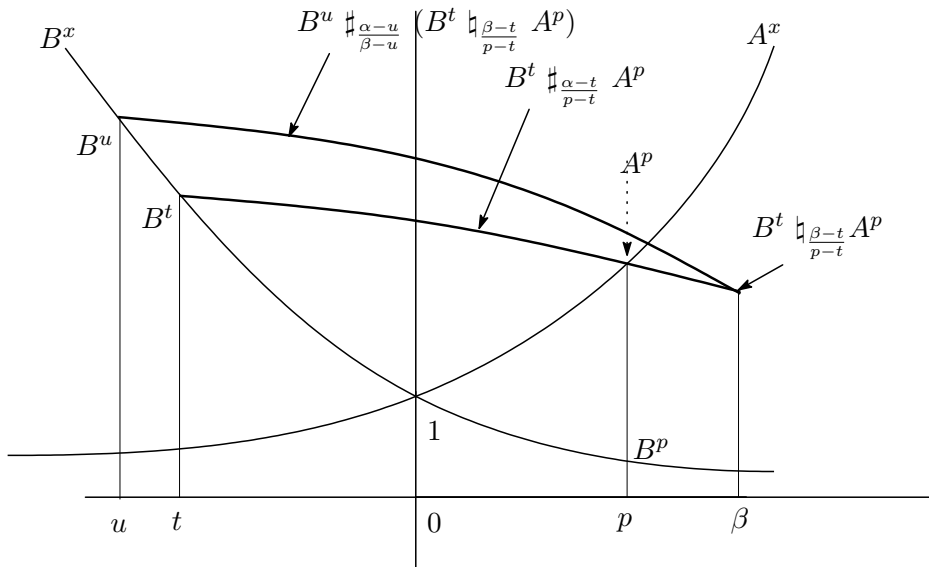


Figure 2

Acknowledgement. The authors would like to express their thanks to the referee for valuable suggestions and to Professor Furuta for giving an opportunity to read [18] before publication.

REFERENCES

- [1] T. ANDO, *On some operator inequality*, Math. Ann., **279**(1987), 157–159.
- [2] T. ANDO AND F. HIAI, *Log majorization and complementary Golden-Thompson type inequalities*, Linear Algebra Appl., 197/198(1994), 113–131.
- [3] M. FUJII, *Furuta's inequality and its mean theoretic approach*, J. Operator theory, **23**(1990), 67–72.
- [4] M. FUJII, T. FURUTA AND E. KAMEI, *Furuta's inequality and its application to Ando's theorem*, Linear Alg. and Its Appl., **179**(1993), 161–169.
- [5] M. FUJII, T. FURUTA AND D. WANG, *An application of the Furuta inequality to operator inequalities on chaotic orders*, Math. Japon., **40**(1994), 317–321.
- [6] M. FUJII, J.-F. JIANG, E. KAMEI AND K. TANAHASHI, *A characterization of chaotic order and a problem*, J. Inequal. Appl., **2**, (1998), 149–156.
- [7] M. FUJII AND E. KAMEI, *Furuta's inequality for the chaotic order I, II*, Math. Japon., **36**(1991), 603–606 and 717–722.
- [8] M. FUJII AND E. KAMEI, *Mean theoretic approach to the grand Furuta inequality*, Proc. Amer. Math. Soc., **124**(1996), 2751–2756.
- [9] M. FUJII, E. KAMEI AND R. NAKAMOTO, *On a question of Furuta on chaotic order*, Linear Alg. and Its Appl, **341**(2002), 119–127.
- [10] M. FUJII, A. MATSUMOTO AND R. NAKAMOTO, *A short proof of the best possibility for the grand Furuta inequality*, J.Inequal. Appl.,**4**(1999), 339–344.
- [11] T. FURUTA, *$A \geq B \geq 0$ assures $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0, p \geq 0, q \geq 1$ with $(1+2r)q \geq p+2r$* , Proc. Amer. Math. Soc., **101**(1987), 85–88.
- [12] T. FURUTA, *Elementary proof of an order preserving inequality*, Proc. Japan Acad., **65**(1989), 126.
- [13] T. FURUTA, *Applications of order preserving operator inequalities*, Operator Theoriey: Advances and Applications, **59**(1992), 180–190.
- [14] T. FURUTA, *Extension of the Furuta inequality and Ando-Hiai log-majorization*, Linear Alg. and Its Appl., **219**(1995), 139–155.
- [15] T. FURUTA, *Simplified proof of an order preserving operator inequality*, Proc. Japan Acad., **74**, Ser. A(1998), 114.
- [16] T. FURUTA, *Operator functions involving order preserving inequalities*, Sci. Math., **1**(1998), 141–147.
- [17] T. FURUTA, *Results under $\log A \geq \log B$ can be derived from ones under $A \geq B \geq 0$ by Uchiyama's method - associated with Furuta and Kantorovich type operator inequalities -*, Math. Inequal. Appl., **3**(2000), 423–436.
- [18] T. FURUTA, *$A \geq B > 0$ ensures $A^{1+r-t} \geq \{A^{\frac{r}{2}}(A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1+r-t}{(p-t)s+r}}$ for $t \in [0, 1], r \geq t, p \geq 1, s \geq 1$ and related inequalities*, Archiv. Inequal., Appl., to appear.
- [19] E. KAMEI, *A satellite to Furuta's inequality*, Math. Japon., **33**(1988), 883–886.
- [20] E. KAMEI, *Chaotic order and Furuta inequality*, Sci. Math., **53**(2001), 221–225.
- [21] E. KAMEI AND M. NAKAMURA, *Remark on chaotic Furuta inequality*, Sci. Math., **53**(2001), 535–539.
- [22] K. TANAHASHI, *Best possibility of the Furuta inequality*, Proc. Amer. Math. Soc., **124**(1996), 141–146.
- [23] K. TANAHASHI, *The best possibility of the grand Furuta inequality*, Proc. Amer. Math. Soc., **128**(2000), 511–519.
- [24] M. UCHIYAMA, *Some exponential operator inequalities*, Math. Inequal. Appl., **2**(1999), 469–471.

- [25] T. YAMAZAKI, *Simplified proof of Tanahashi's result on the best possibility of generalized Furuta inequality*, Math. Inequal. Appl., **2**(1999), 473–477.
- [26] M. YANAGIDA, *Some applications of Tanahashi's result on the best possibility of Furuta inequality*, Math. Inequal. Appl., **2**(1999), 297–305.

MASATOSHI FUJII

DEPARTMENT OF MATHEMATICS

OSAKA KYOIKU UNIVERSITY

ASAHIGAOKA, KASHIWARA, OSAKA 582-8582, JAPAN

e-mail address: mfujii@@cc.osaka-kyoiku.ac.jp

EIZABURO KAMEI

MAEBASHI INSTITUTE OF TECHNOLOGY

KAMISADORI, MAEBASHI, GUNMA 371-0816, JAPAN

e-mail address: kamei@@maebashi-it.ac.jp

RITSUO NAKAMOTO

FACULTY OF ENGINEERING

IBARAKI UNIVERSITY

HITACHI, IBARAKI 316-0033, JAPAN

e-mail address: nakamoto@@base.ibaraki.ac.jp

(Received April 4, 2003)